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Contribution and response functions for Stokes line profiles formed in a magnetic field

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Summary. Expressions defining the contribution functions of the "line depression" Stokes profiles formed in a general magnetic field are derived. This definition is valid for a general atmosphere. Such contribution functions are better suited to determining the heights at which the bound-bound transitions responsible for spectral lines are important than the contribution functions to the "intensity" Stokes parameters defined by Van Ballegooijen (1985). Expressions defining response functions for both "intensity" and "line depression" Stokes parameters are also derived for an arbitrary atmosphere and magnetic field. A code for calculating the various Stokes contribution functions is described and some example calculations are presented. These clearly demonstrate the superior diagnostic value of the contribution functions to the "line depression" Stokes profiles.

In an Appendix an analytical calculation of the contribution functions for a special case (absorption matrix independent of optical depth) is presented. A major result of this analytical calculation is a simple relationship between the contribution functions of the Stokes I and V profiles for weak fields.

Key words: lines: formation – polarization – radiation transfer – stars: magnetic field sun: magnetic field

1. Introduction

Stellar spectral lines are generally analysed in order to probe the atmospheres in which they are formed. Therefore it is important to know just at which depth in the atmosphere a spectral line, or, to be more precise, a specific part of the spectral line is "formed". A rich and, for that matter, controversial literature exists on the subject (e.g. De Jager, 1952; Pecker, 1952; Mein, 1971; Gurtovenko et al., 1974; Beckers and Milkey, 1975; Caccin et al., 1977; Makita, 1977; Gurtovenko and Sheminova, 1983; Magain, 1986). Two different approaches have been proposed to solve the problem, one uses a "Contribution Function", the other a "Response Function". While it seems generally accepted by now that the two concepts are complementary to each other, so that the choice between them should be made according to the kind of information one wants to extract from an observed line profile, the exact form of these functions has been quite controversial until rather recently. We feel that the issue has been settled, at least in principle, by the work of Magain (1986) who developed plausible selection criteria for both functions. Thus it is now possible to obtain unambiguous information on structural details of stellar atmospheres from the analysis of line profiles.

When it comes to the interpretation of the Stokes parameters which emerge at the top of an atmosphere pervaded by a magnetic field the situation is less satisfactory. Until recently only two attempts to assess the relative importance of an atmospheric layer for the creation of a Stokes signal had been undertaken. The first by Wittmann (1973, 1974) who computed a Stokes "contribution function" which actually was a numerical approximation to a magnetic field response function of the Stokes V parameter. Later Landi Degl'Innocenti and Landi Degl'Innocenti (1977) derived an expression for the response functions of the Stokes parameters under unrealistically restrictive assumptions. Considerable progress was made by Van Ballegooijen (1985) who developed a formalism to solve the Unno-Rachkovsky equations (Unno, 1956; Rachkovsky, 1962) which permitted the Stokes parameters to be represented as integrals over depth of certain functions. The latter were identified by the author with the respective Stokes contribution functions based on the analogy to the definition of the intensity contribution function in the absence of a magnetic field. However, as has been shown by Magain (1986) and by others (e.g. Gurtovenko et al., 1974) the *intensity* contributions function is *not* a useful diagnostic tool because it represents, in particular if the line is weak, predominantly the bound-free and free-free processes responsible for the continuous intensity. We believe that Magain's arguments in favour of the relative line depression $R = 1 - I/I_c$ being the quantity whose contribution function represents best the bound-bound interactions of the spectral line and is thus responsible for the shape of the line profile apply equally well to the other three Stokes parameters, because they also owe their existence entirely to the various bound-bound processes. Hence in the present paper we shall extend the work of Van Ballegooijen and, using his technique, compute what might be called the Stokes line depression contribution function (Sect. 2). In addition, in Sect. 3, we shall derive a general expression for the corresponding response functions. In Sect. 4 we shall briefly describe a computer code which we have developed for numerically solving the Unno-

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Rachkovsky equations in a stellar atmosphere in order to determine the emergent Stokes parameters and their appropriate contribution functions. In Sect. 5 a few Stokes parameter profiles will be shown which we have computed together with their contribution functions, calculated according to the different definitions. These plots demonstrate the superior diagnostic quality of the Stokes "line depression" contribution function as compared to Van Ballegooijen's definition.

Finally, in the Appendix we derive analytical expressions for the contribution functions under certain assumptions. Although these expressions are of little diagnostic value they do allow us to to show that for weak fields a simple relation exists between the contribution functions of Stokes I and V. The derivation also given us a clearer insight into why the Jones and Mueller formalisms are analytically but not numerically equivalent when obtaining the formal solution of the Stokes radiative transfer equations. In addition, an analytical solution may be used to check the reliability of numerical procedures of calculating the contribution functions.

As a prelude we list the basic equations of relevance to the rest of the calculations. In Jones calculus, which we use throughout the present paper, the radiative transfer equation for 4 Stokes parameters in the presence of a magnetic field can be written as

$$\frac{d\mathbf{D}}{d\tau_c} = \mathbf{AD} + \mathbf{DA}^* - \mathbf{F}, \tag{1}$$

(Van Ballegooijen, 1985) with

$$\mathbf{D} = \frac{1}{2}\begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}, \tag{2}$$

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} 1+\eta_I+\alpha_Q & \alpha_U+i\alpha_V \\ \alpha_U-i\alpha_V & 1+\eta_I-\alpha_Q \end{pmatrix}, \tag{3}$$

$$\mathbf{F} = \frac{1}{2}\begin{pmatrix} S_c+(\eta_I+\eta_Q)S_L & (\eta_U+i\eta_V)S_L \\ (\eta_U-i\eta_V)S_L & S_c+(\eta_I-\eta_Q)S_L \end{pmatrix}, \tag{4}$$

where

$$\alpha_Q = \eta_Q - i\varrho_Q, \quad \alpha_U = \eta_U - i\varrho_U, \quad \alpha_V = \eta_V - i\varrho_V.$$

Expressions for the various coefficients η and ϱ have been given by e.g. Landi Degl'Innocenti (1976). In Eq. (1) $\mathbf{A}^*$ represents the transpose and complex conjugate of $\mathbf{A}$. Note that τ_c is the continuum optical depth *along the line of sight*. By defining τ_c in this manner, we do not have to introduce $\mu = \cos\theta$, where θ is the heliocentric angle, and as a consequence our transfer equation is also valid for non-plane-parallel atmospheres, which we consider to be closer to reality (e.g. when modelling small magnetic flux concentrations on the sun).

To obtain the formal solution we follow Van Ballegooijen (1985) and introduce the matrix $\mathbf{T}$ which satisfies the differential equation

$$\frac{d\mathbf{T}}{d\tau_c} = \mathbf{AT} \quad \text{and} \quad \mathbf{T}(0) = \mathbf{E}, \tag{5}$$

where $\mathbf{E}$ is the unity matrix. Once $\mathbf{T}(\tau_c)$ is known, the contribution functions of the Stokes parameters C_I, C_Q, C_U, C_V can be obtained in a straightforward manner via

$$\mathbf{C}(\tau_c) = (\mathbf{T})^{-1}\,\mathbf{F}(\mathbf{T}^*)^{-1} \tag{6}$$

and the relations

$$C_I(\tau_c) = C_{11}(\tau_c) + C_{22}(\tau_c),$$
$$C_Q(\tau_c) = C_{11}(\tau_c) - C_{22}(\tau_c),$$
$$C_U(\tau_c) = C_{12}(\tau_c) + C_{21}(\tau_c),$$
$$C_V(\tau_c) = -i(C_{12}(\tau_c) - C_{21}(\tau_c)). \tag{7}$$

The matrix $\mathbf{C}$ is the integrand of the formal solution of Eq. (1). For more information we refer to Van Ballegooijen (1985).

2. Contribution functions to the "line depression" Stokes parameters

In order to derive a diagnostically useful contribution function for the Stokes parameters we follow the procedure outlined by Magain (1986) for unpolarized light, i.e. we first derive the transfer equation for what we call "line depression Stokes parameters" and then find its formal solution. Although the derivation is completely equivalent in both Mueller and Jones formalisms, we present it using the latter since it is of greater practical value for the numerical solution of the transfer equation.

We define the Stokes parameters of the relative line depression at any depth in the atmosphere as

$$R_I(\tau_c) = \frac{I_c(\tau_c) - I(\tau_c)}{I_c(\tau_c)} = 1 - \frac{I(\tau_c)}{I_c(\tau_c)},$$
$$R_Q(\tau_c) = \frac{Q_c(\tau_c) - Q(\tau_c)}{I_c(\tau_c)} = -\frac{Q(\tau_c)}{I_c(\tau_c)},$$
$$R_U(\tau_c) = \frac{U_c(\tau_c) - U(\tau_c)}{I_c(\tau_c)} = -\frac{U(\tau_c)}{I_c(\tau_c)},$$
$$R_V(\tau_c) = \frac{V_c(\tau_c) - V(\tau_c)}{I_c(\tau_c)} = -\frac{V(\tau_c)}{I_c(\tau_c)}, \tag{8}$$

where I_c, Q_c, U_c, and V_c are the Stokes parameters of the continuum radiation. They are functions of the continuum optical depth. We have implicitly assumed that no continuum polarization is present in the stellar atmosphere. For field strengths of 1–10 kG, typical for the magnetic structures in the photospheres of non-degenerate stars, continuum circular polarization is expected to be between 10^{-6} and 10^{-5} in units of I_c (Kemp, 1970) and can therefore be neglected. For fields much stronger than this, the "weak" field limit for the Zeeman effect is no longer valid in any case and the absorption matrix changes its character.

In this section we make no assumptions regarding the matrices $\mathbf{A}$ and $\mathbf{F}$. In analogy to the matrix $\mathbf{D}$, defined in Eq. (2), we introduce the matrix $\mathbf{R}$ such that

$$\mathbf{R} = \frac{1}{2}\begin{pmatrix} R_I+R_Q & R_U+iR_V \\ R_U-iR_V & R_I-R_Q \end{pmatrix} = \frac{1}{2}\mathbf{E} - \frac{1}{I_c}\mathbf{D}. \tag{9}$$

Making use of the identity

$$\frac{d\mathbf{R}}{d\tau_c} = \frac{\mathbf{D}}{I_c^2}\frac{dI_c}{d\tau_c} - \frac{1}{I_c}\frac{d\mathbf{D}}{d\tau_c}$$

and replacing the derivatives of I_c and $\mathbf{D}$ through the right-hand-sides of their respective radiative transfer equations, i.e. the right-hand-side of Eq. (1) and of

$$\frac{dI_c}{d\tau_c} = I_c - S_c, \tag{10}$$

we obtain after a short calculation

$$\frac{d\mathbf{R}}{d\tau_c} = \mathbf{A}_R\mathbf{R} + \mathbf{R}\mathbf{A}_R^* - \mathbf{F}_R, \tag{11}$$

where

$$\mathbf{A}_R = \frac{1}{2}\begin{pmatrix} \frac{S_c}{I_c} + \eta_I + \alpha_Q & \alpha_U + i\alpha_V \\ \alpha_U - i\alpha_V & \frac{S_c}{I_c} + \eta_I - \alpha_Q \end{pmatrix}$$

$$= \mathbf{A} + \frac{1}{2}\left(\frac{S_c}{I_c} - 1\right)\mathbf{E}, \tag{12}$$

$$\mathbf{F}_R = \frac{1}{2}\left(1 - \frac{S_L}{I_c}\right)\begin{pmatrix} \eta_I + \eta_Q & \eta_U + i\eta_V \\ \eta_U - i\eta_V & \eta_I - \eta_Q \end{pmatrix}$$

$$= \frac{1}{S_L}\left(1 - \frac{S_L}{I_c}\right)\left(\mathbf{F} - \frac{S_c}{2}\mathbf{E}\right). \tag{13}$$

The formal solution of Eq. (11) is analogous to the one proposed by Van Ballegooijen (1985) for **D**. We introduce the matrix $\mathbf{T}_R$ which satisfies the differential equation

$$\frac{d\mathbf{T}_R}{d\tau_c} = \mathbf{A}_R \mathbf{T}_R \quad \text{with} \quad \mathbf{T}_R(0) = \mathbf{E}. \tag{14}$$

The contribution matrix for the line depression is then determined by

$$\mathbf{C}_R(\tau_c) = \mathbf{T}_R^{-1}\,\mathbf{F}_R\,(\mathbf{T}_R^*)^{-1}, \tag{15}$$

while the contribution functions for R_I, R_U, R_Q, and R_V can be determined from $\mathbf{C}_R$ through relations similar to Eq. (7). The emergent value of **R** can be obtained by integrating over $\mathbf{C}_R(\tau_c)$. Alternatively, using Eq. (9) we can directly determine $\mathbf{D}(0)$ if so required:

$$\mathbf{D}(0) = I_c(0)\left(\frac{1}{2}\mathbf{E} - \int_0^\infty \mathbf{T}_R^{-1}\,\mathbf{F}_R\,(\mathbf{T}_R^*)^{-1}\,d\tau_c\right). \tag{16}$$

3. Response functions for the Stokes parameters

In some cases the response function of a spectral line is of greater interest than its contribution function, e.g. for the empirical diagnostics of given atmospheric parameters, such as the temperature, magnetic field, or the velocity. Examples of diagnostic techniques for the temperature and for velocity gradients in solar magnetic fluxtubes involving response functions have been given by Landi Degl'Innocenti and Landolfi (1982, 1983), respectively.

We generalize the derivation given by Caccin et al. (1977) for the case of $B = 0$ to the case of an arbitrary magnetic field. We first apply this approach to derive the response functions of the unnormalized Stokes parameters and later also present an expression for the response functions of the "line depression" Stokes parameters.

Let β be the (scalar) physical parameter of the atmosphere which we are interested in, with $\beta = \beta(\tau_c)$ being a general function of optical depth. Then a perturbation in β of the form

$$\beta \to \beta + \delta\beta$$

will affect **A**, **D**, **F**, and κ_c (the continuum absorption coefficient, hidden in τ_c) in the transfer Eq. (1). The perturbed quantities, denoted in the following by an index $\delta\beta$, can be expressed through the unperturbed ones by Taylor expansions according to $\delta\beta$.

$$\mathbf{A}_{\delta\beta} = \sum_{n=0}^{\infty} \frac{(\delta\beta)^n}{n!}\,\frac{\partial^n \mathbf{A}}{\partial\beta^n},$$

$$\mathbf{D}_{\delta\beta} = \sum_{n=0}^{\infty} \frac{(\delta\beta)^n}{n!}\,\frac{\partial^n \mathbf{D}}{\partial\beta^n},$$

$$\mathbf{F}_{\delta\beta} = \sum_{n=0}^{\infty} \frac{(\delta\beta)^n}{n!}\,\frac{\partial^n \mathbf{F}}{\partial\beta^n},$$

$$\kappa_{c,\delta\beta} = \sum_{n=0}^{\infty} \frac{(\delta\beta)^n}{n!}\,\frac{\partial^n \kappa_c}{\partial\beta^n}. \tag{17}$$

For sufficiently small perturbations $\delta\beta$ we can neglect second and higher order terms. If we introduce these expressions into the perturbed transfer equation (multiplied by $-\kappa_{c,\delta\beta}$)

$$\frac{d\mathbf{D}_{\delta\beta}}{dz} = -\kappa_{c,\delta\beta}\left(\mathbf{A}_{\delta\beta}\mathbf{D}_{\delta\beta} + \mathbf{D}_{\delta\beta}\mathbf{A}_{\delta\beta}^* - \mathbf{F}_{\delta\beta}\right) \tag{18}$$

and equate terms with equal powers of $\delta\beta$ we obtain the original unperturbed transfer equation, Eq. (1), for terms of zero-th order in $\delta\beta$. For terms linear in $\delta\beta$ we get the following equation for $\delta\mathbf{D}$ which has the same general form as the transfer equation:

$$\frac{d\delta\mathbf{D}}{d\tau_c} = \mathbf{A}\,\delta\mathbf{D} + \delta\mathbf{D}\mathbf{A}^* - \delta\mathbf{F}, \tag{19}$$

where

$$\delta\mathbf{D} = \delta\beta\frac{\partial\mathbf{D}}{\partial\beta} \tag{20}$$

and

$$\delta\mathbf{F} = \delta\beta\left(\frac{\partial\mathbf{F}}{\partial\beta} - \frac{1}{\kappa_c}\frac{\partial\kappa_c}{\partial\beta}(\mathbf{AD} + \mathbf{DA}^* - \mathbf{F}) - \frac{\partial\mathbf{A}}{\partial\beta}\mathbf{D} - \mathbf{D}\frac{\partial\mathbf{A}^*}{\partial\beta}\right). \tag{21}$$

Since in Eq. (21) **D**, the solution of Eq. (1), can be taken to be a known quantity, Eq. (19) for $\delta\mathbf{D}$ can accordingly be solved using Van Ballegooijen's (1985) technique,

$$\delta\mathbf{D}(0) = \int_0^\infty (\mathbf{T})^{-1}\,\delta\mathbf{F}\,(\mathbf{T}^*)^{-1}\,d\tau. \tag{22}$$

The response function matrix of **D** with respect to β, which we denote as $\mathbf{P}_\beta$, is essentially the integrand of Eq. (22):

$$\mathbf{P}_\beta = \mathbf{T}^{-1}\left(\frac{\partial\mathbf{F}}{\partial\beta} - \frac{1}{\kappa_c}\frac{\partial\kappa_c}{\partial\beta}(\mathbf{AD} + \mathbf{DA}^* - \mathbf{F}) - \frac{\partial\mathbf{A}}{\partial\beta}\mathbf{D} - \mathbf{D}\frac{\partial\mathbf{A}^*}{\partial\beta}\right) \cdot (\mathbf{T}^*)^{-1}. \tag{23}$$

T is once more the solution of Eq. (5), so that no new set of differential equations must be solved in order to evaluate the response functions. The response function for the four Stokes parameters can then be written, in analogy to Eq. (7), as

$$P_{\beta,I}(\tau_c) = P_{\beta,11}(\tau_c) + P_{\beta,22}(\tau_c),$$
$$P_{\beta,Q}(\tau_c) = P_{\beta,11}(\tau_c) - P_{\beta,22}(\tau_c),$$
$$P_{\beta,U}(\tau_c) = P_{\beta,12}(\tau_c) + P_{\beta,21}(\tau_c),$$
$$P_{\beta,V}(\tau_c) = -i\,(P_{\beta,12}(\tau_c) - P_{\beta,21}(\tau_c)).$$

Note that for $B = 0$, the Stokes I response function reduces to the non-magnetic response function derived by Caccin et al. (1977). For the response function, $\mathbf{P}_{\beta,R}$, of the "line depression" Stokes parameters, **R**, we obtain exactly the same expression as Eq. (23), with **A**, **D**, **F**, and **T** replaced by their counterparts, $\mathbf{A}_R$, **R**, $\mathbf{F}_R$ and $\mathbf{T}_T$, given in Sect. 2:

$$\mathbf{P}_{\beta,R}=\mathbf{T}_R^{-1}\left(\frac{\partial \mathbf{F}_R}{\partial\beta}-\frac{1}{\kappa_c}\frac{\partial\kappa_c}{\partial\beta}(\mathbf{A}_R\mathbf{R}+\mathbf{R}\mathbf{A}_R^*-\mathbf{F}_R)\right.$$
$$\left.-\frac{\partial \mathbf{A}_R}{\partial\beta}\mathbf{R}-\mathbf{R}\frac{\partial \mathbf{A}_R^*}{\partial\beta}\right)(\mathbf{T}_R^*)^{-1}. \quad (25)$$

This is not surprising, since the transfer equation for **R**, Eq. (11), is formally the same as Eq. (1).

4. Numerical computation of the Stokes parameters and their contribution functions

We have developed a FORTRAN code for the numerical solution of the Unno-Rachkovsky equations using the Jones calculus as proposed by Van Ballegooijen. Several servions of the program exist. Their final result is always the set of "normalized" Stokes parameters (SP), i.e. the SP divided by I_c, the continuous intensity. The versions differ by the nature of the employed contribution functions (CF). The first version computes the CF of the SP themselves and obtains the SP by integration of the CF [Eqs. (1)–(7)]. I_c is then calculated in the same fashion as Stokes I at a wavelength far from line center and the SP are normalized by division through this quantity. The second version computes the CF of the *normalized* SP. Integration then yields the normalized SP directly. In the third version the "line depression" CF are computed and integrated [Eqs. (9)–(16)]. The normalized SP are then obtained by negating the results (and adding 1 in the case of Stokes I). In versions 2 and 3 I_c has to be computed explicitly in the program as a function of continuum optical depth.

The program consists of three parts. In the first part all variables that depend on atomic parameters alone are determined, in particular the relative strength of the Zeeman sublevels involved. In the second part the height dependent atmospheric quantities are computed for each atmospheric grid point. We use $x=\log\tau_{c5}$, the logarithm of the continuous opacity at 500 nm, as independent variable. This part includes the determination of the electron density by the iterative solution of a set of Saha equations. As a result we obtain total line opacity, continuous opacity and the parameters which govern the profile of the line. Input to part two are temperature, total pressure, magnetic field vector, bulk velocity and microturbulent velocity at each grid point. The third part of the program is version dependent. It computes for each wavelength point the respective CF as function of depth by a Runge-Kutta integration of Eq. (5) or Eq. (14) which are actually sets of 8 coupled (real) first order differential equations. Then the appropriate matrix equations are solved [e.g. Eq. (15)]. Finally the values of the emergent SP (or related quantities) are obtained by integration of the CF over depth.

For a detailed description of the relevant quantities involved in the computation the reader is referred to Landi Degl'Innocenti's description of his own computer program (Landi Degl'Innocenti, 1976).

An advantage of Van Ballegooijen's matrix formalism is the simplicity of the boundary condition for integration of Eq. (5): $T(0)$ (Actually $T(x_0)$) is equal to the unity matrix. This is considerably easier to deal with than the somewhat cumbersome conditions which have to be met in the conventional integration of the Unno-Rachkovsky equations (Landi Degl'Innocenti, 1976).

Various tests were performed in order to ascertain the validity of our program. They were all made using the Fe I 5250.2 Å line. First we compared Stokes I for vanishing magnetic field with values obtained by direct integration of the transfer equation of unpolarized light. We also compared it with values computed by H. Schleicher (private communication) who used his own radiative transfer program. The results of the two direct integrations differed by less than 1%. The results of the Van Ballegooijen procedure were found to deviate from these values by amounts which depended on the spacing of the grid points of the model atmosphere which in the tests extended from $x=-4.0$ to $x=1.0$. With a grid width of $\Delta x=0.1$ the deviations varied from 0.1% to 12%, for $\Delta x=0.05$ from 0.1% to 6%, and for $\Delta x=0.01$ the deviations hardly ever exceeded 1%. From this we concluded that for vanishing magnetic fields our code contains no significant error. Similar results were obtained with the third version of the program [Eqs. (9)–(16)]: In order to achieve an accuracy of about 1% the grid width has to be of the order 0.01. For a grid width of 0.1 inaccuracies of the order of 10% occur.

In order to test the program's performance for the case of non-vanishing magnetic field we used the analytical solution of the Unno-Rachkovsky equations for the special case of a longitudinal and uniform magnetic field and a normal Zeeman triplet:

$$I=\frac{1}{2}\left[f(\lambda+\Delta\lambda)+f(\lambda-\Delta\lambda)\right]$$

$$V=\frac{1}{2}\left[f(\lambda-\Delta\lambda)-f(\lambda+\Delta\lambda)\right],$$

where $f(\lambda)=I(\lambda)$ for vanishing magnetic field and $\Delta\lambda$ is the Zeeman shift. We found that for a grid width of 0.1 these equations were fulfilled to an accuracy of 3% or better.

5. Sample calculations

We have used the code described in Sect. 4 to calculate Stokes profiles and contribution functions. Some examples shall be discussed in this section with the aim of illustrating the difference between the contribution functions to the intensity Stokes profiles (C_I, C_Q, C_U, C_V) and to the line depression Stokes profiles ($C_{R,I}$, $C_{R,Q}$, $C_{R,U}$, $C_{R,V}$). Figure 1 shows the results of such an LTE calculation of Fe I 5250.2 Å for a height independent field of strength $B=1200$ G and a quiet sun atmosphere (Schleicher, private communication) which is similar to model C of Vernazza et al. (1981) in the upper layers and follows Holweger and Müller (1974) in the deeper layers. The angle between the line of sight and the magnetic field B is 135°, the azimuthal angle of the field is 90°, the microturbulence velocity 1.2 km s^{-1} and a Van der Waals enhancement factor of 3.0 has been assumed (cf. Holweger, 1979). The emergent Stokes I, Q, U, V profiles are plotted in Fig. 1a. Figure 1b to d show their contribution functions (dashed curves: C_I, C_Q, C_U, C_V, solid curves: $C_{R,I}$, $C_{R,Q}$, $C_{R,U}$, $C_{R,V}$), at wavelengths $\Delta\lambda=0$ mÅ, 50 mÅ, and 100 mÅ from line centre respectively. The maxima of all contribution functions have been normalized to unity to make their comparison easier.

Note the totally different shapes of the intensity contribution functions from the line depression contribution functions. In particular, the intensity contribution functions are dominated by the contribution to the continuum as soon as one moves out towards the line wings. They change only little for $\Delta\lambda \gtrsim 50$ mÅ. In contrast, the C_R change considerably between $\Delta\lambda=50$ mÅ and 100 mÅ. Also note that even for $\Delta\lambda \geqq 100$ mÅ, where the emergent Stokes profiles approach their continuum values, the C_R are still quite different from the intensity contribution functions. This illustrates quite clearly that the bound-bound transitions giving rise to a spectral line may occur preferably at a hight quite

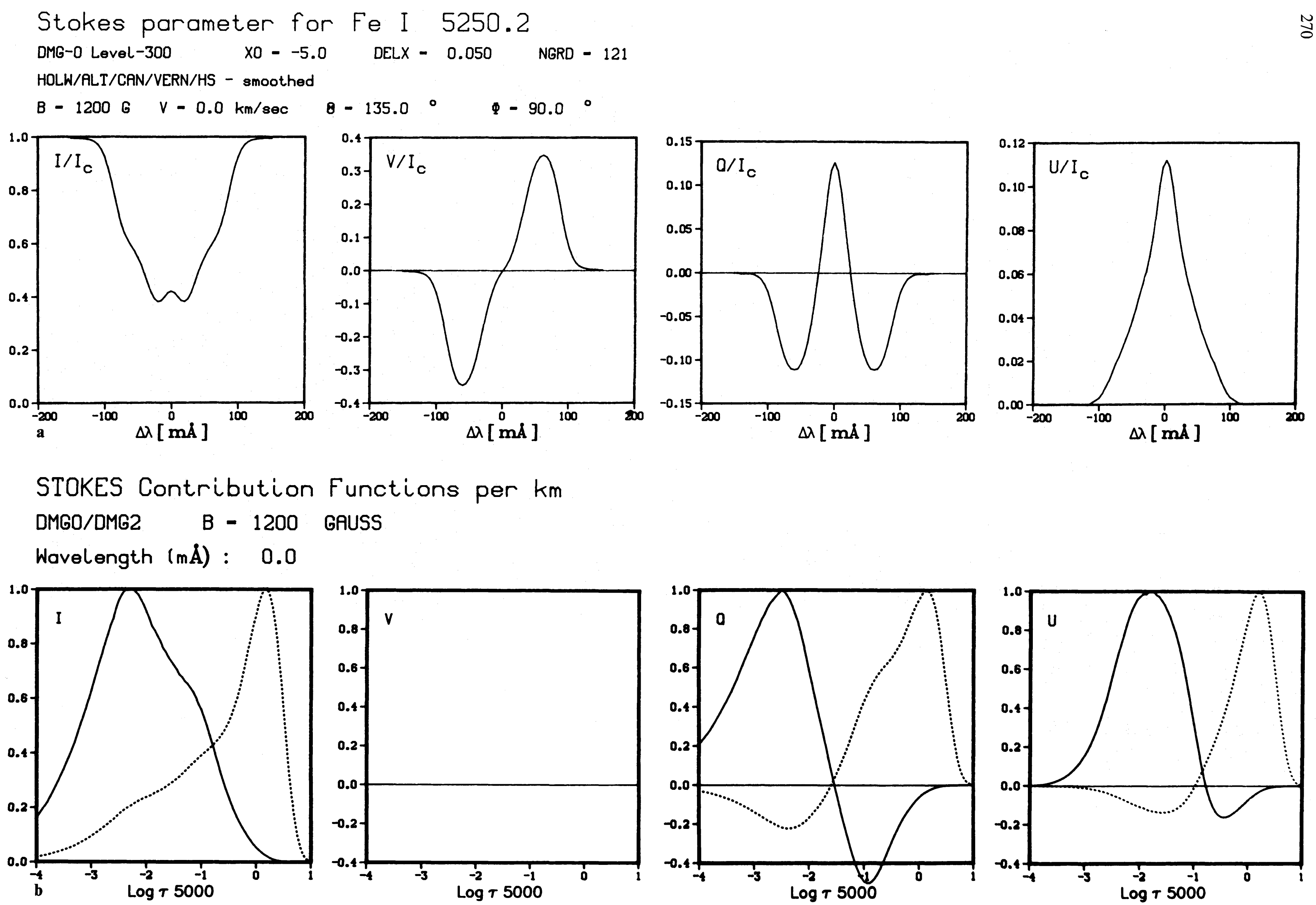

Fig. 1a–d. Fe I 5250.2 Å line calculated in a quiet sun atmosphere with a height-independent field of strength $B = 1200$ G. The angle between the line of sight and the field, $\theta = 135°$, the azimuthal angle of the field, $\phi = 90°$. **a** Emergent Stokes I, Q, U, V profiles. **b** Contribution functions to the intensity Stokes profiles (C_I, C_Q, C_U, C_V, dashed) and the line depression Stokes profiles ($C_{R,I}$, $C_{R,Q}$, $C_{R,U}$, $C_{R,V}$, solid) at line centre, i.e. for $\Delta\lambda = 0$ mÅ. **c** Same as **b**, except for $\Delta\lambda = 50$ mÅ. **d** Same as **b**, except for $\Delta\lambda = 100$ mÅ

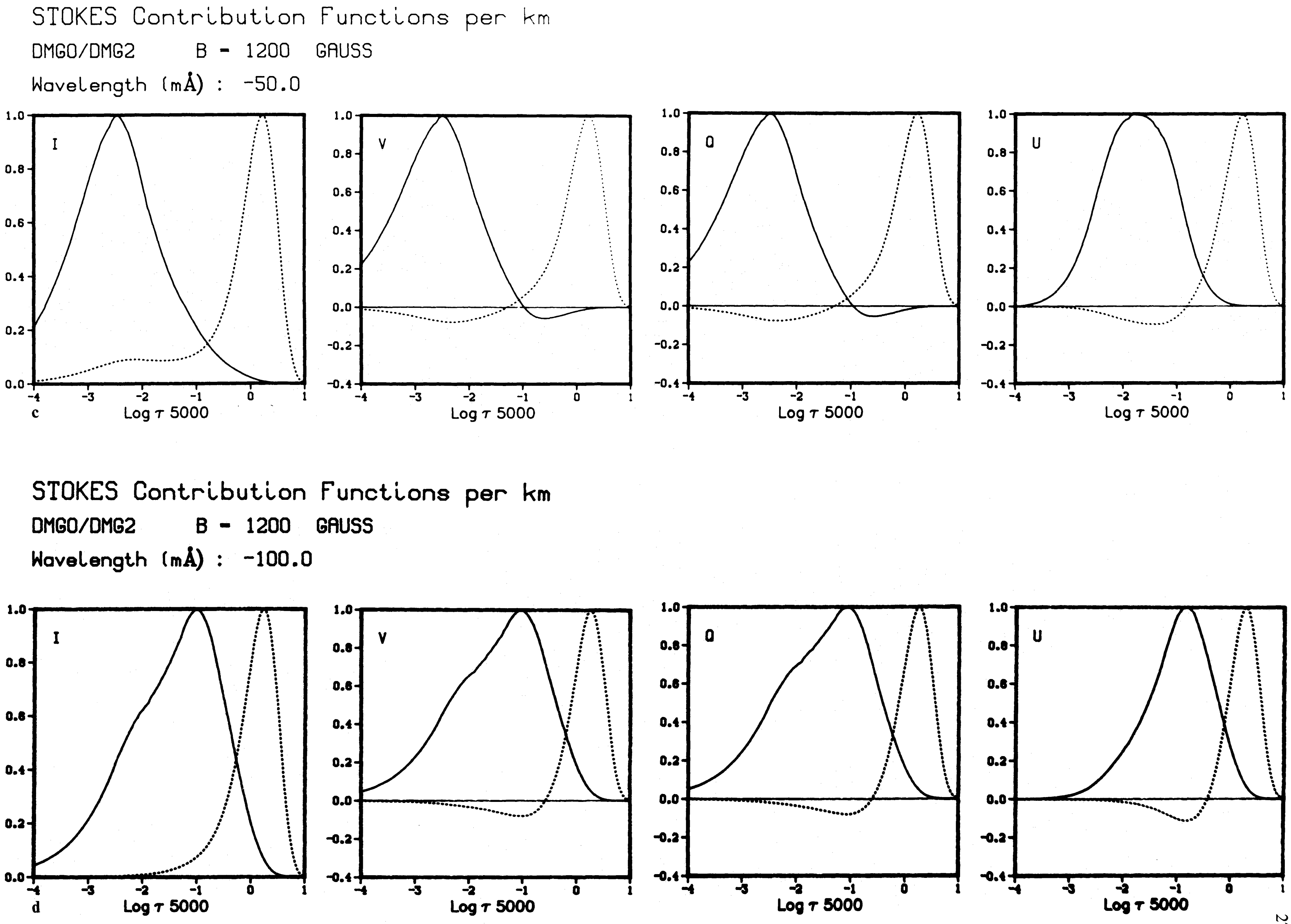

Fig. 1 (continued)

different from the free-free and bound-free transitions responsible for the continuum, even if the line is very weak at the wavelength in question. This basic difference between the line depression and continuum formation height is mirrored in the non-magnetic case which has been clearly and convincingly discussed by Magain (1986).

Thus it becomes apparent that the C_R are theoretically "cleaner" than the intensity contribution functions, in the sense that while the latter mix information on the continuum and the lines, the former give us information on the lines only. However, there remains the important open question of the sensitivity of the two contribution functions to atmospheric parameters, a vital aspect if they are to be used for the diagnosis of solar and stellar magnetic features. Once more the C_R are superior to the C as Fig. 2 demonstrates. In Fig. 2a two emergent Stokes I profiles are plotted which were calculated for the same atmosphere as the profiles in Fig. 1, but for different field strengths, namely $B = 0$ G (solid) and $B = 2000$ G (dashed). In Fig. 2b the C_I at $\Delta\lambda = 80$ mÅ for these two profiles are plotted. Note the minute difference between the two curves. Finally, Fig. 2c shows the $C_{R,I}$ at the same $\Delta\lambda$. Here the difference is easily visible. At $B = 2000$ G, the σ-components of the Stokes I profile have their minima close to $\Delta\lambda = 80$ mÅ, so that we would intuitively expect the line at this wavelength to be formed higher in the atmosphere for $B = 2000$ G than for $B = 0$ G. $C_{R,I}$ is therefore not only more sensitive to changes in B, but its dependence on B is also plausible and may be interpreted in a straightforward manner. The same is also true for the contribution functions to the other "line depression" Stokes parameters.

A more systematic analysis of the contribution functions to the Stokes profiles as a function of different atmospheric and atomic parameters will be published separately. Such a study will later allow us to use Stokes contribution functions as an integral part of the diagnostics of solar and stellar polarimetric observations (e.g. those of Stenflo et al., 1984), but also of models of magnetic features (e.g. Knölker et al., 1988; Steiner et al., 1986).

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Appendix: analytical derivation of Stokes contribution functions

In this Appendix we use the algorithm of Van Ballegooijen (1985) to derive the intensity contribution functions to the Stokes parameters of a Zeeman split line *analytically* assuming an atmosphere with an absorption matrix independent of optical depth. We also assume LTE and neglect magnetooptical effects. The last two assumptions are not necessary. They only simplify the calculations and should not affect the conclusions reached in this section. This set of assumptions is more general than that adopted by Unno (1956), since we do not require the Planck function to be linear in τ and the line need not be a Zeeman triplet, any splitting pattern is allowed. Note that for the absorption matrix to be independent of height, both the magnetic field strength and the total Doppler broadening must be constant.

In contrast to Landi Degl'Innocenti and Landi Degl'Innocenti (1985), who present the formal solution for the Stokes profiles in Mueller calculus involving 4 × 4 matrices, and without giving an explicit expression for the contribution function, we shall derive

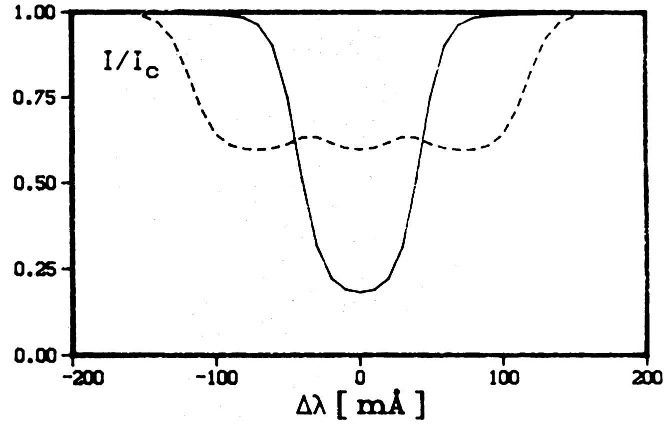

Fig. 2a. Emergent Stokes I profiles of Fe I 5250.2 Å calculated in a quiet sun atmosphere with $\theta = 135°$ and $\phi = 90°$. Dashed curve: $B = 2000$ G, solid curve: $B = 0$ G

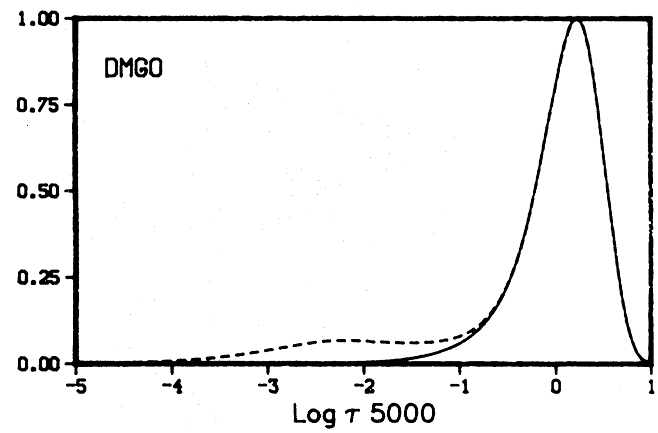

Fig. 2b. C_I at $\Delta\lambda = 80$ mÅ to the two profiles plotted in Fig. 2a

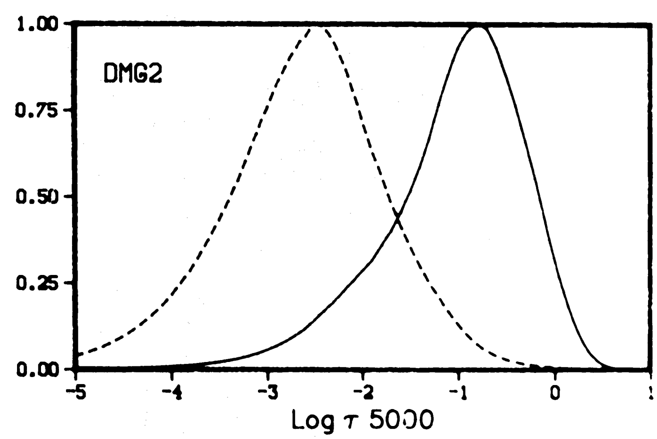

Fig. 2c. $C_{R,I}$ at $\Delta\lambda = 80$ mÅ to the two profiles plotted in Fig. 2a

an explicit expression in Jones calculus which involves 2×2 complex matrices.

As a first step we must solve Eq. (5), which at first sight appears to be a set of 4 complex coupled differential equations. However, closer inspection reveals that it is composed of two independent pairs of coupled differential equations; one pair for T_{11} and T_{21} and one for T_{12} and T_{22} respectively. The differential equations for T_{11} and T_{21} read

$$\frac{dT_{11}}{d\tau_c}=\frac{1}{2}\left[(1+\eta_I+\alpha_Q)\,T_{11}+(\alpha_U+i\alpha_V)\,T_{21}\right],$$

$$\frac{dT_{21}}{d\tau_c}=\frac{1}{2}\left[(\alpha_U-i\alpha_V)\,T_{11}+(1+\eta_I-\alpha_Q)\,T_{21}\right].$$

The differential equations for T_{12} and T_{22} are identical to these if we replace T_{11} by T_{12} and T_{21} by T_{22}. This is a direct result of the definition of matrix multiplication. Any set of differential equations which can be written in the form of Eq. (5), where **A** and **T** are arbitrary $n\times n$ matrices, can be reduced to a set of coupled equations for the first column of **T** only, the equations for the other columns being identical to those for the first column. This property, which implies that instead of $n\times n$ only n coupled equations have to be solved, means that both the traditional Mueller formalism and the Jones formalism are in principle equivalent. We shall return to this point later.

Now we make use of the assumptions listed at the beginning of the section ($S_L=S_c=B_\nu$, $\varrho_Q=\varrho_U=\varrho_V=0$). Then the matrices **A** and **F** reduce to:

$$\mathbf{A}=\frac{1}{2}\begin{pmatrix}1+\eta_I+\eta_Q & \eta_U+i\eta_V\\ \eta_U-i\eta_V & 1+\eta_I-\eta_Q\end{pmatrix}=\frac{1}{B_\nu}\mathbf{F}. \qquad (A1)$$

By transforming Eq. (5) into the basis of the eigenvectors of τ-independent **A** we can diagonalise **A** and solve for T_{11} and T_{21} (or equivalently T_{12} and T_{22}). The general solution, after retransformation into the original basis, reads

$$T_{11}=T_{12}=\frac{1}{\eta_U-i\eta_V}\left(K_1\left(\eta_Q+\sqrt{\eta_Q^2+\eta_U^2+\eta_V^2}\right)e^{\varepsilon_1\tau_c}+K_2\left(\eta_Q-\sqrt{\eta_Q^2+\eta_U^2+\eta_V^2}\right)e^{\varepsilon_2\tau_c}\right), \qquad (A2)$$

$$T_{21}=T_{22}=K_1\,e^{\varepsilon_1\tau_c}+K_2\,e^{\varepsilon_2\tau_c}.$$

In Eq. (A2) K_1 and K_2 are constants of integration, while ε_1 and ε_2 are the eigenvalues of **A**

$$\varepsilon_{1,2}=\tfrac{1}{2}\left(1+\eta_I\pm\sqrt{\eta_Q^2+\eta_U^2+\eta_V^2}\right). \qquad (A3)$$

The boundary condition $\mathbf{T}(0)=\mathbf{E}$ ensures that **T** is not singular. After taking the boundary conditions into account we obtain the final solution

$$T_{11}=\frac{1}{2\eta_p}\left((\eta_Q+\eta_p)\,e^{\varepsilon_1\tau_c}-(\eta_Q-\eta_p)\,e^{\varepsilon_2\tau_c}\right),$$

$$T_{21}=\frac{\eta_U-i\eta_V}{2\eta_p}\left(e^{\varepsilon_1\tau_c}-e^{\varepsilon_2\tau_c}\right),$$

$$T_{12}=\frac{\eta_U+i\eta_V}{\eta_U-i\eta_V}\,T_{21},$$

$$T_{22}=T_{11}-\frac{2\eta_Q}{\eta_U-i\eta_V}\,T_{21}, \qquad (A4)$$

where $\eta_p=\sqrt{\eta_Q^2+\eta_U^2+\eta_V^2}$.

The main difference between the analytical and numerical methods of solving Eq. (5) is that while analytically we can first determine a general solution (which is the same for all the columns of **T**) and have to consider the boundary conditions only at the end, numerically each element of **T** has to be solved for individually, since the boundary conditions must be taken into account from the very beginning. This is also the reason why, although in principle equivalent, the Mueller matrix formalism is numerically inferior by a factor of approximately 2 to the Jones formalism when calculating the Stokes profiles via the formal solution of the transfer equation (as first pointed out by Van Ballegooijen, 1985). This means that the solution of the Stokes radiative transfer equation does not violate the generally accepted basic equivalence of the two formalisms, a reassuring result.

Once we have obtained **T** we can determine the contribution functions of the Stokes parameters via Eqs. (6), (7), and (A1). The calculation is tedious but straightforward and the resulting expressions read:

$$C_I(\tau_c)=\left(\varepsilon_1\,e^{-2\varepsilon_1\tau_c}+\varepsilon_2\,e^{-2\varepsilon_2\tau_c}\right)B_\nu(\tau_c),$$

$$C_Q(\tau_c)=\frac{\eta_Q}{\eta_p}\,B_\nu(\tau_c)\left(\varepsilon_1\,e^{-2\varepsilon_1\tau_c}-\varepsilon_2\,e^{-2\varepsilon_2\tau_c}\right),$$

$$C_U(\tau_c)=\frac{\eta_U}{\eta_Q}\,C_Q,$$

$$C_V(\tau_c)=\frac{\eta_V}{\eta_Q}\,C_Q. \qquad (A5)$$

Expressions for $\varepsilon_{1,2}$ are given in Eq. (A3).

For the case of $B=0$ (no magnetic field), Eq. (A5) reduces to

$$C_{I,0}(\tau_c)=B_\nu(1+\eta_0)\,e^{-(1+\eta_0)\tau_c},$$
$$C_{Q,0}=C_{U,0}=C_{V,0}=0, \qquad (A6)$$

which is the well known emission contribution function of the unpolarized line profile. η_0 is the absorption coefficient of the Stokes I profile of the Zeeman unsplit line. If we assume that $B_\nu=B_{\nu_0}(1+\beta\tau_c)$, i.e. the Planck function is linear in τ_c, when we can obtain the emergent Stokes vector by integrating Eq. (A5). The result is the classical Unno solution of the radiative transfer equation for polarized light (Unno, 1956).

Let us next consider the case of a magnetic field aligned along the line of sight. Then $\eta_U=\eta_Q=0$, so that C_I and C_V reduce to

$$C_I=\frac{B_\nu}{2}\left((1+\eta_+)\,e^{-(1+\eta_+)\tau_c}+(1+\eta_-)\,e^{-(1+\eta_-)\tau_c}\right)\equiv\frac{C_++C_-}{2}$$

$$C_V=\frac{B_\nu}{2}\left((1+\eta_+)\,e^{-(1+\eta_+)\tau_c}-(1+\eta_-)\,e^{-(1+\eta_-)\tau_c}\right)\equiv\frac{C_+-C_-}{2}, \qquad (A7)$$

where $\eta_\pm=\eta_I\pm\eta_V$ and $C_\pm$ are the contribution functions of right and left circularly polarized light, respectively.

From the well known relation, valid in a constant field for a Zeeman triplet (e.g. Unno, 1956; Mathys, 1988)

$$\eta_\pm(\lambda)=\eta_0(\lambda\mp\Delta\lambda_H)$$

it follows directly that

$$C_\pm(\lambda)=C_{I,0}(\lambda\mp\Delta\lambda_H), \qquad (A8)$$

where $\Delta\lambda_H$ is the Zeeman splitting in wavelength units. We have assumed that B_ν *at a given* τ_c does not vary over the width of the spectral line. In this special case it is possible to determine C_I and C_V from the contribution function of the magnetically unsplit line,

$C_{I,0}$. If we further assume weak fields, i.e. $\Delta\lambda_{\rm H} \ll \Delta\lambda_{\rm D}$, where $\Delta\lambda_{\rm D}$ is the Doppler width of the line, then due to Eq. (A8) we can write $C_{\pm}$ in terms of a Taylor expansion of $C_{I,0}$. Neglecting terms of the order of $\Delta\lambda_{\rm H}^3$ we obtain with the help of Eq. (A7)

$$C_I = C_{I,0} + \frac{\Delta\lambda_{\rm H}^2}{2}\,\frac{d^2 C_{I,0}}{d\lambda^2},$$

$$C_V = -\Delta\lambda_{\rm H}\,\frac{dC_{I,0}}{d\lambda}. \tag{A9}$$

The remarkable relationship between V and the unsplit I profile I_0

$$V = -\Delta\lambda_{\rm H}\,\frac{dI_0}{d\lambda}, \tag{A10}$$

known to be valid in the weak field case for the absorption coefficients (i.e. unsaturated weak lines, Stenflo et al., 1984) and for the line profile including saturation (Solanki et al., 1987), is therefore seen to hold for the intensity contribution functions of Stokes I and V for weak and constant magnetic fields as well.

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