Abstract

The four Cluster spacecraft will form a tetrahedron, which ideally should be a regular one: equal spacing between all pairs of vertices. In reality, this will not be the case. A number of parameters exist to specify how badly off the true figure is. This paper presents some mathematics of tetrahedrons, describes the two parameters that are to be used in the Cluster Science Data System, and gives a Fortran program for their calculation.

Introduction

Four points in space define a tetrahedron. If the separations between each pair of points are equal, then it is a regular tetrahedron. The four Cluster spacecraft will form a tetrahedron, which in general is not regular. How can we specify the degree to which regularity is achieved?

The “Glassmeier” Parameter

The parameter proposed by vom Stein, Glassmeier, and Dunlop (1992) is defined as

\[ Q_G = \frac{\text{True Vol.}}{\text{Ideal Vol.}} + \frac{\text{True Surf.}}{\text{Ideal Surf.}} + 1 \]  

and takes on values between 1 and 3. It tends to describe the dimensionality of the figure, as listed in Table 1. The ideal volume and surface are calculated for a regular tetrahedron with a side length equal to the average of the 6 distances between the 4 points.

<table>
<thead>
<tr>
<th>( Q_G )</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>The four points are colinear</td>
</tr>
<tr>
<td>2.0</td>
<td>The points all lie in a plane</td>
</tr>
<tr>
<td>3.0</td>
<td>A regular tetrahedron is formed</td>
</tr>
</tbody>
</table>

The “Robert/Roux” Parameter

In their paper on tetrahedron shape, Robert and Roux (1993) present 17 different parameters, as ratios of various volumes, sizes, areas. Of these, the CSDS community has decided to adopt one as its second quality parameter for the auxiliary data. It is defined as

\[ Q_R = \mathcal{N} \cdot \left( \frac{\text{True Vol.}}{\text{Sphere Vol.}} \right)^{\frac{1}{2}} \]  

where the sphere is that circumscribing the tetrahedron (all four points on its surface) and \( \mathcal{N} \) is a
Mathematics of a Tetrahedron

Consider four points in space and the figure formed by joining them with lines (Figure 1). The points are numbered 0 to 3, and their vectors are \( r_0, r_1, r_2, r_3 \). Without any loss of generality, we may consider only the differences \( d_i = r_i - r_0 \) in describing the points.

Area of a Side

The area of a parallelogram bounded by two vectors \( d_1 \) and \( d_2 \) is given by the magnitude of their cross product; any triangle is half of a parallelogram, so its area is

\[
S = \frac{1}{2} |d_1 \times d_2|
\]

where \( d_1 \) and \( d_2 \) are the vectors for any two sides of the triangle (Figure 2).

For the four sides of the tetrahedron, specify side \( n \) to be that one that does not contain point \( n \) at any of its vertices. Thus:

\[
S_1 = \frac{1}{2} |d_2 \times d_3| \quad \text{(3)}
\]

\[
S_2 = \frac{1}{2} |d_1 \times d_3| \quad \text{(4)}
\]

Volume of a Tetrahedron

The volume of a figure bounded by three vectors in space is the triple product of those vectors. Any tetrahedron is \( \frac{1}{6} \) of such a figure, hence

\[
V = \frac{1}{6} |d_1 \cdot d_2 \times d_3| \quad \text{(7)}
\]

\[
= \frac{1}{6} \begin{vmatrix} d_{1x} & d_{1y} & d_{1z} \\ d_{2x} & d_{2y} & d_{2z} \\ d_{3x} & d_{3y} & d_{3z} \end{vmatrix} \quad \text{(8)}
\]

Center of Circumscribed Sphere

To find the circumscribed sphere, we need the point that is equidistant from all four vertices, i.e. we want \( r \) such that

\[
(r - r_n) \cdot (r - r_n) = \rho^2; \quad \forall n = 0, 3
\]

\[
r^2 - 2r \cdot r_n + r_n^2 = \rho^2
\]

If we take point 0 as the origin, that is, if we use the \( d_n \) vectors in place of the \( r_n \), then \( r^2 = \rho^2 \), the sphere radius, and the above 4 equations reduce to

\[
2r \cdot d_n = d_n^2; \quad n = 1, 3
\]
Values for the regular tetrahedron of unit side length are listed in Table 2.

### Calculating the Quality Factors

The quality factors in equations 1 and 2 can now be found with the help of these formulas. For $Q_G$, we average the 6 distances between the 4 points to get the side $L$ of the “ideal” regular tetrahedron, with volume $L^3 \sqrt{2}/12$ and surface $L^2 \sqrt{3}$. The true volume and surface are found from equations 7 and 3–6.

For $Q_R$, the radius of the circumscribing sphere is calculated from equation 9. The actual volume of the sphere need not be calculated, for all the factors just go into the normalizing $N$.

$$Q_R = \left( \frac{9\sqrt{3}}{8} V \right)^{\frac{1}{3}} \rho^{-1}$$

A Fortran program at the end of this paper calculates both these parameters.

### References


A Fortran Subroutine to Calculate the Parameters

subroutine TETRAQ(r,qg,qr)

To calculate the Glassmeier (QG) and Robert/Roux (QR) quality factors for a tetrahedron.

Application: CLUSTER SCIENCE DATA SYSTEM
(These are to be two auxiliary parameters)

Input: R(3,4) = positions of the 4 points

Output: QG = Glassmeier factor
QR = Robert/Roux factor (their factor number 10)

Inputs and outputs are single precision, internal calculations are double precision

True volume True surface
QG = ------------ + ------------ + 1.
Ideal vol Ideal surf

where the ideals are volume and surface of a regular tetrahedron of side length equal to the mean of the 6 sides.

| True volume |(1/3)
QR = Factor * | -- |
| Sphere vol |

where the sphere is that circumscribing the tetrahedron (all 4 points on the surface), and the factor is such that QR=1 for a regular tetrahedron.

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1994 June 7
*********************************************************************

implicit none
real r(3,4),qg,qr
double precision d(3,3),c(3,3),s1,s2,s3,s0,s11,s12,s13,vol
double precision v(3),w,smean,vmean,lmean,rc
double precision dot
integer n,m,k

c
Find the differences

do n=1,3
   w=dble(r(n,1))
do m=1,3

THE CSDS TETRAHEDRON FACTORS

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d(n,m)=dble(r(n,m+1))-w
enddo
c
Find the average side length of all 6 sides
c
l1=dsqrt(dot(d(1,1),d(1,1)))
l2=dsqrt(dot(d(1,2),d(1,2)))
l3=dsqrt(dot(d(1,3),d(1,3)))
w= l1 + l2 + l3
do n=1,3
  v(n)=d(n,2)-d(n,1)
endo
dw=dsqrt(dot(v,v))
do n=1,3
  v(n)=d(n,3)-d(n,1)
endo
dw=dsqrt(dot(v,v))
do n=1,3
  v(n)=d(n,3)-d(n,2)
endo
dw=dsqrt(dot(v,v))
lmean=w/6.d0
c
Find the cross products
c
do n=1,3
  m=mod(n,3) + 1
  k=mod(m,3) + 1
  call cross(d(1,m),d(1,k),c(1,n))
endo
c
Find the volume of the tetrahedron
c
vol=dabs(dot(d(1,1),c(1,1)))/6.d0
c
Find the area of the 4 surfaces and their sum
c
s1=0.5d0*dsqrt(dot(c(1,1),c(1,1)))
s2=0.5d0*dsqrt(dot(c(1,2),c(1,2)))
s3=0.5d0*dsqrt(dot(c(1,3),c(1,3)))
do n=1,3
  v(n)=c(n,1) + c(n,2) + c(n,3)
endo
s0=0.5d0*dsqrt(dot(v,v))
s=s0 + s1 + s2 + s3
c
Find the volume and total area for reg tetrahedron with mean side
c
vmean=dsqrt(2.d0)*lmean*lmean*lmean/1.2d1
smean=dsqrt(3.d0)*lmean*lmean
```
Calculate the Glassmeier factor
\[ w = \frac{\text{vol}}{\text{vmean}} + \frac{s}{\text{smean}} + 1.0 \]
\[ qg = \text{sgn}(w) \]

Find the center of the circumscribed circle
\[ w = \frac{1.0}{(1.2d1 \times \text{vol})} \]
\[ v(1) = w \times (c(1,1) \times l1 \times l1 + c(1,2) \times l2 \times l2 + c(1,3) \times l3 \times l3) \]
\[ v(2) = w \times (c(2,1) \times l1 \times l1 + c(2,2) \times l2 \times l2 + c(2,3) \times l3 \times l3) \]
\[ v(3) = w \times (c(3,1) \times l1 \times l1 + c(3,2) \times l2 \times l2 + c(3,3) \times l3 \times l3) \]
\[ r_c = \text{dsqrt}(\text{dot}(v,v)) \]

Calculate the Robert/Roux factor
\[ w = \frac{9.0 \times \text{dsqrt}(3.0)}{8.0} \]
\[ w = (w \times \text{vol})^{(1.0/3.0)} / r_c \]
\[ q_r = \text{sgn}(w) \]

return
end

function DOT(v,w)
To return the dot product of vectors V and W

implicit none
double precision dot,v(3),w(3)
\[ \text{dot} = v(1) \times w(1) + v(2) \times w(2) + v(3) \times w(3) \]
return
end

subroutine CROSS(v,w,x)
To return X as the cross product of vectors V and W

implicit none
double precision x(3),w(3),v(3)
\[ x(1) = v(2) \times w(3) - v(3) \times w(2) \]
\[ x(2) = v(3) \times w(1) - v(1) \times w(3) \]
\[ x(3) = v(1) \times w(2) - v(2) \times w(1) \]
return
end