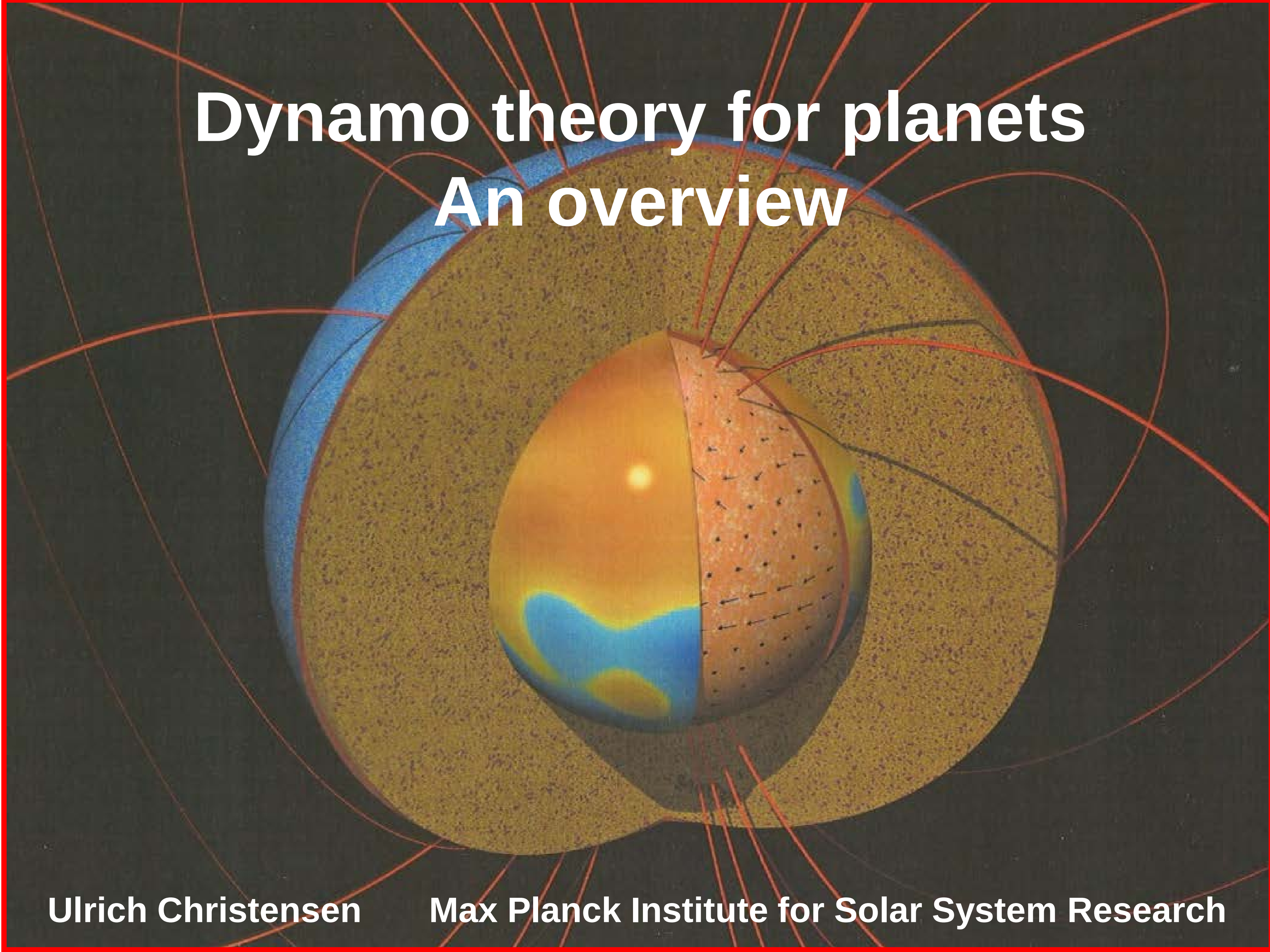
Dynamo theory for planets
An overview



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What is a dynamo ?

Wikipedia: Dynamo theory

Dynamo theory describes the process through which a rotating, convecting, and electrically conducting fluid acts to maintain a magnetic field

The disc dynamo

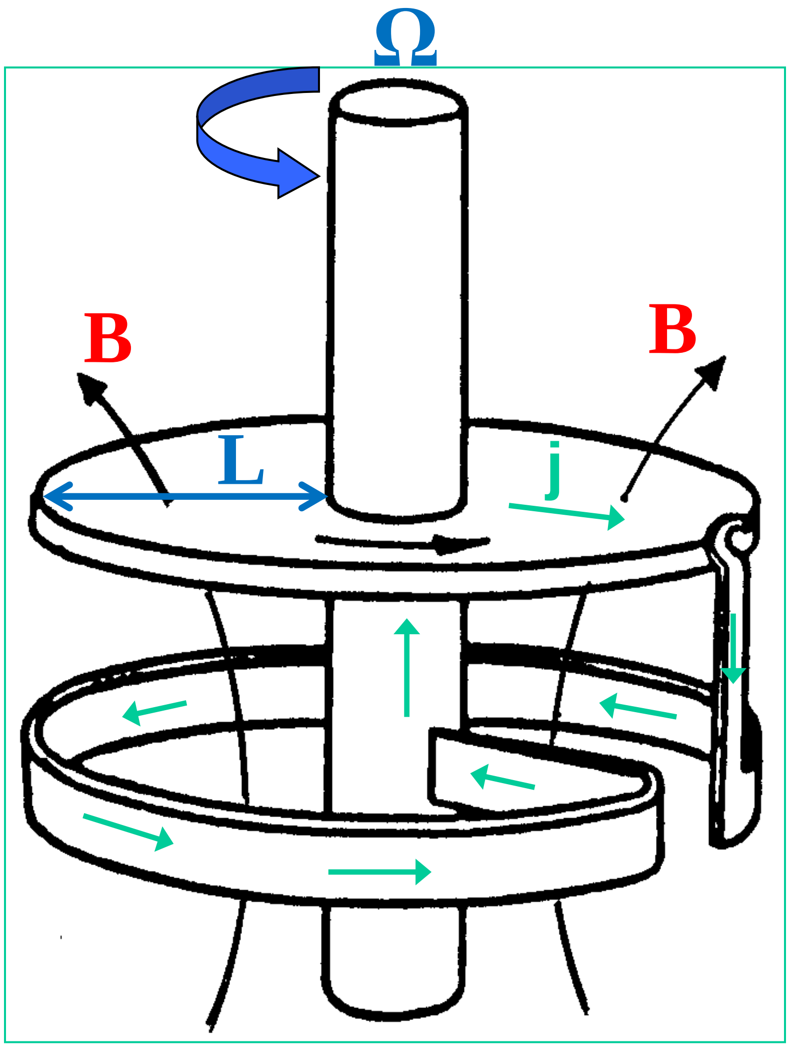
Self-sustained dynamo if some conditions are satisfied.

Helpful are:
• Fast rotation **Ω**
• High electrical conductivity **σ**
• Large size (length scale) **L**
⇒ Critical condition on $\sigma\Omega L^2$

Can start from tiny seed field B

Works in the same way when B is replaced by –B: two polarity states

When **Ω** kept constant, B will grow (or decay) exponentially.
When torque constant, Lorentz force opposes rotation (Lenz's rule) and leads to saturation of field strength



How can a dynamo work in planetary cores ?

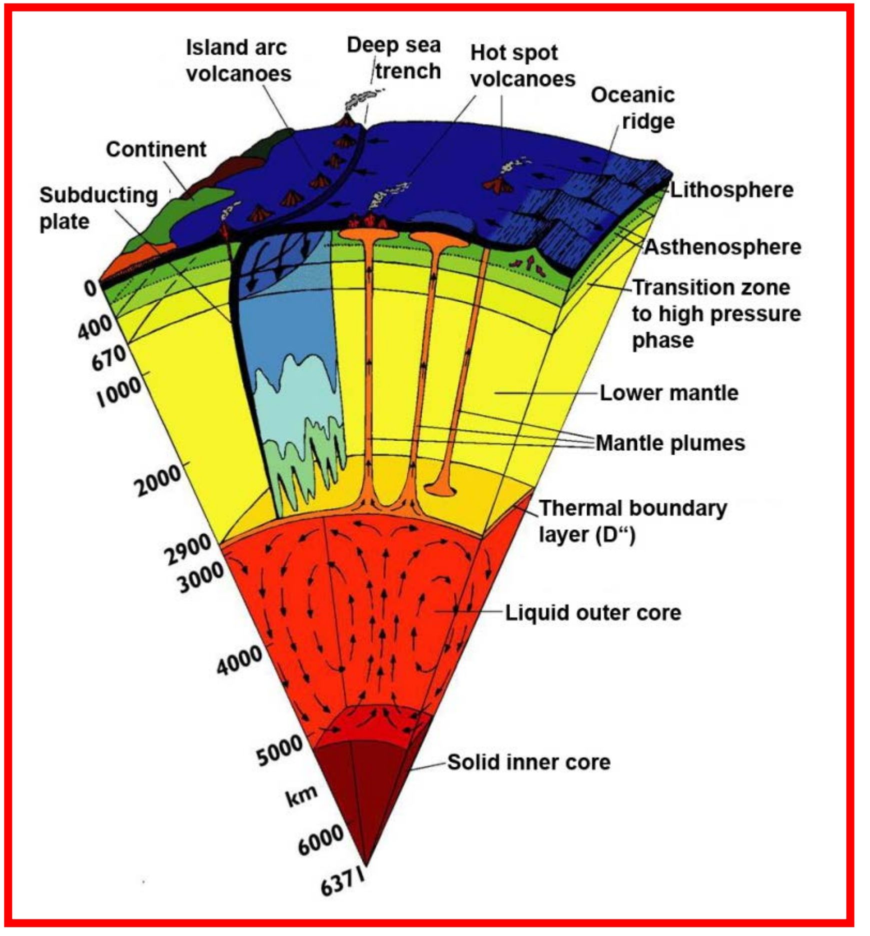
Some basic requirements satisfied:

- Fluid electrical conductor
- Set into motion by thermal or compositional convection

But:

Almost homogeneous sphere without wires and sliding contacts
⇒ „shortcut“ between different parts

For homogeneous dynamo to work, the complex geometry of electrical conductors in a technical dynamo must be replaced by a complex (but ordered) structure of the fluid flow



Planetary rotation (Coriolis force) helps to set up such structure

Magnetic induction equation

How does a magnetic field **B** evolve in a fluid with conductivity σ for a known velocity field **u** ?

(Pre-) Maxwell equations

$$\nabla \times \mathbf{B} = \mu \mathbf{j} + \cancel{1/c^2 \, \partial \mathbf{E}/\partial t} \qquad \textbf{(1) Ampère's law}$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \qquad \textbf{(2) Faraday's law}$$

$$\mathbf{j} = \sigma \mathbf{E}'$$

Ohm's law

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$

Lorentz transform

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad \textbf{(3)}$$

Ohm's law in moving medium

Substitute (3) into (1), take the curl, and eliminate E by using eqn. (2):

$$\partial \mathbf{B}/\partial t = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{1}{\mu\sigma} \nabla \times \nabla \times \mathbf{B}$$

[Assumptions made: u << c ; μ,σ isotropic and constant]

Magnetic induction equation II

$$\partial \mathbf{B} / \partial t = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{1}{\mu\sigma} \nabla \times \nabla \times \mathbf{B}$$

(1) with $\nabla \cdot \mathbf{B}=0$: $\nabla \times \nabla \times \mathbf{B} = - \nabla^2 \mathbf{B}$

(2) introduce magnetic diffusivity $\lambda = 1/(\mu\sigma)$

(3) incompressible flow ($\nabla \cdot \mathbf{u}=0$) : $\nabla \times (\mathbf{u} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{B}$

$$\partial \mathbf{B} / \partial t + (\mathbf{u} \cdot \nabla)\mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{u} + \lambda \nabla^2 \mathbf{B}$$

advection induction diffusion

Compare advection-diffusion equation for temperature:

$$\partial T / \partial t + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T$$

Magnetic Reynolds number

$$\partial \mathbf{B} / \partial t + (\mathbf{u} \cdot \nabla)\mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{u} + \lambda \nabla^2 \mathbf{B}$$

Introduce non-dimensional variables (underlined) by scaling with characteristic values for length L, velocity U_o and magnetic field strength B_o: $\underline{\mathbf{u}} = \mathbf{u}/U_o$, $\underline{\nabla} = L\nabla$, $\underline{t} = t\, U_o/L$, $\underline{B} = B/B_o$.

The non-dimensional induction equation contains one non-dimensional parameter, the magnetic Reynolds number Rm.

$$\partial \underline{\mathbf{B}} / \partial \underline{t} + (\underline{\mathbf{u}} \cdot \underline{\nabla})\underline{\mathbf{B}} = (\underline{\mathbf{B}} \cdot \underline{\nabla})\underline{\mathbf{u}} + \mathrm{Rm}^{-1} \underline{\nabla}^2 \underline{\mathbf{B}}$$

$$\mathrm{Rm} = \frac{U_0 L}{\lambda}$$

For Rm < 1, diffusion dominates; for Rm > 1 advection and induction dominates

Kinematic dynamo problem

$$\partial \underline{\mathbf{B}} / \partial \underline{t} + (\underline{\mathbf{u}} \cdot \underline{\nabla})\underline{\mathbf{B}} = (\underline{\mathbf{B}} \cdot \underline{\nabla})\underline{\mathbf{u}} + \mathrm{Rm}^{-1} \underline{\nabla}^2 \underline{\mathbf{B}}$$

Assume flow pattern **u** and seek solutions of magn. induction eqn.

Field structure: eigenmodes that grow or decrease with time

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_n(\mathbf{r}) \exp(\sigma_n t)$$

Seek growing solutions with Re(σ)>0. For small Rm only decaying solutions will exist. What is, for a given flow pattern, the minimum value of Rm for which growing solutions exist?

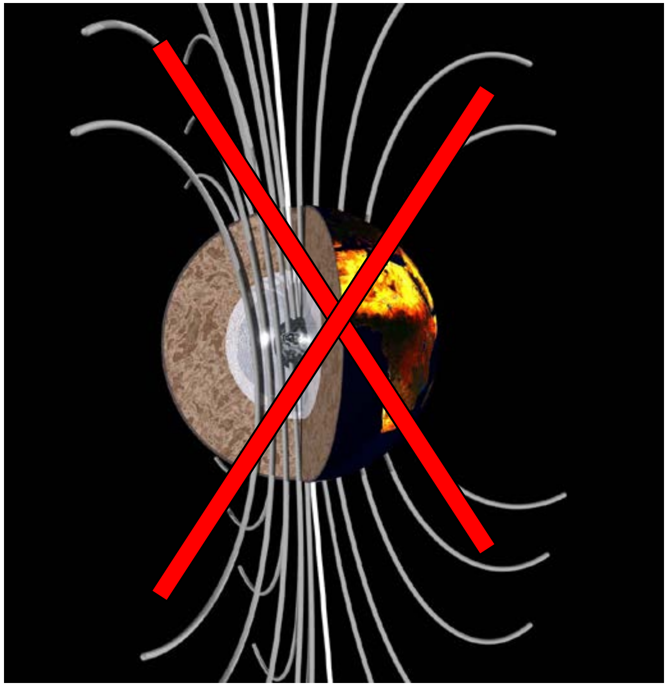
In simplest case with u=0 only decaying solutions. Free decay modes (mode with slowest decay in a sphere of radius R is a dipole field with $\sigma = -\pi^2\lambda/R^2$)

Kinematic dynamo theory: Cowling‘s theorem

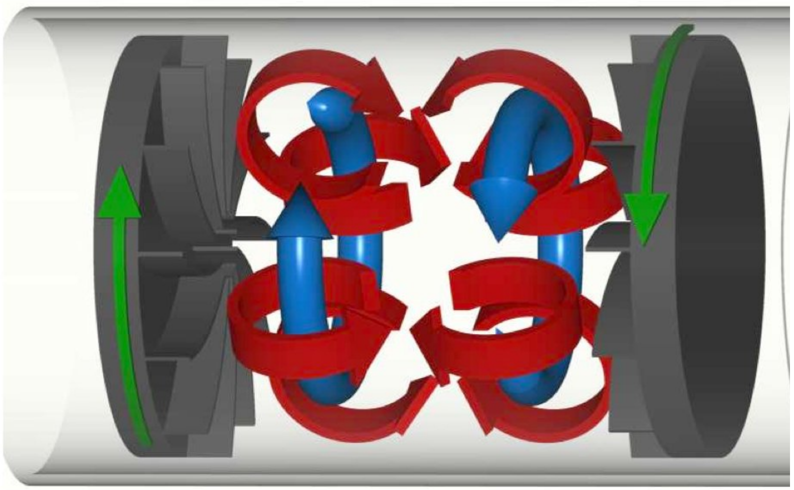
A (strictly) axisymmetric magnetic field cannot be generated by a dynamo

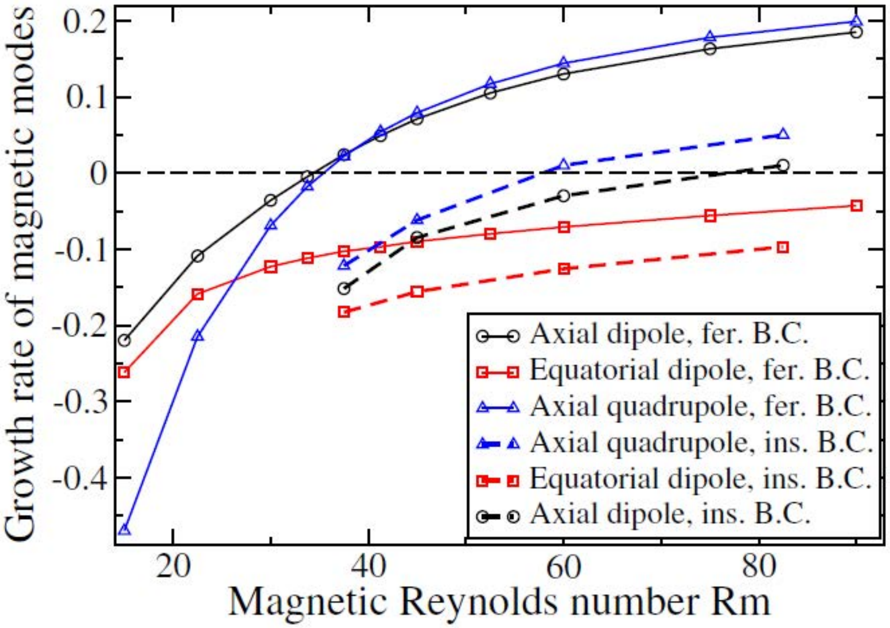
(Example for an „antidynamo-theorem“, Cowling, 1934)

Implication: It is not possible to study dynamos in two dimensions as a simplifying step. Any numerical dynamo model must be 3D.



Kinematic dynamo: an example





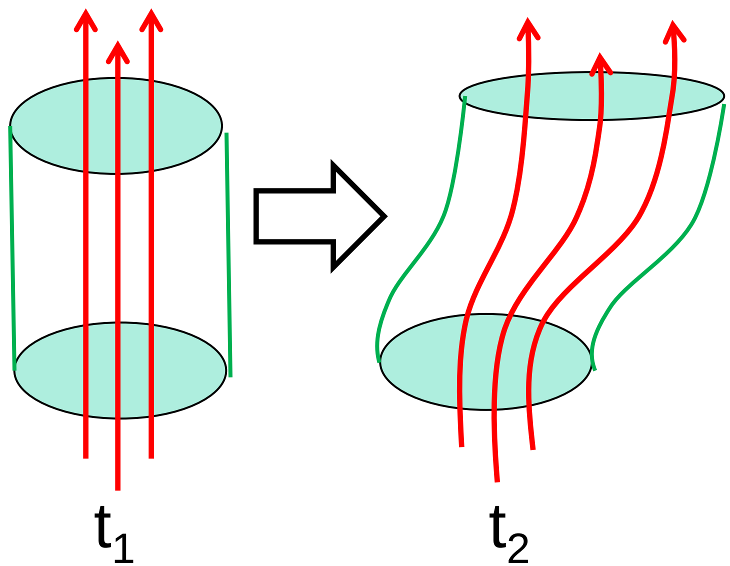
Schematic flow pattern in VKS dynamo experiment, where two propellers drive a flow of liquid sodium in a cylindrical container

Kinematic dynamo simulation for VKS flow for different magnetic field modes and different boundary conditions (Gissinger, EPL, 2009)

Alfvén's theorem (frozen flux)

Consider case of negligible diffusion (Rm $\to \infty$)

Magnetic flux $\Phi = \iint \mathbf{B} \cdot \mathbf{dS}$ passing through a bounded material surface that moves and deforms with the flow does not change.



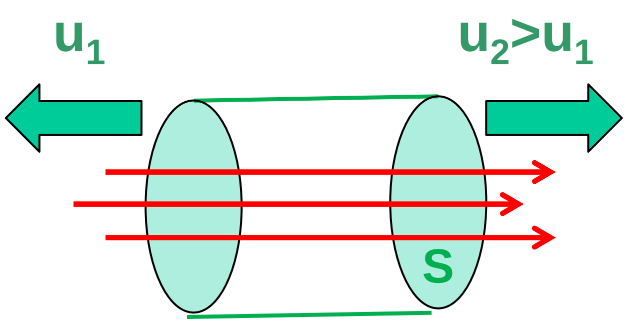
A related statement is that, if a magnetic field line passes at a given time through a certain chain of fluid particles, it will always pass through the same droplets, irrespective of how they move (frozen field line).

For large Rm>>1 it holds approximately and is a useful concept to understand how the magnetic field evolves.

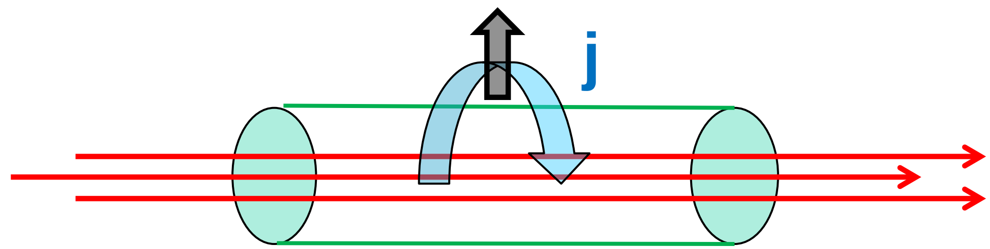
Field line stretching

$$\partial \mathbf{B} / \partial t + ... \;=\; (\mathbf{B} \cdot \nabla)\mathbf{u} + ...$$

Induction term positive when fluid stretches in direction of field lines



F = j x B (Lorentz force)



Flux through cylinder face S conserved; S shrinks $\Rightarrow$ B grows

Magnetic energy density $\mathbf{B}^2/(2\mu_o)$ increases (volume conserved)

Contraction of cylinder requires work done against Lorentz force that points radially outward

Hydrodynamic flow in rotating reference system

Navier-Stokes equation with Coriolis term

$$\rho(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla p - 2\rho(\Omega\times\mathbf{u}) + \rho\nu\nabla^2\mathbf{u} + \mathbf{F}_{body}$$

inertia Coriolis viscosity

Non-dimensionalize by scaling u, ∇, t as before and scale p by $\rho\Omega U_o L$

$$\mathrm{Ro}(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla p - 2(\mathbf{e}_z\times\mathbf{u}) + \mathrm{E}\nabla^2\mathbf{u} + \mathbf{F}_{body}$$

		Earth core value
Rossby number	**Ro = $U_o/(L\Omega)$**	**10^{-6}**
Ekman number	**E = $\nu/(L^2\Omega)$**	**10^{-15}**

ρ: density p: pressure $\mathbf{\Omega} = \Omega\mathbf{e}_z$: rotation vector ν: kinematic viscosity

Proudman-Taylor theorem

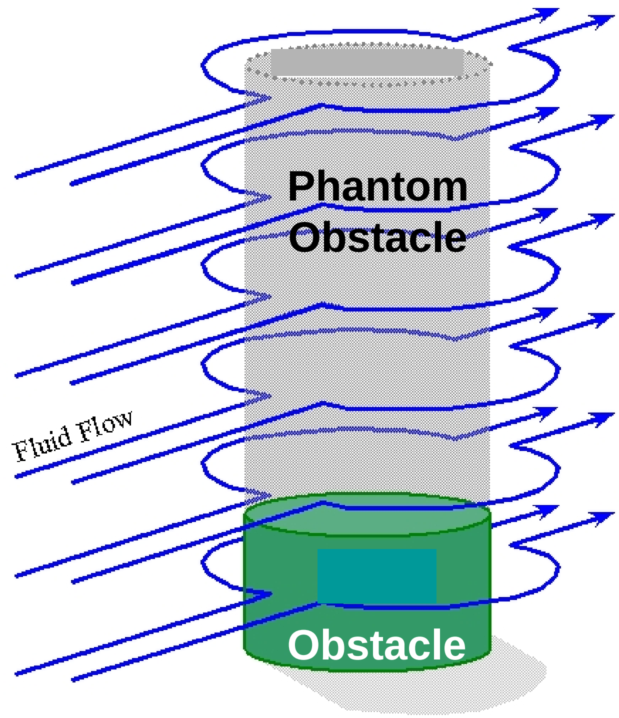
Ro << 1 and E << 1 and ignore F_{body}

$2\mathbf{e}_z \times \mathbf{u} + \nabla p = 0$ **Geostrophic flow** (follows isobars)

take curl $\Rightarrow$ $\partial \mathbf{u}/\partial z = 0$

Velocity vector does not change in direction of rotation axis !

Cylindrical water tank on rotating table with small off-center obstacle on bottom



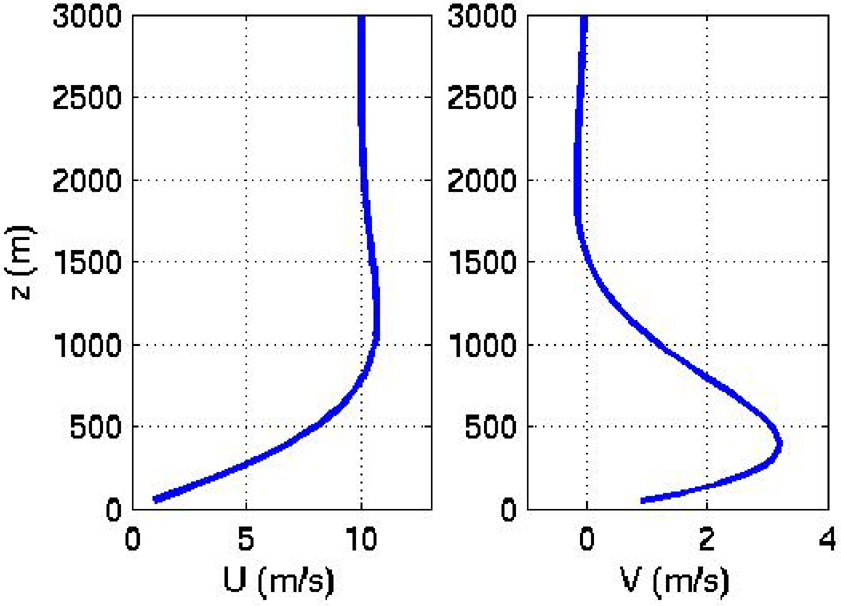
Ekman boundary layer

When fluid meets a rigid wall in z-direction, where u=0, Proudman-Taylor theorem cannot be strictly satisfied. In a thin boundary layer of width δ, viscosity enters the force balance

$$2(\mathbf{e}_z \times \mathbf{u}) \approx \mathrm{E}\nabla^2\mathbf{u} \qquad \mathrm{E}\delta^{-2} \approx 1 \qquad \delta \approx \mathrm{E}^{1/2}$$

For Earth's core, $\delta \approx 10^{-7.5}$
in dimensional terms $d_{Ekman} = 0.1$ m

Flow velocity not only decreases in the Ekman layer, but also changes direction. Close to the rigid boundary, it moves at a 45 degree angle to the isobars



Rotating MHD equation

MHD = magnetohydrodynamic

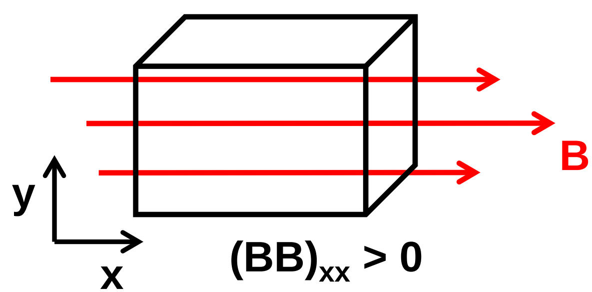
$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p - 2\rho(\Omega\times\mathbf{u}) + \rho\nu\nabla^2\mathbf{u} + \frac{1}{\mu}(\nabla\times\mathbf{B})\times\mathbf{B} + \rho\mathbf{g}$$

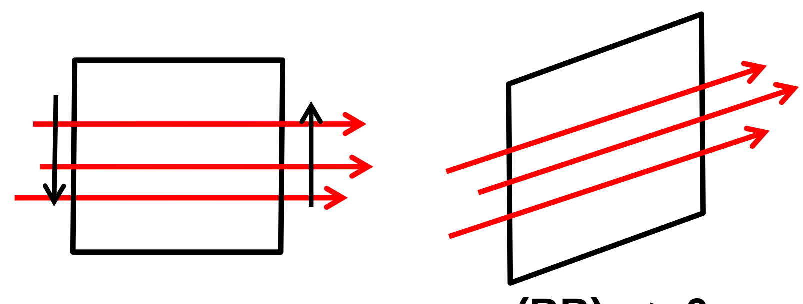
inertia — Coriolis — viscosity — Lorentz — gravity

Lorentz force can be expressed as gradient of a pressure-like term and divergence of a (stress) tensor. Tensor component $(\mathbf{BB})_{ij} = B_i B_j$

$$\frac{1}{\mu}(\nabla\times\mathbf{B})\times\mathbf{B} = -\nabla\left(\frac{B^2}{2\mu}\right) + \nabla\cdot\left(\frac{\mathbf{BB}}{\mu}\right)$$

magnetic pressure — Maxwell stress tensor





Magnetic field lines in a good conductor (Rm>>1) can be compared with elastic strings that resist stretching and bending

Full dynamo equations

Non-dimensional, for convection-driven flow in a rotating system

$$\mathrm{E}\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p' - 2(\mathbf{e}_z\times\mathbf{u}) + \mathrm{E}\nabla^2\mathbf{u} + \frac{1}{\mathrm{Pm}}(\nabla\times\mathbf{B})\times\mathbf{B} + \frac{\mathrm{RaE}}{\mathrm{Pr}}\frac{g(r)}{g_0}\mathrm{T}$$

$$\partial\mathbf{B}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{u} + \frac{1}{\mathrm{Pm}}\nabla^2\mathbf{B}$$

$$\partial\mathrm{T}/\partial t + (\mathbf{u}\cdot\nabla)\mathrm{T} = \frac{1}{\mathrm{Pr}}\nabla^2\mathrm{T}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \nabla\cdot\mathbf{B} = 0$$

Scaling of variables: length scale L, time scale L^2/ν, velocity scale ν/L, magnetic field scale $(\rho\mu\lambda\Omega)^{1/2}$, temperature scale ΔT. p' is non-hydrostatic pressure.

Boussinesq-approximation: ρ = constant, except in the gravity term (buoyancy term) of the momentum equation, where $\rho=\rho_o(1-\alpha T)$

Non-dimensional control parameters

Definition	Name	Balance	Earth value	Model values
$Ra = \frac{\alpha g_o \Delta T L^3}{\kappa \nu}$	Rayleigh number	Buoyancy Diffusion	10^4 x critical ?	1 – 100 x critical
$E = \nu/\Omega L^2$	Ekman number	Viscosity Coriolis force	10^{-15}	$\geq 10^{-6}$
$Pr = \nu/\kappa$	Prandtl number	Viscosity Thermal diffusion	0.1 - 1	0.1 – 10
$Pm = \nu/\lambda$	Magnetic Prandtl #	Viscosity Magnetic diffus.	10^{-6}	0.06 - 20

α: thermal expansion coefficient, g_o: gravity (reference value) ΔT: temperature contrast

Non-dimensional diagnostic parameters

	Name	Ratio of	Earth	Models
Re = UL/ν	Reynolds number	Nonlinear inertia / Viscosity	10^8	10 - 2000
Rm = UL/λ	Magnetic Reynold#	Advection / Magnet. diffus.	10^3	40 - 3000
Ro = $U/\Omega L$	Rossby number	Nonlinear inertia / Coriolis	10^{-5}	10^{-4} - 1
Nu = q/q_{con}	Nusselt number	Total heat flow / Conductive heat	? (>>1)	1 - 30
$\Lambda = B^2/\rho\mu\lambda\Omega$	Elsasser number	Lorentz force / Coriolis force	1 - 10	0.03 - 100

Geometry and boundary conditions

Rotating fluid spherical shell

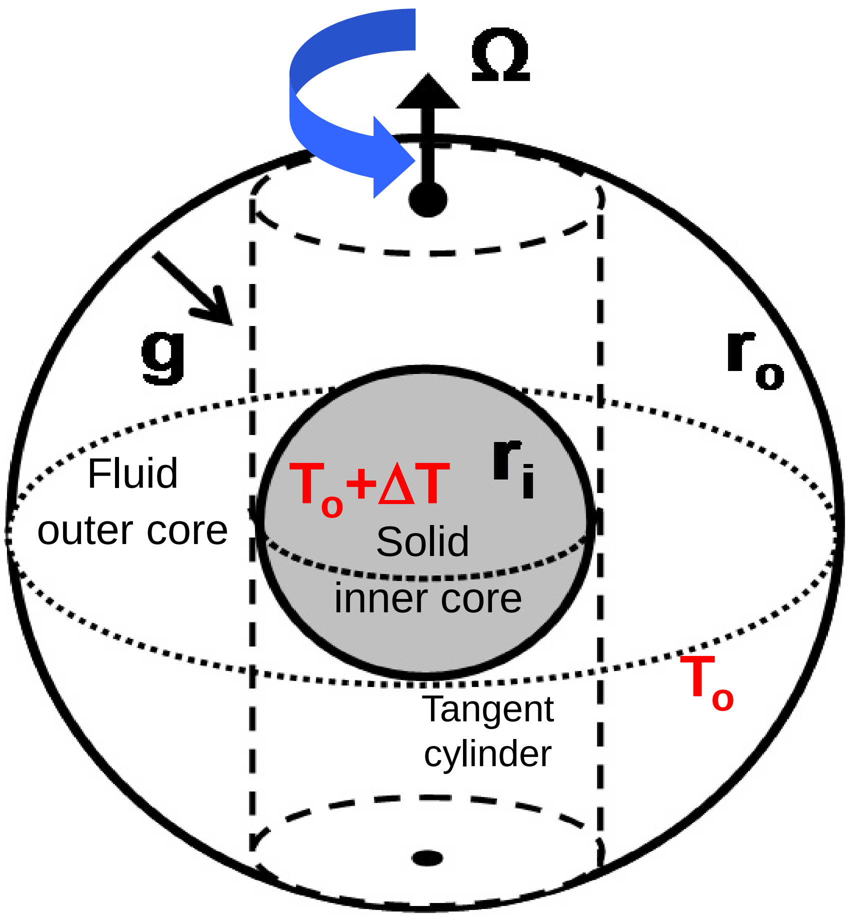
Impenetrable boundaries

No slip or free-slip boundaries

Fixed temperatures or fixed heat flux boundaries

Magnetic field: match at boundary to potential field in the outside that decays with radius (at least $\propto r^{-3}$)

Tangent cylinder divides shell into dynamically distinct regions

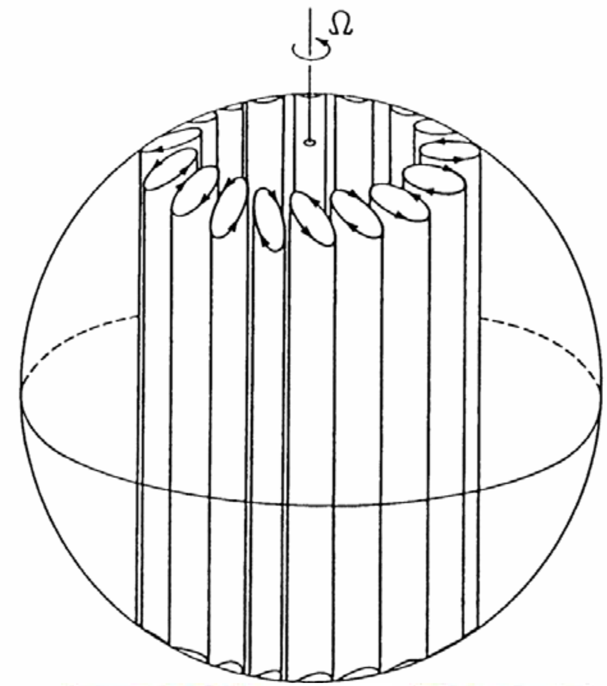


Convection in rotating sphere

Rayleigh # must exceed critical value Ra_{crit}

At low E, convection starts in form of narrow elongated columns parallel to Ω (Taylor / Busse rolls) [PT-theorem]

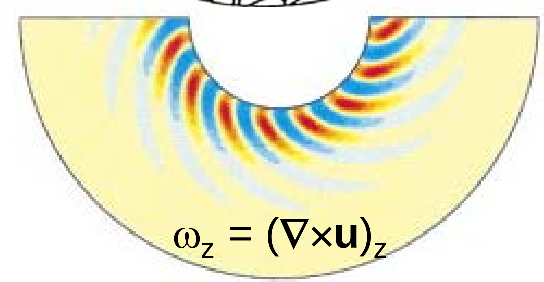
The columns end at sloping boundaries
$\Rightarrow$ Proudman-Taylor not strictly satisfied



For E << 1:

$$Ra_{crit} \propto E^{-4/3}$$

$$m_{crit} \propto E^{-1/3} \text{ (wave \#)}$$

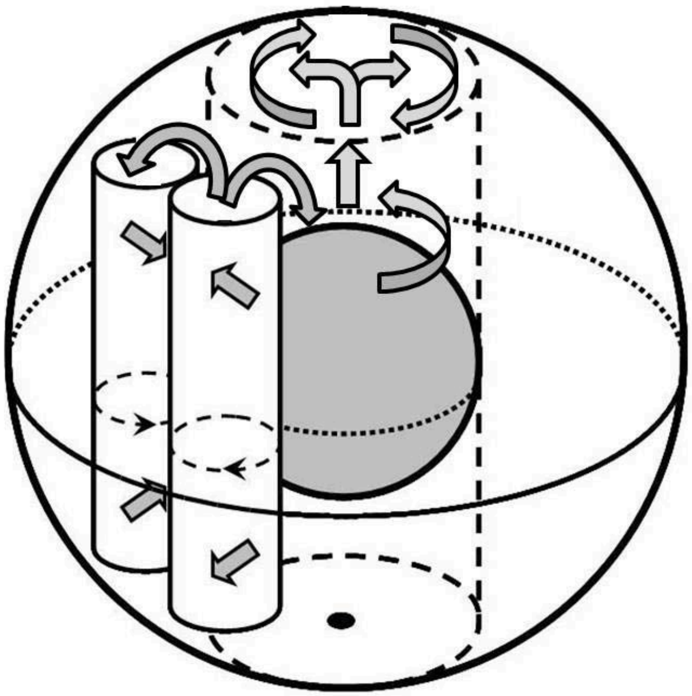


Columns array near tangent cylinder (TC) boundary, outside the TC.
Convection inside the TC occurs only when Ra significantly > Ra_{crit}

Helical columnar convection

Primary motion is around column axis

A secondary motion is along the column axis, diverging from the equatorial plane in columns with clockwise motion (as seen from North) and converging towards equator in anticlockwise columns. The net motion is spiralling or *helical*.



Helicity: $H = \mathbf{u} \cdot (\nabla \times \mathbf{u})$

Helicity is consistently negative in the northern hemisphere and positive in the southern hemisphere

Note: When Ra >> Ra_{crit}, flow structures are no longer well ordered, but many properties persist in a statistical sense

Toroidal-poloidal decomposition

Any vector field B with $\nabla\cdot B=0$ can be written as a combination of a toroidal part B_T and poloidal part B_P:

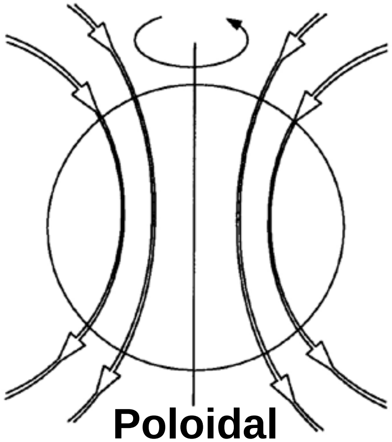
$$B = B_T + B_P = \nabla\times T e_r + \nabla\times(\nabla\times P e_r)$$

T **and** P **are scalar fields.**

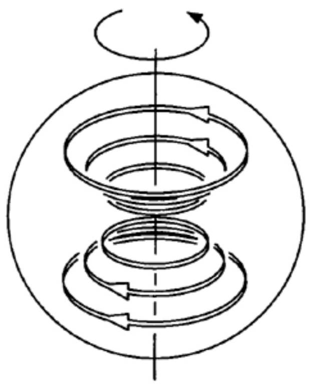
B_T has no radial component . B_T is confined to inside the dynamo and „invisible“ from outside

The current associated with B_P ($j\sim\nabla\times B_P$) has no radial component

A dynamo cannot generate a purely toroidal or a purely poloidal magnetic field



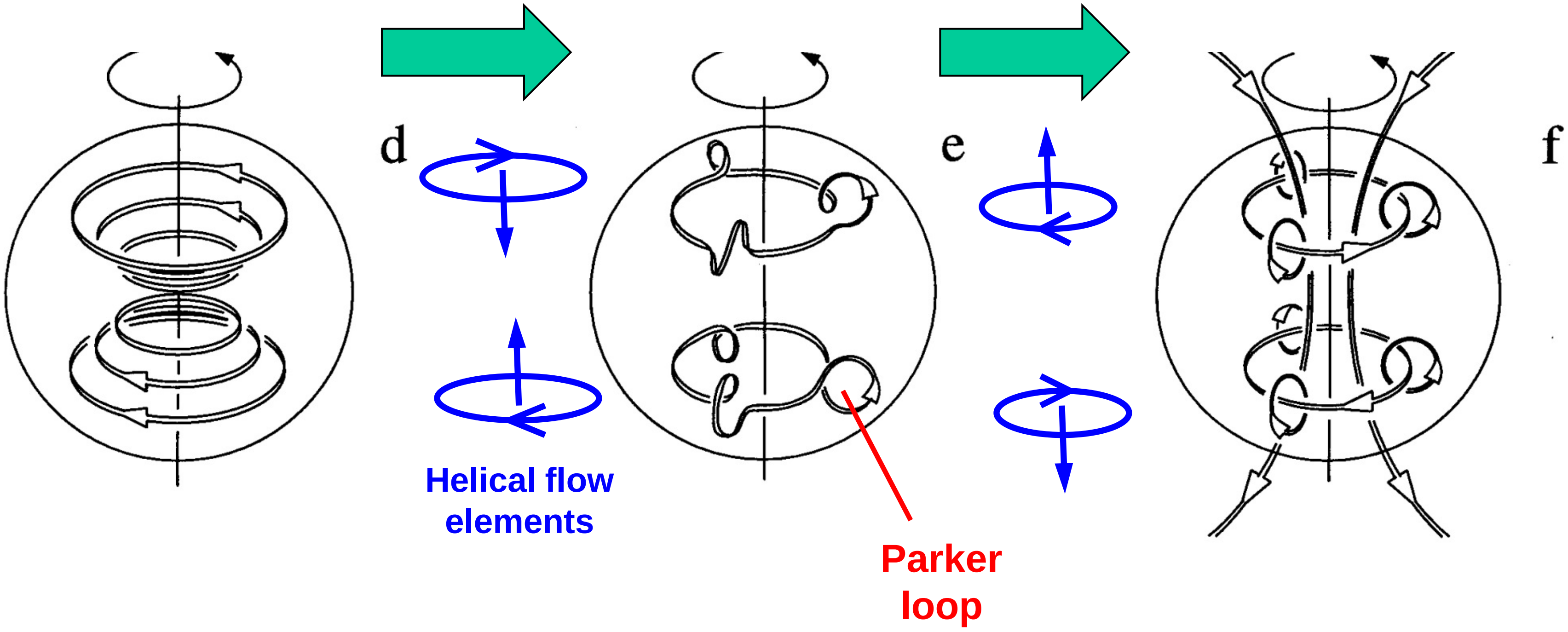
Poloidal



Toroidal

α - effect

How can an axisymmetric poloidal field be (re-)created from an axi-symmetric toroidal field by induction processes that may involve small-scale flow and small-scale magnetic fields ?

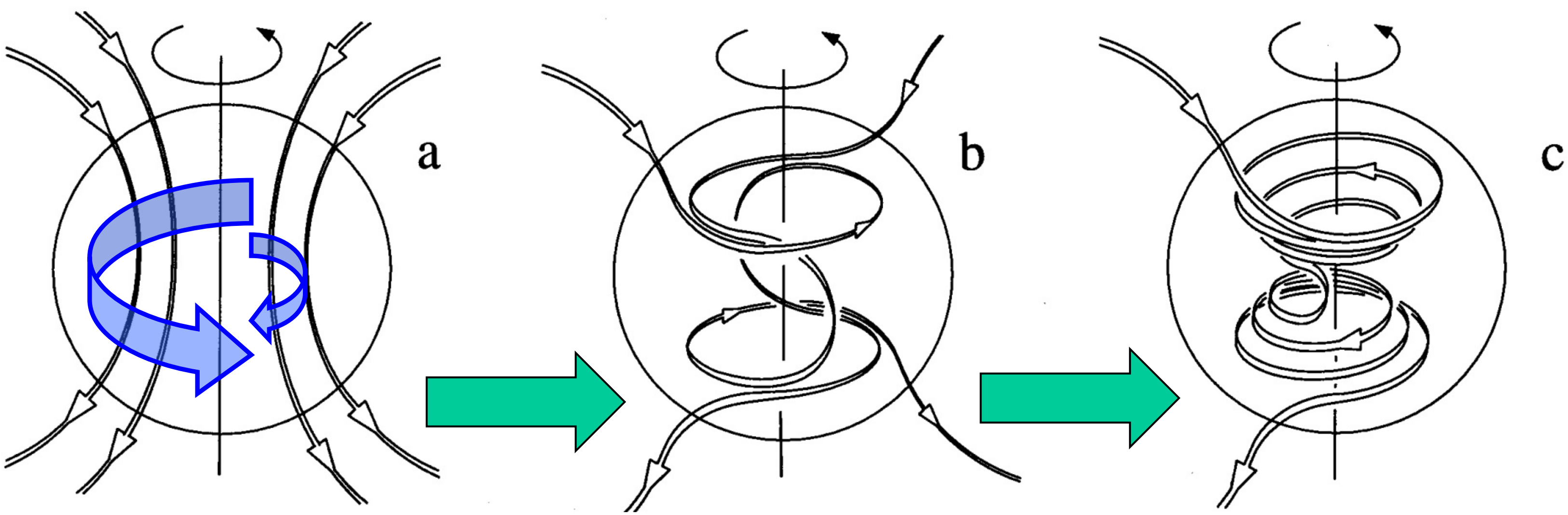


ω - effect

How can an axisymmetric toroidal field be (re-)created from an axi-symmetric poloidal field ?

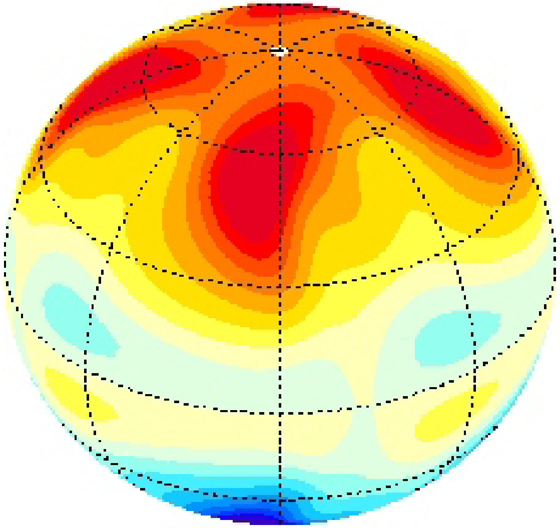
(1) The α-effect can also work in this direction

(2) Another mechanism is through shearing of poloidal field lines by axisymmetric toroidal flow (differential rotation) – the ω-effect

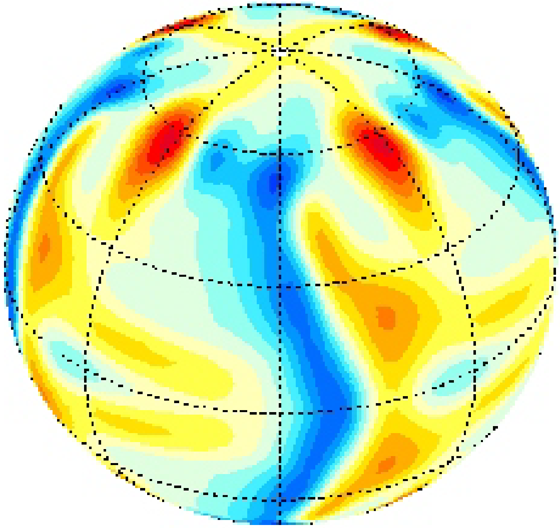


A simple dynamo model

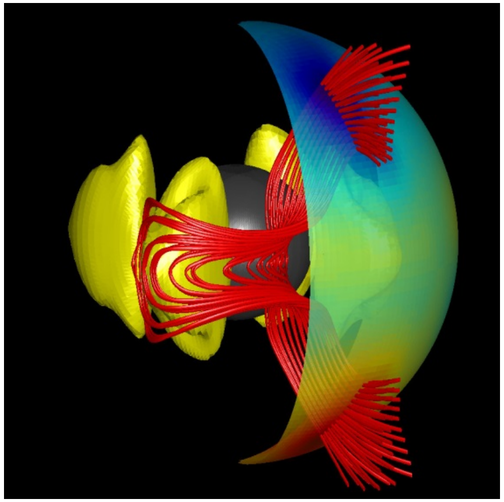
The benchmark dynamo



B_r at $r=r_o$



u_r at $r=0.9r_o$



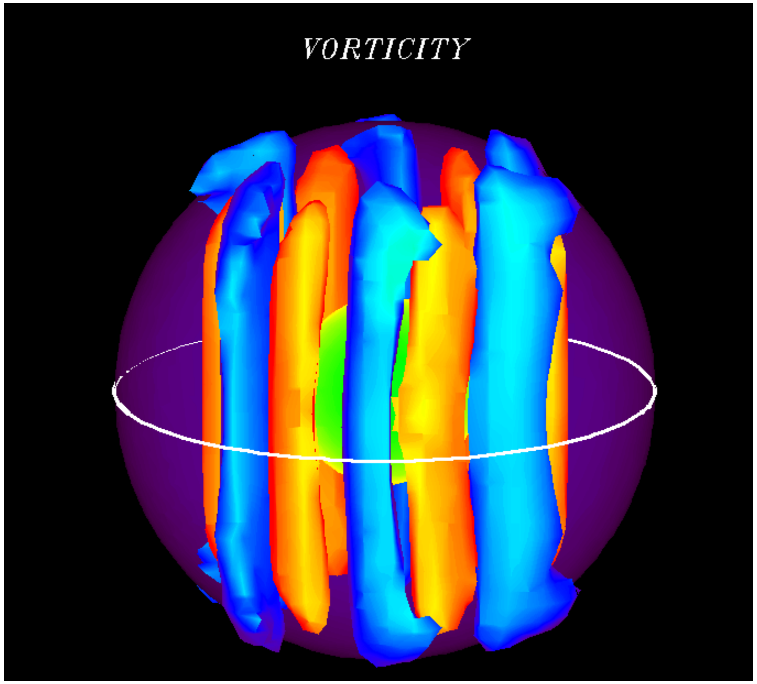
Magnetic field lines

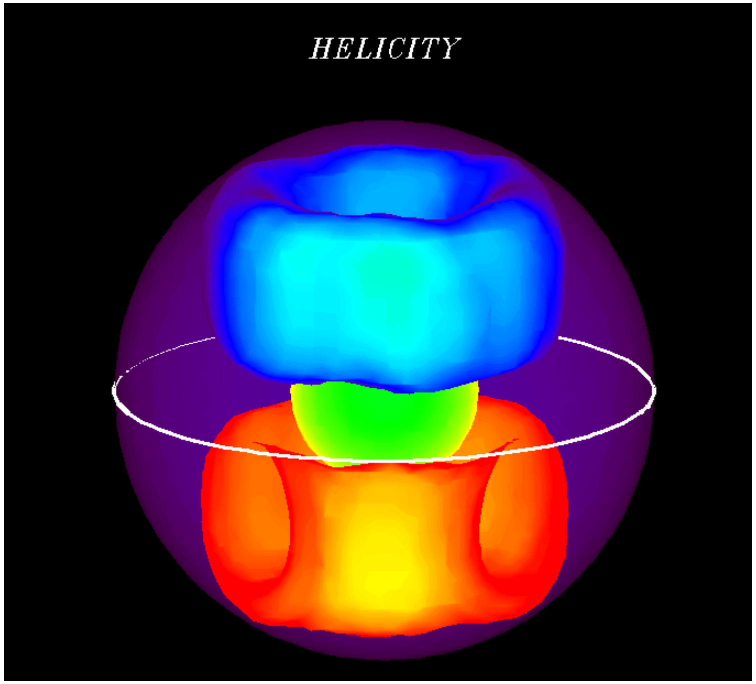
$Ra=10^5$, $E=10^{-3}$, $Pr=1$, $Pm=5$.

Solution is stationary, aside from a drift in longitude

Rm=39 (near minimum required for a dynamo)

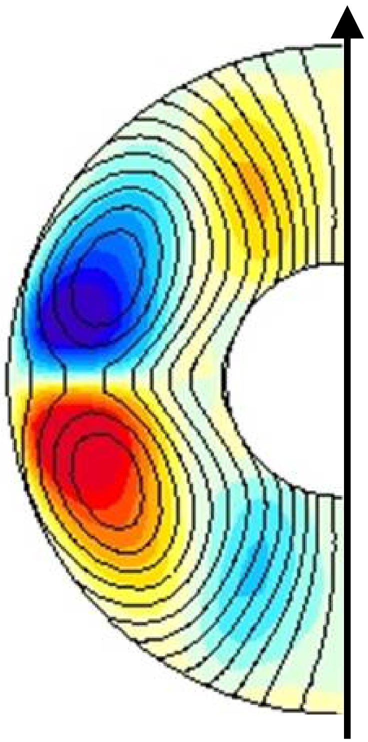
Flow structure



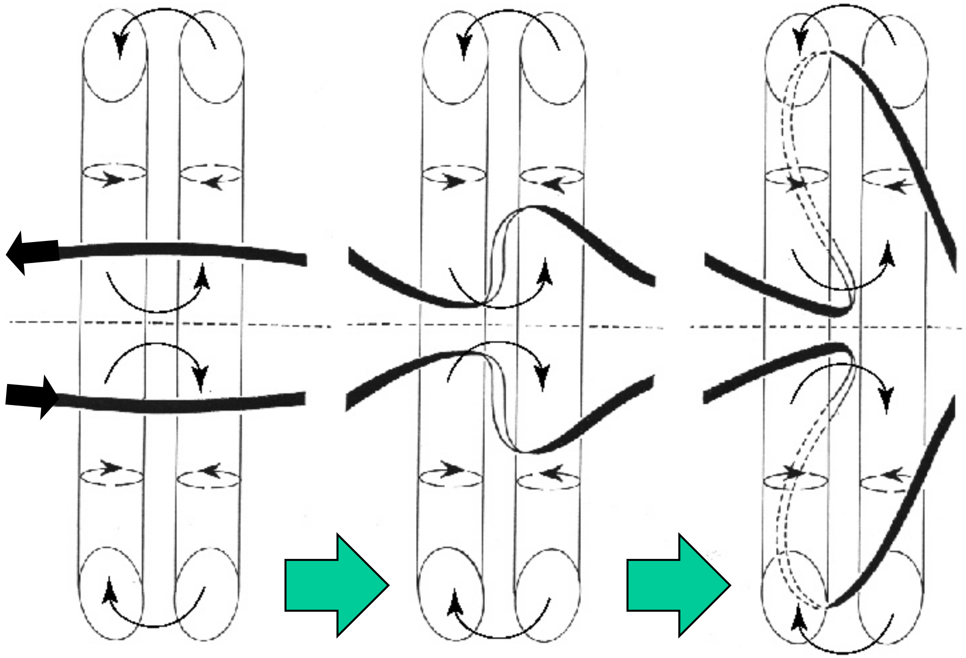


Vorticity $\nabla \times \mathbf{u}$ **Helicity** $\mathbf{u} \cdot (\nabla \times \mathbf{u})$

Toroidal - poloidal conversion in benchmark dynamo



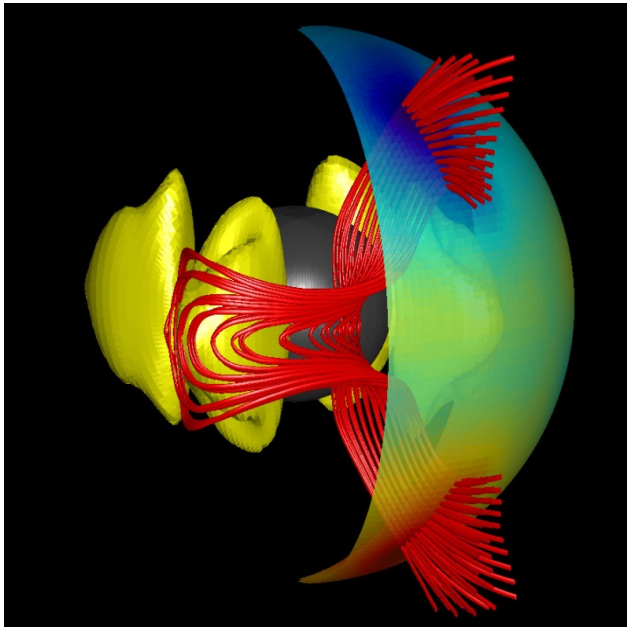
Axisymmetric field. Toroi-dal: color, poloidal: lines



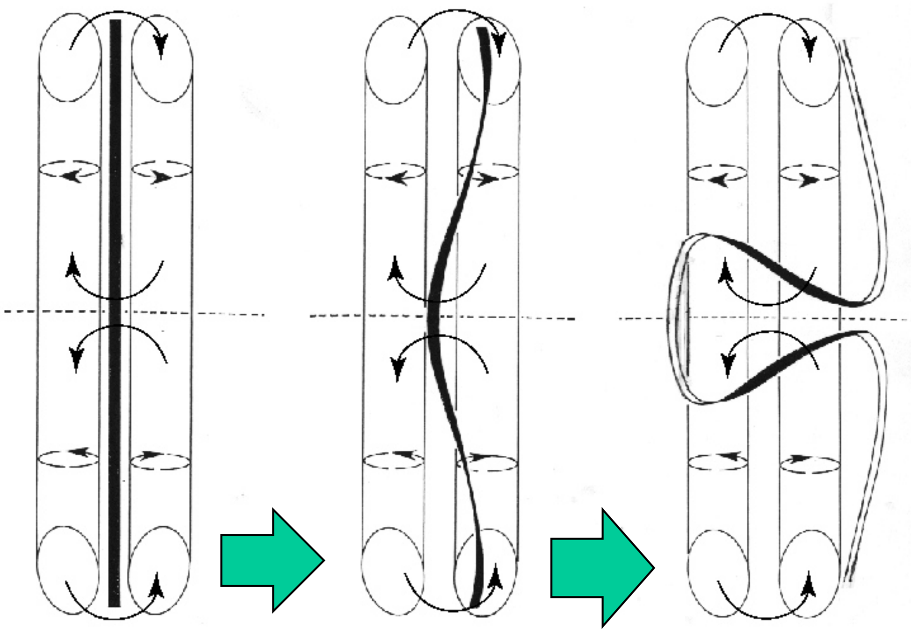
Advection and twisting of toroidal field lines (assuming frozen flux) by helical columar flow, transforming them into configuration with strong poloidal dipole part

The dipole field is generated from the toroidal axisymmetric field by a macroscopic α-effect in the helical convection columns.

Poloidal - toroidal conversion in benchmark dynamo



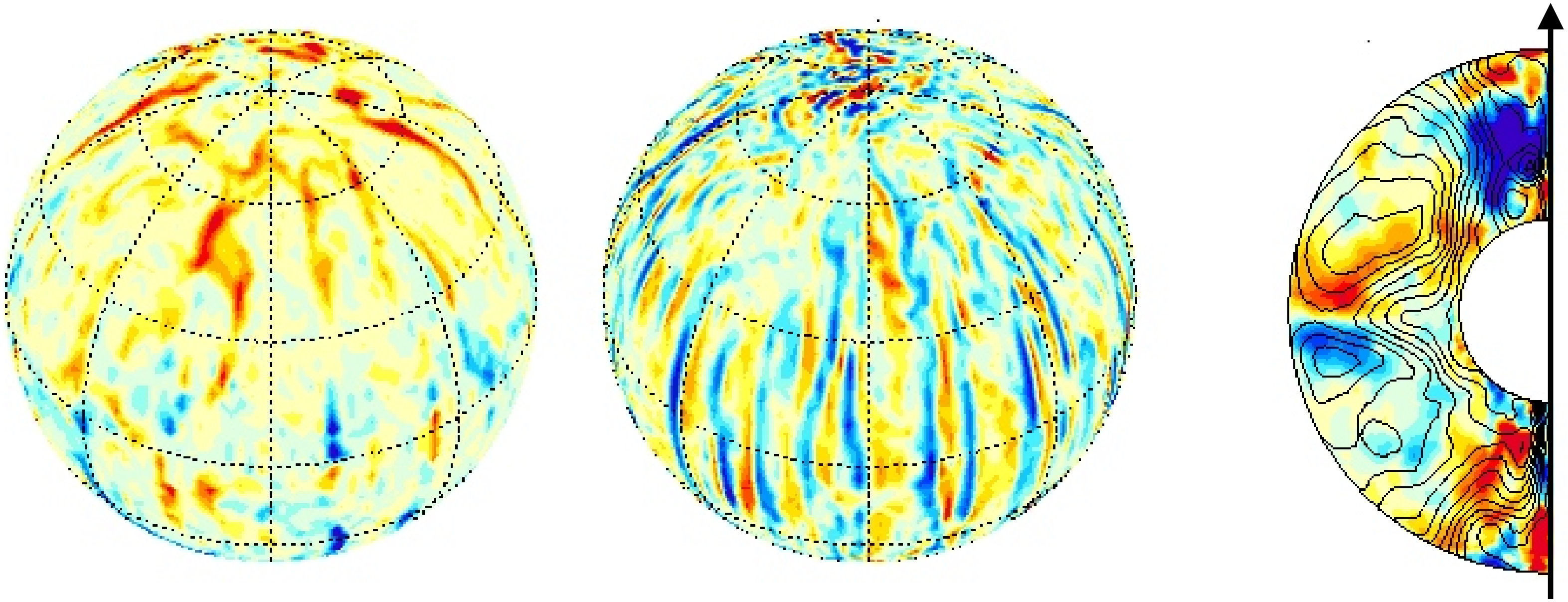
Bundle of field lines. Yellow: anticyclonic vortices.



Evolution of a poloidal field line starting close to the tangent cylinder.

Axisymmetric toroidal field generated by α-effect in helical columns. In this case ω-effect does not play a constructive role

More extreme parameters



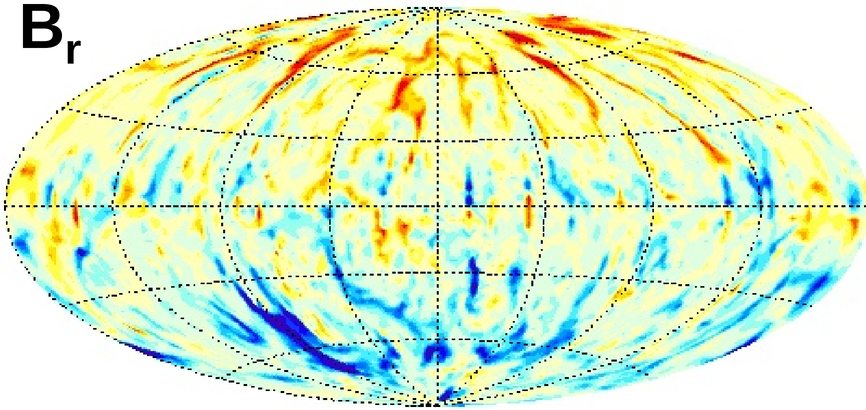
Radial magnetic field **Radial velocity** **Axisymmetric field**

$Ra = 1.2 \times 10^8$ $Pr = 1$ $E = 3 \times 10^{-5}$ $Pm = 2.5$ $Rm = 925$

Small scale magnetic flux concentrations, but dipole still dominant.
N-S-alignment of flow structures ⇒ (imperfect) convection columns
In addition strong convection inside tangent cylinder (TC).
Similar toroidal field as in simple model outside TC ⇒ same mechanism
Strong toroidal flux rings inside TC generated by ω-effect

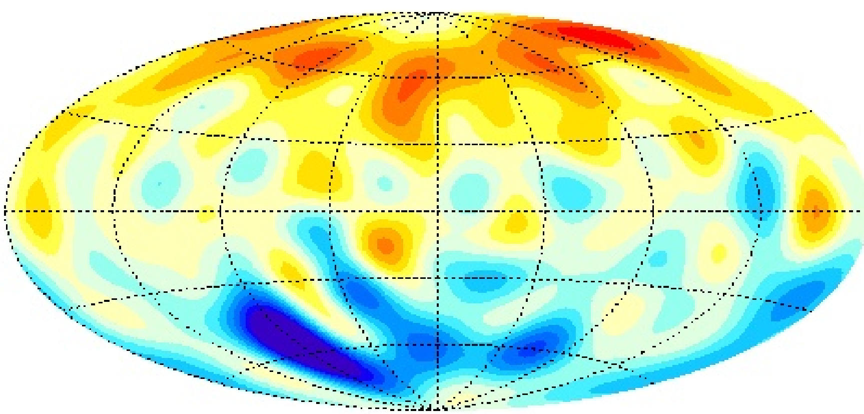
Comparison with Earth‘s field at core-mantle boundary

B_r

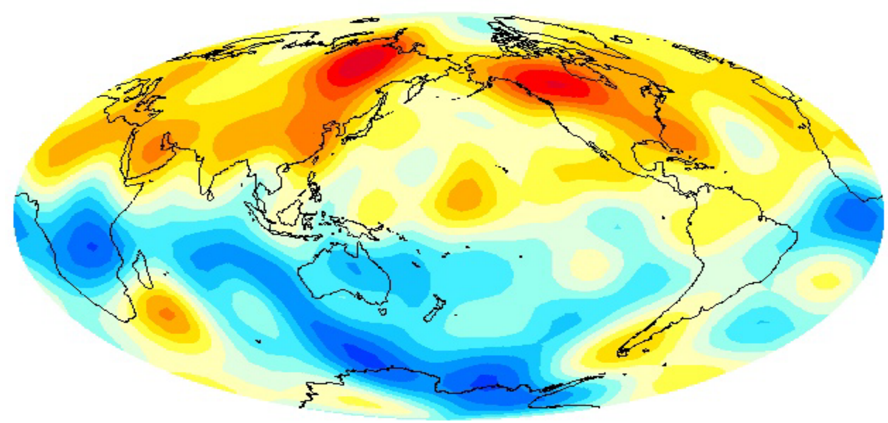


Dynamo model, full resolution

Because in the observed field the contribution of crustal origin dominates at degrees n>13 , we do not know the small-scale core field. Comparison is best made at the same spatial resolution



Dynamo model, filtered to n<13



Earth‘s CMB field