Parameter identification for the acoustic wave equation in helioseismology

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Abstract

The goal of helioseismology is to infer properties of the Sun interior using observations of solar oscillations on its surface. It requires a good knowledge of the wave propagation inside the Sun (forward model), of the noise properties of the observations and a reliable inversion method. In this paper, a simplified model (scalar acoustic wave equation) that captures most of the propagating aspects of the physics will be used. The goal is then to identify some parameters of this PDE that characterize the medium (density, sound speed) by using linear and nonlinear inversions.

Keywords: inverse problem, helioseismology, acoustic wave equation

1 Introduction

Helioseismology aims at recovering some properties of the solar interior from observations of the line-of-sight velocity $\psi(\mathbf{r}, t)$ where $\mathbf{r}$ are points on the surface and t is the time. From this time-serie, one generally computes the time $\tau(\mathbf{r}_1, \mathbf{r}_2)$ it takes for the wave to go between two points $\mathbf{r}_1$ and $\mathbf{r}_2$ at the solar surface. These quantities are the basic input of time-distance helioseismology [1]. In order to recover some properties q of the solar interior, they have to be linked to the observations. A simplified forward model that represents wave propagation in the Sun (PDE satisfied by ψ) is presented in Section 2. The observation operator that links travel-time to ψ is given in Section 3 and finally different inversion methods are compared in Section 4.

2 Forward problem

We consider that ψ satisfies an acoustic wave equation in the Sun Ω with homogeneous Dirichlet boundary conditions on $\partial\Omega$. The medium is assumed to be steady and is characterized by its density ρ and sound speed c. The source S is stationary and stochastic with zero mean and known covariance. The problem decouples for all frequencies ω and is given by

$$\begin{cases} \mathcal{L}\psi := -\sigma^2\psi - 2\mathrm{i}\omega\mathbf{u}\cdot\nabla\psi + H\psi = S & \text{in } \Omega \\ \psi = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

with $\sigma = \omega + \mathrm{i}\gamma$ and

$$H\psi = -c\nabla\cdot\left(\frac{1}{\rho}\nabla(\rho c\psi)\right). \tag{2}$$

The waves are damped by γ and are subject to a flow $\mathbf{u}$. Without flow and if the coefficients ρ and c are constant, then Eq. 1 is simply the Helmholtz equation. In the Sun, these coefficients vary strongly close to the boundary (several orders of magnitude) and care has to be taken in the numerical resolution. We use the Montjoie code [1] that solves Eq. 1 with finite elements. Details about the numerical scheme can be found in [3] where it is also shown that even if Eq. 1 is highly simplified, it captures most of the propagating aspects of the physics.

3 Observation operator

In order to link travel-time to the observations, let us first define the cross-covariances $C_{12}(\omega) = C(\mathbf{r}_1, \mathbf{r}_2, \omega)$ in the Fourier space between pairs of points $(\mathbf{r}_1, \mathbf{r}_2)$ on the solar surface by

$$C_{12}(\omega) = \psi^*(\mathbf{r}_1, \omega)\psi(\mathbf{r}_2, \omega). \tag{3}$$

The travel times are linearly dependent of the cross-covariance

$$\tau_{12} = \int W_{12}(\omega)^*\Big(C_{12}(\omega) - C_{12}^{\mathrm{ref}}(\omega)\Big)d\omega \tag{4}$$

with C^{ref} representing a reference cross-covariance that can come from a solar model or averaged observations [2] and W is a given function that depends on C^{ref}. We denote T the (quadratic) operator that maps the observations to ψ

$$\tau = T(\psi). \tag{5}$$

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4 Inversion

The observations τ are linked to some internal properties of the Sun q by a nonlinear inverse problem

$$F(q) = \tau, \qquad (6)$$

where F is defined implicitely by $F(q) = (T \circ \mathcal{L}_q^{-1})\psi$ with $\mathcal{L}_q$ given by Eq. 1 and the observation operator T by Eq. 5. Then the inverse problem can be stated:

The inverse problem (IP). *Knowing the observations τ^{obs}, the problem is to find the optimal parameter q solution of the nonlinear inverse problem (Eq. 6).*

4.1 Linear inversion

A classical approach to solve (IP) in helioseismology is to consider only first order perturbations by using the first Born approximation (single scattering approximation). In this case the perturbations are linearly linked to the observations

$$\mathbb{E}[\tau] = \sum_q \int_\Omega K_q(\mathbf{r})\delta q(\mathbf{r}) dV. \qquad (7)$$

The kernels are obtained by differentiating Eq. 4 and computing $\delta\psi$ at first order

$$\mathcal{L}_q[\delta\psi] = -\delta\mathcal{L}_q[\psi] + \delta S, \qquad (8)$$

with $\delta\mathcal{L}_q$ computed by deriving formally Eq. 1. For the different perturbations q, the sensitivity kernels K_q can be written as a function of G, C and of the operators H and σ. The exact expression of the kernels can be found in [3].

Eq. 7 can be solved for example by Tikhonov regularization

$$\min_{\delta q} \left(\|K_q \delta q - \mathbb{E}[\tau]\|^2 + \|L\delta q\|^2 \right), \qquad (9)$$

where L can be the identity or a discrete version of a gradient or a Laplacian in order to impose smoothness of the solution. (IP) can also be solved by the adjoint method [4] which employs techniques close to nonlinear inversions.

4.2 Nonlinear inversion

In order to find the optimal q by nonlinear methods, we need to be able to evaluate the forward operator $F(q_k)$, its derivative $F'[q_k]\delta q$ and the adjoint of the derivative $F'[q_k]^\dagger \delta C$. These three ingredients are required for all types of nonlinear inversions and can be computed by solving the same PDEs (the forward operator and its adjoint) but with different right hand side. For example, the update $\psi_{k+1} = \psi_k + \delta\psi$ is obtained from $F'[q_k]\delta q = T'[\psi_k]\delta\psi$ where $\delta\psi$ is the solution of

$$\mathcal{L}_q[\delta\psi] = -\delta\mathcal{L}_q[\psi_k](\delta q). \qquad (10)$$

An efficient method to solve (IP) is to use the conjugate gradient applied to the normal equation. We solve a quadratic least square problem to find δq_k that minimizes

$$\left\| F'[q_k]\delta q + F(q_k) - \tau^{\mathrm{obs}} \right\|^2, \qquad (11)$$

and the regularization is made by choosing an early stopping criterion at each iteration [5].

A comparison of the inversion methods will be presented showing which types of perturbations can be recovered with linear inversions and when nonlinear methods become necessary.

References

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