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1+1+2 gravitational perturbations on LRS class II spacetimes: II. Decoupling gravito-electromagnetic 2-vector and scalar harmonic amplitudes

R B Burston

Max Planck Institute for Solar System Research, 37191 Katlenburg-Lindau, Germany

E-mail: burston@mps.mpg.de

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Abstract
This is the second paper in a series that considers first-order, gauge-invariant and covariant, gravitational perturbations to *locally rotationally symmetric* (LRS) class II vacuum spacetimes. Focusing on the 1+1+2 *gravito-electromagnetic* (GEM) formalism, the first paper used linear algebra techniques to derive four decoupled equations that govern four specific combinations of the GEM 2-tensor harmonic amplitudes. This paper completes the decoupling of the 1+1+2 GEM system by showing how to derive seven new decoupled quantities. Four of these arise when considering the GEM 2-vector harmonic amplitudes and it is found that decoupling is achieved by combining these with the (2/3-sheet) shear 2-tensor harmonic amplitudes. The remaining three arise from the 1+1+2 GEM scalars. Two of which concern the 2-gradient of the gravito-electric scalar that must also be combined with shear 2-tensor amplitudes, whereas the other involves the gravito-magnetic scalar only.

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1. Introduction

Clarkson and Barrett's, gauge-invariant and covariant, 1+1+2 formalism [2, 3] is very well suited for describing *locally rotationally symmetric* (LRS) spacetimes [4–6]. This formalism was first developed in [2] for an analysis of vacuum gravitational perturbations to a covariantly defined Schwarzschild spacetime, and was later used to study first-order *electromagnetic* (EM) perturbations to both LRS class II spacetimes in [7–9] and LRS spacetimes in [10, 11].

In [9], we expressed Maxwell's equations in a new 1+1+2 complex form that was convenient for decoupling the first-order 1+1+2 EM 2-vectors and scalars. It was discovered that there exists four specific combinations of the EM 2-vector amplitudes that decouple and these were separated into polar $\{\mathscr{E}_\mathrm{v} + \bar{\mathscr{B}}_\mathrm{v}, \mathscr{E}_\mathrm{v} - \bar{\mathscr{B}}_\mathrm{v}\}$ and axial $\{\mathscr{B}_\mathrm{v} + \bar{\mathscr{E}}_\mathrm{v}, \mathscr{B}_\mathrm{v} - \bar{\mathscr{E}}_\mathrm{v}\}$ perturbations.

Subsequent to this, the first paper in this series [1], hereafter paper I, presented an analysis of 1+1+2 first-order gravitational and energy–momentum perturbations to vacuum LRS class II spacetimes. The focus was on demonstrating how to decouple specific components of the *gravito-electromagnetic* (GEM) 2-tensors $\mathcal{E}_{\mu\nu}$ and $\mathcal{H}_{\mu\nu}$. It is well established that the GEM formalism has remarkably similar features when compared to the EM formalism [12, 13]. Thus, based on the success of the methods used in [9], we applied the same eigenvector analysis to the 1+1+2 GEM formalism. This ultimately leads to the discovery that there are four specific combinations of the GEM 2-tensor amplitudes that decouple and these may also be categorized into polar $\{\mathcal{E}_{\rm T}+\bar{\mathcal{H}}_{\rm T}, \mathcal{E}_{\rm T}-\bar{\mathcal{H}}_{\rm T}\}$ and axial $\{\mathcal{H}_{\rm T}+\bar{\mathcal{E}}_{\rm T}, \mathcal{H}_{\rm T}-\bar{\mathcal{E}}_{\rm T}\}$ perturbations. This once again illustrates the strong similarities between the GEM and EM formalisms when one makes the direct correspondence between $\mathcal{E}_{\rm T} \leftrightarrow \mathscr{E}_{\rm v}$ and $\mathcal{H}_{\rm T} \leftrightarrow \mathscr{B}_{\rm v}$.

To complete the decoupling of Maxwell's equations it is also possible to decouple equations for the EM scalars, $\mathscr{E}$ and $\mathscr{B}$, and their corresponding scalar harmonic amplitudes, $\mathscr{E}_{\rm s}$ and $\mathscr{B}_{\rm s}$. This was initially demonstrated in [7], and later reproduced and analyzed further by using the new 1+1+2 complex formalism in [9]. The purpose of this paper is to complete the decoupling of the first-order 1+1+2 GEM formalism, as presented in paper I, by showing how to decouple an additional seven new quantities. This is achieved by constructing specific combinations of the GEM 2-vectors and scalars with the 2/3-sheet shear 2-tensors. Four of these quantities concern the GEM 2-vectors whereas the remaining three govern the GEM scalars. Furthermore, this analysis also demonstrates the decoupling between the gravito-magnetic and gravito-electric scalars.

In this series, the decoupling has been achieved in two levels. Paper I first demonstrated how to decouple the complex GEM 2-tensor, $\Phi_{\mu\nu} := \mathcal{E}_{\mu\nu} + {\rm i}\mathcal{H}_{\mu\nu}$, which was useful for inferring that the GEM 2-tensors clearly decouple from the remaining GEM 2-vectors and scalars. General harmonic expansions [1–3, 7] of the GEM 2-tensors were then useful for pushing the decoupling further for a more pragmatic result, i.e. the four real decoupled equations as mentioned above. This paper follows a similar trend by first decoupling newly defined 2-vectors, and subsequently, uses harmonic expansions to ultimately result in seven decoupled quantities. The use of harmonic expansions to render equations into a more pragmatic form has been traditionally used throughout the literature. Spherical harmonic expansions were used to derive the famous Regge–Wheeler (RW) [14] equation that govern the metric perturbations of a Schwarzschild spacetime (also see [15] for a collection of some important developments). Other examples include perturbations of higher-dimensional spacetimes by Kodama *et al* [16, 17] who used the harmonic expansions defined by [18–20] to derive decoupled equations for their perturbation variables, essentially generalizing four-dimensional cases by [21, 22].

Some typical vacuum LRS class II applications include the Schwarzschild spacetime, which can be easily expressed using a variety of different choices for the frame vectors. Such as the traditional static metric, freely falling observers, or even the recently discovered 'generalized Painleve–Gullstrand' (GPG) coordinate systems [23]. LRS class II also applies to some spatially homogeneous and anisotropic vacuum Bianchi models. For example, the Kasner line element, ${\rm d}s^2 = {\rm d}t^2 - t^{2p_1}\,{\rm d}x^2 - t^{2p_2}\,{\rm d}y^2 - t^{2p_3}\,{\rm d}z^2$ where the constants p_1, p_2 and p_3 satisfy $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$, describes the vacuum Bianchi I models [24], which are in general not LRS. However, it is possible to make specific choices for the constants such that it reduces to LRS class II and hence a subset of vacuum Bianchi I [25].

In section 2, a brief qualitative review of the background LRS class II vacuum spacetime is given as well as the corresponding first-order gravitational and energy–momentum perturbation variables. There is also a summary of the important results arising from paper I. Section 3 explains the immediate complications that arise when attempting to use the 1+1+2 complex

GEM system for further decoupling, and how a specific choice of new dependent variables overcomes these issues. Sections 4 and 5 then deliver a treatment of the GEM 2-vectors and scalars respectively. A summary then collates the results obtained here with the results from paper I. Finally, we adhere to the notations and conventions of paper I and [2].

2. Preliminaries

This section presents a qualitative introduction to the notation used throughout this paper and also a brief summary of the main results from paper I. For further information, refer to [2] for the initial derivation of Clarkson and Barrett's 1+1+2 formalism and see [1, 3, 7, 10, 11] for later developments.

2.1. Background LRS class II spacetimes

The background comprises the most general vacuum LRS class II spacetime where there exists a preferred spatial direction indicated by the radial vector n^μ [2]. These spacetimes are defined by six non-vanishing scalars,

$$\text{LRS class II} : \{\mathcal{A}, \theta, \phi, \Sigma, \mathcal{E}, \Lambda\}. \tag{1}$$

Here, $\mathcal{A}$ is the radial acceleration of the 4-velocity, θ and ϕ are respectively the expansions of the 3-sheets and 2-sheets, and Σ is the radial part of the shear of the 3-sheet. The radial part of the gravito-electric tensor is $\mathcal{E}$, and finally, Λ is the cosmological constant.

The background 1+1+2 Ricci identities, for both the timelike 4-velocity (u^μ) and the spacelike radial vector, and the Bianchi identities yield a set of evolution, transportation and constraint equations governing these scalars. Those specific to this analysis can be seen in paper I and they can also be found in [3, 7–10].

2.2. First-order gravitational and energy–momentum perturbations

The gravitational and energy–momentum perturbations on the background LRS class II spacetimes are quantities of first-order (ϵ) and are reproduced from paper I,

$$\text{first-order scalars:} \quad \{\xi, \Omega, \mathcal{H}, \mu, p, \mathcal{Q}, \Pi\} = \mathcal{O}(\epsilon), \tag{2}$$

$$\begin{aligned}\text{first-order 2-vectors:} \quad &\{a^\mu, \alpha^\mu, \mathcal{A}^\mu, \Omega^\mu, \Sigma^\mu, \mathcal{E}^\mu, \mathcal{H}^\mu, V^\mu, W^\mu, X^\mu, Y^\mu, Z^\mu, \mathcal{Q}^\mu, \Pi^\mu\}\\ &= \mathcal{O}(\epsilon),\end{aligned} \tag{3}$$

$$\text{first-order 2-tensors:} \quad \{\mathcal{E}_{\mu\nu}, \mathcal{H}_{\mu\nu}, \zeta_{\mu\nu}, \Sigma_{\mu\nu}, \Pi_{\mu\nu}\} = \mathcal{O}(\epsilon). \tag{4}$$

The first-order scalars can be described as follows; ξ is the twisting of the 2-sheet, Ω is the radial part of the vorticity of the 3-sheet and the radial part of the gravito-magnetic tensor is $\mathcal{H}$. The energy–momentum quantities, mass–energy density, pressure, radial heat flux and radial anisotropic stress are denoted respectively μ, p, $\mathcal{Q}$ and Π. The first-order 2-vectors are the acceleration a^μ, the projection onto 2-sheets of the 'dot' derivative of the radial vector produces α^μ, and similarly the projection of the 'dot' derivative of the 4-velocity gives $\mathcal{A}^\mu$. The vorticity 2-vector is Ω^μ, whereas Σ^μ, $\mathcal{E}^\mu$ and $\mathcal{H}^\mu$ arise from the 1+1+2 decomposition of the shear of the 3-sheet and the GEM tensors respectively. The next five terms were defined in paper I in order to maintain a gauge-invariant theory (according to the Sachs–Stewart–Walker lemma [26, 27]) and this was initiated in [2]. They consist of 2-gradients, δ_μ is the covariant 2-derivative associated with the 2-sheets, according to

$V_\mu := \delta_\mu\left(\Sigma + \frac{1}{3}\theta\right)$, $W_\mu := \delta_\mu\left(\Sigma - \frac{2}{3}\theta\right)$, $X_\mu := \delta_\mu\mathcal{E}$, $Y_\mu := \delta_\mu\phi$ and $Z_\mu := \delta_\mu\mathcal{A}$. Finally, the energy–momentum 2-vectors are $\mathcal{Q}^\mu$ and Π^μ which are related to flux and anisotropic stresses respectively. The first-order GEM 2-tensors are $\mathcal{E}_{\mu\nu}$ and $\mathcal{H}_{\mu\nu}$, the 2-tensors describing the shear of the 2/3-sheets are respectively $\zeta_{\mu\nu}$ and $\Sigma_{\mu\nu}$, and finally, $\Pi_{\mu\nu}$ is the anisotropic stress which has been projected onto the 2-sheets. In general, the quantities (2)–(4) will not be frame invariant as they will depend on the choice of first-order 4-velocity and radial vector and this issue is discussed in greater detail in [2].

Finally, the first-order 1+1+2 Ricci identities and twice contracted Bianchi identities give a system of differential equations governing the gauge-invariant perturbation quantities. They are not reproduced here but should be referred to in paper I along with important commutation relationships between various derivative operators (they can also be seen in [3] for LRS spacetimes).

2.3. Summary of paper I

The primary results from paper I involved using an eigenvector and eigenvalue analysis to rewrite the 1+1+2 GEM system in a new complex form according to

$$\left(\mathcal{L}_n + \frac{3}{2}\phi\right)\mathcal{C}_{\bar{\mu}} + \delta_\mu\delta^\alpha\Phi_\alpha + \frac{3}{2}\mathcal{E}\left[Y_\mu - \phi a_\mu - 2\left(\Sigma - \frac{2}{3}\theta\right)\epsilon_\mu{}^\alpha\Omega_\alpha + \mathrm{i}2\delta_\mu\Omega\right] = \delta_\mu\mathcal{G}, \tag{5}$$

$$\left(\mathcal{L}_u - \frac{3}{2}\Sigma + \theta\right)\mathcal{C}_{\bar{\mu}} + \mathrm{i}\delta_\mu(\epsilon^{\alpha\beta}\delta_\alpha\Phi_\beta) - \frac{3}{2}\mathcal{E}\left[\mathcal{A}_\mu\left(\Sigma - \frac{2}{3}\theta\right)\right.$$
$$\left. + \phi(\Sigma_\mu - \epsilon_\mu{}^\alpha\Omega_\alpha + \alpha_\mu) + W_\mu - \mathrm{i}2\delta_\mu\xi\right] = \delta_\mu\mathcal{F}, \tag{6}$$

$$(\mathcal{L}_n + \phi)\Phi_{\bar{\mu}} + \delta^\alpha\Phi_{\mu\alpha} - \frac{1}{2}\mathcal{C}_\mu - \mathrm{i}\frac{3}{2}\Sigma\epsilon_\mu{}^\alpha\Phi_\alpha + \frac{3}{2}\mathcal{E}\Lambda_\mu = \mathcal{G}_\mu, \tag{7}$$

$$\left(\mathcal{L}_u - \Sigma + \frac{2}{3}\theta\right)\Phi_{\bar{\mu}} + \mathrm{i}\epsilon_\mu{}^\alpha\delta^\beta\Phi_{\alpha\beta} + \mathrm{i}\frac{1}{2}\epsilon_\mu{}^\alpha\left[\mathcal{C}_\alpha - (2\mathcal{A} - \phi)\Phi_\alpha\right] + \frac{3}{2}\mathcal{E}\Upsilon_\mu = \mathcal{F}_\mu, \tag{8}$$

$$\left(\mathcal{L}_u + \frac{5}{2}\Sigma + \frac{1}{3}\theta\right)\Phi_{\bar{\mu}\bar{\nu}} - \mathrm{i}\epsilon_{(\mu}{}^\alpha\left(\mathcal{L}_n + 2\mathcal{A} - \frac{1}{2}\phi\right)\Phi_{\bar{\nu})\alpha} + \mathrm{i}\epsilon_{\{\mu}{}^\alpha\delta_{|\alpha|}\Phi_{\nu\}} + \frac{3}{2}\mathcal{E}\Lambda_{\mu\nu} = \mathcal{F}_{\mu\nu}, \tag{9}$$

where complex variables were defined

$$\mathcal{C}_\mu := X_\mu + \mathrm{i}\delta_\mu\mathcal{H}, \qquad \Phi_\mu := \mathcal{E}_\mu + \mathrm{i}\mathcal{H}_\mu \qquad \text{and} \qquad \Phi_{\mu\nu} := \mathcal{E}_{\mu\nu} + \mathrm{i}\mathcal{H}_{\mu\nu}. \tag{10}$$

Various other natural definitions also arose,

$$\Upsilon_\mu := \alpha_\mu + \mathrm{i}\epsilon_\mu{}^\alpha\mathcal{A}_\alpha, \qquad \Lambda_\mu := a_\mu + \mathrm{i}\epsilon_\mu{}^\alpha\left(\Sigma_\alpha + \epsilon_\alpha{}^\beta\Omega_\beta\right)$$
$$\text{and} \qquad \Lambda_{\mu\nu} := \Sigma_{\mu\nu} + \mathrm{i}\epsilon_{(\mu}{}^\alpha\zeta_{\nu)\alpha}. \tag{11}$$

Furthermore, the following quantities are all explicitly defined in paper I; $\epsilon_{\mu\nu}$ is the Levi-Civita 2-tensor, $\mathcal{L}_u$ and $\mathcal{L}_n$ are Lie derivatives with respect to u^μ and n^μ respectively, and the first-order complex energy–momentum sources are $\mathcal{G}$, $\mathcal{F}$, $\mathcal{G}_\mu$, $\mathcal{F}_\mu$ and $\mathcal{F}_{\mu\nu}$. Using (5)–(9), along with the 1+1+2 Ricci identities and commutation relationships from paper I, a relatively simple decoupled equation was achieved,

$$[(\mathcal{L}_u + \theta)\mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + \phi)\mathcal{L}_n - V]\Phi_{\mu\nu}$$
$$- \mathrm{i}\,\epsilon_{(\mu}{}^\alpha\left[(4\mathcal{A} - 2\phi)\mathcal{L}_u - 6\Sigma\mathcal{L}_n + U\right]\Phi_{\nu)\alpha} = \mathcal{M}_{\mu\nu}, \tag{12}$$

where the potentials V and U and source $\mathcal{M}_{\mu\nu}$ are also in paper I. This clearly indicated the decoupling of the complex GEM 2-tensor, $\Phi_{\mu\nu}$, from the remaining 1+1+2 GEM 2-vectors and scalars. The GEM 2-tensors (and energy–momentum terms) were then expanded using generic tensor harmonics,

$$\mathcal{E}_{\mu\nu} = \mathcal{E}_{\mathrm{T}} Q_{\mu\nu} + \bar{\mathcal{E}}_{\mathrm{T}} \bar{Q}_{\mu\nu} \qquad \text{and} \qquad \mathcal{H}_{\mu\nu} = \mathcal{H}_{\mathrm{T}} Q_{\mu\nu} + \bar{\mathcal{H}}_{\mathrm{T}} \bar{Q}_{\mu\nu}, \tag{13}$$

where $Q_{\mu\nu}$ and $\bar{Q}_{\mu\nu}$ are generic even and odd 2-tensor harmonics as defined in paper I.[1] Furthermore, $\{\mathcal{E}_{\mathrm{T}}, \bar{\mathcal{H}}_{\mathrm{T}}\}$ and $\{\mathcal{H}_{\mathrm{T}}, \bar{\mathcal{E}}_{\mathrm{T}}\}$ are the corresponding polar and axial perturbation amplitudes respectively. Ultimately, four specific combinations of these amplitudes were shown to decouple, namely,

$$\text{Decoupled polar perturbations:} \quad \{\mathcal{E}_{\mathrm{T}} + \bar{\mathcal{H}}_{\mathrm{T}}, \mathcal{E}_{\mathrm{T}} - \bar{\mathcal{H}}_{\mathrm{T}}\}, \tag{14}$$

$$\text{Decoupled axial perturbations:} \quad \{\mathcal{H}_{\mathrm{T}} + \bar{\mathcal{E}}_{\mathrm{T}}, \mathcal{H}_{\mathrm{T}} - \bar{\mathcal{E}}_{\mathrm{T}}\}. \tag{15}$$

The remainder of the paper focuses on how to decouple an additional seven quantities.

3. Choosing new 1+1+2 GEM variables

In two recent papers, we successfully showed how to fully decouple EM perturbations to LRS class II spacetimes in [9] and to LRS spacetimes in [10]. There were two primary reasons that full decoupling could be achieved in these cases. First we expressed the 1+1+2 EM system in a complex form that was conducive to decoupling. Second, the first-order EM fields were the only first-order quantities in the system; other than first-order energy–momentum sources. This is in contrast to the 1+1+2 complex GEM system (5)–(9) as it does not only depend on the 1+1+2 GEM quantities. By inspection, it is clearly coupled to the 1+1+2 Ricci identities (see paper I and [3]) through first-order terms such as Ω, ξ, Λ_μ, Υ_μ, W_μ, Y_μ and $\Lambda_{\mu\nu}$. This additional complication compared to the EM system did not hinder the process of decoupling $\Phi_{\mu\nu}$ in paper I, as the 1+1+2 Ricci identities and twice contracted Bianchi identities were used to successfully remove these miscellaneous terms.

However, immediate difficulties arise when attempting to construct a decoupled equation for Φ_μ by taking the Lie derivative with respect to u^μ of (8). It follows that one must have an equation to eliminate $\mathcal{L}_u \mathcal{A}_\mu$, but no such equation arises naturally from the 1+1+2 Ricci identities. Therefore, new dependent variables are chosen and higher-order derivatives are constructed to incorporate these miscellaneous terms thereby alleviating this complication.

A new operator is defined to be $(\delta^2 + K)$, where $\delta^2 := \delta^\alpha \delta_\alpha$ is the 2-Laplacian and K is the Gaussian curvature as in [1, 7],

$$K = \tfrac{1}{4}\phi^2 - \tfrac{1}{4}\left(\Sigma - \tfrac{2}{3}\theta\right)^2 - \mathcal{E} + \tfrac{1}{3}\Lambda, \tag{16}$$

$$(\mathcal{L}_n + \phi)K = 0 \qquad \text{and} \qquad \left(\mathcal{L}_u - \Sigma + \tfrac{2}{3}\theta\right)K = 0. \tag{17}$$

Furthermore, for textual simplicity we introduce new notation according to

$$(\delta^2 + K)\Phi_{\mu\cdots\nu} := \check{\Phi}_{\mu\cdots\nu}, \tag{18}$$

where $\Phi_{\mu\cdots\nu}$ represents any quantity that has been entirely projected onto the 2-sheets. It is also important to specify the commutation relationships for this operator,

$$(\delta^2 + K)\mathcal{L}_u \Phi_{\bar{\mu}\cdots\bar{\nu}} = \left(\mathcal{L}_u - \Sigma + \tfrac{2}{3}\theta\right)\check{\Phi}_{\bar{\mu}\cdots\bar{\nu}}, \tag{19}$$

$$(\delta^2 + K)\mathcal{L}_n \Phi_{\bar{\mu}\cdots\bar{\nu}} = (\mathcal{L}_n + \phi)\check{\Phi}_{\bar{\mu}\cdots\bar{\nu}}. \tag{20}$$

[1] The definition of harmonic functions used here and in paper I follows from those defined by [2, 7].

It is also required to know how the Lie derivatives commute with the 2-divergence,

$$\delta^{\alpha}\mathcal{L}_u\Phi_{\alpha\cdots\bar{\mu}} = \left(\mathcal{L}_u - \Sigma + \tfrac{2}{3}\theta\right)\delta^{\alpha}\Phi_{\alpha\cdots\bar{\mu}}, \tag{21}$$

$$\delta^{\alpha}\mathcal{L}_n\Phi_{\alpha\cdots\bar{\mu}} = (\mathcal{L}_n + \phi)\delta^{\alpha}\Phi_{\alpha\cdots\bar{\mu}}. \tag{22}$$

In paper I, we specified the evolution and propagation equations for $\Lambda_{\mu\nu}$ and here we take the 2-divergence of those results to yield

$$(\mathcal{L}_n + \phi)(\epsilon_{\bar{\mu}}{}^{\alpha}\delta^{\beta}\Lambda_{\alpha\beta}) - \mathrm{i}\delta^{\alpha}\Phi_{\mu\alpha} - \tfrac{1}{2}\phi\epsilon_{\mu}{}^{\alpha}\delta^{\beta}\Sigma_{\alpha\beta} - \tfrac{3}{2}\Sigma\epsilon_{\mu}{}^{\alpha}\delta^{\beta}\zeta_{\alpha\beta} + \mathrm{i}\left(\Sigma + \tfrac{1}{3}\theta\right)\delta^{\alpha}\Sigma_{\mu\alpha} + \mathrm{i}\tfrac{1}{2}\check{\Lambda}_{\mu} = \mathrm{i}\tfrac{1}{2}\delta^{\alpha}\Pi_{\mu\alpha}, \tag{23}$$

$$\left(\mathcal{L}_u - \Sigma + \tfrac{2}{3}\theta\right)\delta^{\alpha}\Lambda_{\bar{\mu}\alpha} + \delta^{\alpha}\Phi_{\mu\alpha} - \mathrm{i}\mathcal{A}\epsilon_{\mu}{}^{\alpha}\delta^{\beta}\Lambda_{\alpha\beta} + \mathrm{i}\tfrac{1}{2}\phi\epsilon_{\mu}{}^{\alpha}\delta^{\beta}\Sigma_{\alpha\beta} + \mathrm{i}\tfrac{1}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)\epsilon_{\mu}{}^{\alpha}\delta^{\beta}\zeta_{\alpha\beta} - \mathrm{i}\tfrac{1}{2}\epsilon_{\mu}{}^{\alpha}\check{\Upsilon}_{\alpha} = \tfrac{1}{2}\delta^{\alpha}\Pi_{\mu\alpha}. \tag{24}$$

Now operate on the propagation and evolution equations for $\mathcal{C}_{\mu}$, i.e. (5) and (6) respectively, with the new operator and make use of the commutation relationships (19) and (20). In order to remove miscellaneous terms, it was then necessary to combine the results into a single transportation equation (25). Then by operating on (7), (8) and using (23), (24) yields (26), (27) respectively. Finally, by taking the 2-divergence of (9) and using the commutation relationships (21), (22) results in (28),

$$\left(\mathcal{L}_u - \tfrac{5}{2}\Sigma + \tfrac{5}{3}\theta\right)\Xi_{\mu} + \mathrm{i}\epsilon_{\mu}{}^{\alpha}\left(\mathcal{L}_n + \tfrac{5}{2}\phi\right)\Xi_{\alpha} + \mathrm{i}\epsilon_{\mu}{}^{\alpha}(\delta^2 - K - 3\mathcal{E})\Psi_{\alpha} - 3\mathcal{E}\left[\left(\Sigma - \tfrac{2}{3}\theta\right)\Gamma_{\mu} + \mathrm{i}\phi\epsilon_{\mu}{}^{\alpha}\Gamma_{\alpha}\right] = T_{\mu}, \tag{25}$$

$$(\mathcal{L}_n + 2\phi)\Psi_{\bar{\mu}} - \mathrm{i}\tfrac{3}{2}\Sigma\epsilon_{\mu}{}^{\alpha}\Psi_{\alpha} - \tfrac{1}{2}\Xi_{\mu} + (\delta^2 + K + 3\mathcal{E})\Gamma_{\mu} = \check{\mathcal{G}}_{\mu}, \tag{26}$$

$$\left(\mathcal{L}_u - 2\Sigma + \tfrac{4}{3}\theta\right)\Psi_{\mu} - \mathrm{i}\epsilon_{\mu}{}^{\alpha}\left(\mathcal{A} - \tfrac{1}{2}\phi\right)\Psi_{\alpha} + \mathrm{i}\tfrac{1}{2}\epsilon_{\mu}{}^{\alpha}\Xi_{\alpha} + \mathrm{i}(\delta^2 + K + 3\mathcal{E})\epsilon_{\mu}{}^{\alpha}\Gamma_{\alpha} = \check{\mathcal{F}}_{\mu}, \tag{27}$$

$$\left(\mathcal{L}_u + \tfrac{3}{2}\Sigma + \theta\right)\Gamma_{\bar{\mu}} - \mathrm{i}\epsilon_{\mu}{}^{\alpha}\left(\mathcal{L}_n + 2\mathcal{A} + \tfrac{1}{2}\phi\right)\Gamma_{\alpha} + \mathrm{i}\tfrac{1}{2}\epsilon_{\mu}{}^{\alpha}\Psi_{\alpha} = \delta^{\alpha}\mathcal{F}_{\mu\alpha}. \tag{28}$$

The new dependent variables have been defined

$$\Xi_{\mu} := \check{\mathcal{C}}_{\mu} - 3\mathcal{E}\left[\phi\delta^{\alpha}\zeta_{\mu\alpha} + \left(\Sigma - \tfrac{2}{3}\theta\right)\delta^{\alpha}\Sigma_{\mu\alpha} + \delta^{\alpha}\Pi_{\mu\alpha}\right], \tag{29}$$

$$\Psi_{\mu} := \check{\Phi}_{\mu} + \mathrm{i}3\mathcal{E}\epsilon_{\mu}{}^{\alpha}\delta^{\beta}\Lambda_{\alpha\beta}, \tag{30}$$

$$\Gamma_{\mu} := \delta^{\alpha}\Phi_{\mu\alpha}, \tag{31}$$

and the new energy–momentum source term is

$$T_{\mu} := \left(\delta_{\mu}\mathcal{F} + \mathrm{i}\epsilon_{\mu}{}^{\alpha}\delta_{\alpha}\mathcal{G}\right)^{\vee} + 6\mathcal{E}\delta^{\alpha}\mathcal{F}_{\mu\alpha}. \tag{32}$$

Thus, the new 1+1+2 GEM system (25)–(28) now involves only three first-order quantities Ξ_{μ}, Ψ_{μ} and Γ_{μ} (and energy–momentum sources). The first paper (paper I) has already demonstrated how to decouple $\Phi_{\mu\nu}$ and thus decoupling Γ_{μ} is not of primary interest here, as this result may be readily obtained. The purpose of the remainder of this paper is to illustrate how to decouple Ψ_{μ} and Ξ_{μ}.

The process used here of constructing higher-order derivatives to simplify the GEM system in order to decouple particular quantities is traditionally standard practice. It is a 'textbook' example in Minkowski spacetime to decouple the electric and magnetic fields by constructing decoupled second-order differential equations from the first-order differential coupled system.

Furthermore, as already pointed out, the EM decoupling problem has now been solved for the generalized cases of LRS class II spacetimes in [7, 9] and for LRS spacetimes in [10]. Since the original 1+1+2 GEM system (25)–(28) is substantially more complicated here, we have had to construct even higher-order derivatives by operating with the 2-Laplacian. Moreover, the properties of the 2-Laplacian are very well known and is comfortably manipulated using harmonic expansions in the following sections.

As a final note, the 2-divergence of anisotropic stress 2-tensor ($\delta^\alpha \Pi_{\mu\alpha}$) is combined with the new definition of Ξ_μ in (29). We consider $\Pi_{\mu\nu}$ to be an energy–momentum source and thus would usually place this term on the right-hand side of the equations to indicate this. However, the way in which it arises in the definition for Ξ_μ is very natural and so as an exception we choose to leave it on the left-hand side.

4. Decoupling Ψ_μ and its vector harmonic amplitudes

It is now possible to use the new GEM system (25)–(28) to derive a decoupled equation governing Ψ_μ. Take the Lie derivative with respect to u^μ of (27) and follow a similar process to that described in paper I for decoupling the complex GEM 2-tensor. However, one additional step is required to achieve full decoupling by operating one final time with $(\delta^2 + K + 3\mathcal{E})$ to obtain,

$$\begin{aligned}\{\delta^2 + K + 3\mathcal{E}\}\big\{[(\mathcal{L}_u - 3\Sigma + 3\theta)\mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + 4\phi)\mathcal{L}_n - A]\Psi_{\bar{\mu}}\\ + \mathrm{i}\epsilon_\mu{}^\alpha[-(2\mathcal{A} - \phi)\mathcal{L}_u + 3\Sigma\mathcal{L}_n - B]\Psi_\alpha\big\} - \tfrac{3}{2}\mathcal{E}\big[\phi N_\mu{}^\alpha + \mathrm{i}\big(\Sigma - \tfrac{2}{3}\theta\big)\epsilon_\mu{}^\alpha\big]\\ \times\big[\mathrm{i}\epsilon_\alpha{}^\beta\big(\mathcal{L}_u - \tfrac{7}{2}\Sigma + \tfrac{4}{3}\theta\big)\Psi_\beta + \big(\mathcal{L}_n - \mathcal{A} + \tfrac{5}{2}\phi\big)\Psi_\alpha\big] = S_\mu,\end{aligned} \tag{33}$$

where the terms related to the potentials are

$$A := \delta^2 - \mathcal{A}^2 + \tfrac{11}{4}\phi^2 + \mathcal{A}\phi - 4\mathcal{E} - \tfrac{3}{4}\Sigma^2 + 4\Sigma\theta - \tfrac{4}{3}\theta^2 - \Lambda, \tag{34}$$

$$B := \dot{\mathcal{A}} - \tfrac{3}{2}\hat{\Sigma} - 4\mathcal{A}\big(\Sigma - \tfrac{2}{3}\theta\big) - \phi\big(\tfrac{19}{4}\Sigma + \tfrac{4}{3}\theta\big), \tag{35}$$

the energy–momentum source is

$$\begin{aligned}S_\mu := \{\delta^2 + K + 3\mathcal{E}\}\big\{-\mathrm{i}2\epsilon_\mu{}^\alpha(\delta^2 + K + 3\mathcal{E})\delta^\beta\mathcal{F}_{\alpha\beta}\\ + \big[N_\mu{}^\alpha\big(\mathcal{L}_u + \tfrac{1}{2}\Sigma + \tfrac{5}{3}\theta\big) - \mathrm{i}\epsilon_\mu{}^\alpha\big(\mathcal{L}_n + 2\mathcal{A} + \tfrac{3}{2}\phi\big)\big]\big[\check{\mathcal{G}}_\alpha + \mathrm{i}\epsilon_\alpha{}^\gamma\check{\mathcal{F}}_\gamma\big]\big\}\\ - \tfrac{3}{2}\mathcal{E}\big[\phi N_\mu{}^\alpha + \mathrm{i}\big(\Sigma - \tfrac{2}{3}\theta\big)\epsilon_\mu{}^\alpha\big]\big[\check{\mathcal{G}}_\alpha + \mathrm{i}\epsilon_\alpha{}^\beta\check{\mathcal{F}}_\beta\big],\end{aligned} \tag{36}$$

and $N_{\mu\nu}$ is the projection tensor for the 2-sheets.

Whilst (33) appears rather complicated in its present state, it does clearly demonstrate the decoupling of Ψ_μ from the remaining quantities. It is also important to observe that this is both gauge-invariant and covariant, and includes a full description of energy–momentum sources. Furthermore, (33) will become more manageable once 2-vector harmonic expansions have been made in the following section resulting in four real decoupled equations.

4.1. Decoupling harmonic amplitudes of Ψ_μ

In order to decouple (33) further, we use the arbitrary 2-vector harmonic expansion as given in paper I,

$$\Psi_\mu = \Psi_{\mathrm{v}} Q_\mu + \bar{\Psi}_{\mathrm{v}}\bar{Q}_\mu \qquad \text{and} \qquad S_\mu = S_{\mathrm{v}} Q_\mu + \bar{S}_{\mathrm{v}}\bar{Q}_\mu, \tag{37}$$

where Q_μ and $\bar{Q}_\mu$ are the even and odd 2-vector harmonics initially described in [2, 7], and $\Psi_{\rm v}$ and $\bar{\Psi}_{\rm v}$ are the corresponding harmonic amplitudes. Using this expansion, (33) yields two complex equations,

$$f\left\{\left[\mathcal{L}_u - 4\Sigma + \frac{11}{3}\theta - \frac{3}{2f}\mathcal{E}\left(\Sigma - \frac{2}{3}\theta\right)\right]\mathcal{L}_u - \left[\mathcal{L}_n + 5\phi + \mathcal{A} + \frac{3}{2f}\mathcal{E}\phi\right]\mathcal{L}_n - \tilde{A}\right\}\Psi_{\rm v} + {\rm i}f\left\{\left[2\mathcal{A} - \phi + \frac{3}{2f}\mathcal{E}\phi\right]\mathcal{L}_u - \left[3\Sigma - \frac{3}{2f}\mathcal{E}\left(\Sigma - \frac{2}{3}\theta\right)\right]\mathcal{L}_n + \tilde{B}\right\}\bar{\Psi}_{\rm v} = S_{\rm v}, \tag{38}$$

$$f\left\{\left[\mathcal{L}_u - 4\Sigma + \frac{11}{3}\theta - \frac{3}{2f}\mathcal{E}\left(\Sigma - \frac{2}{3}\theta\right)\right]\mathcal{L}_u - \left[\mathcal{L}_n + 5\phi + \mathcal{A} + \frac{3}{2f}\mathcal{E}\phi\right]\mathcal{L}_n - \tilde{A}\right\}\bar{\Psi}_{\rm v} - {\rm i}f\left\{\left[2\mathcal{A} - \phi + \frac{3}{2f}\mathcal{E}\phi\right]\mathcal{L}_u - \left[3\Sigma - \frac{3}{2f}\mathcal{E}\left(\Sigma - \frac{2}{3}\theta\right)\right]\mathcal{L}_n + \tilde{B}\right\}\Psi_{\rm v} = \bar{S}_{\rm v} \tag{39}$$

where

$$f := -\frac{k^2}{r^2} + 5K + 3\mathcal{E}, \tag{40}$$

$$\tilde{A} := f - \mathcal{A}^2 + \frac{19}{4}\phi^2 + \mathcal{A}\phi - 5\mathcal{E} - \frac{7}{4}\Sigma^2 + \frac{19}{3}\theta\Sigma - \frac{22}{9}\theta^2, + \frac{3}{2f}\mathcal{E}\left[\phi(3\phi - \mathcal{A}) - \left(\Sigma - \frac{2}{3}\theta\right)\left(4\Sigma - \frac{5}{3}\theta\right)\right] - \frac{5}{3}\Lambda, \tag{41}$$

$$\tilde{B} := \mathcal{L}_u\mathcal{A} - \frac{3}{2}\mathcal{L}_n\Sigma - 5\mathcal{A}\left(\Sigma - \frac{2}{3}\theta\right) - \phi\left(\frac{27}{4}\Sigma + \theta\right) - \frac{3}{2f}\mathcal{E}\left[\mathcal{A}\left(\Sigma - \frac{2}{3}\theta\right) + \phi\left(\Sigma + \frac{1}{3}\theta\right)\right]. \tag{42}$$

The scalar function r, and k, both arise from the harmonic expansions of the 2-Laplacian as defined [1, 7],

$$\left(\mathcal{L}_n - \tfrac{1}{2}\phi\right)r = 0, \qquad \left(\mathcal{L}_u + \tfrac{1}{2}\Sigma - \tfrac{1}{3}\theta\right)r = 0 \qquad \text{and} \qquad \delta_\mu r = 0. \tag{43}$$

These two complex equations (38), (39) are much more manageable as the 2-Laplacians in (33) have been resolved into harmonics leaving a scalar function, f, in its place. However, they still form a coupled system. Thus, similar to the analysis of the 2-tensor harmonics in paper I, (38), (39) are invariant under the simultaneous transformation of the form $\Psi_{\rm v} \to \bar{\Psi}_{\rm v}$ and $\bar{\Psi}_{\rm v} \to -\Psi_{\rm v}$ (when the energy–momentum sources vanish or transform accordingly). Therefore, the eigenvector/eigenvalue analysis we presented in [9] also applies here and this shows that in order to decouple this system one needs to construct the complex combinations

$$\Psi_\pm := \Psi_{\rm v} \pm {\rm i}\bar{\Psi}_{\rm v} \qquad \text{and} \qquad S_\pm := S_{\rm v} \pm {\rm i}\bar{S}_{\rm v}. \tag{44}$$

Thus, (38), (39) result in decoupled complex equations of the form

$$f\left\{\left[\mathcal{L}_u - 4\Sigma + \frac{11}{3}\theta + (2\mathcal{A} - \phi) - \frac{3\mathcal{E}}{2f}\left(\Sigma - \frac{2}{3}\theta - \phi\right)\right]\mathcal{L}_u - \left[\mathcal{L}_n + 5\phi + \mathcal{A} + 3\Sigma - \frac{3\mathcal{E}}{2f}\left(\Sigma - \frac{2}{3}\theta - \phi\right)\right]\mathcal{L}_n - \tilde{A} + \tilde{B}\right\}\Psi_+ = S_+, \tag{45}$$

$$f\left\{\left[\mathcal{L}_u - 4\Sigma + \frac{11}{3}\theta - (2\mathcal{A} - \phi) - \frac{3\mathcal{E}}{2f}\left(\Sigma - \frac{2}{3}\theta + \phi\right)\right]\mathcal{L}_u \right.$$
$$\left. - \left[\mathcal{L}_n + 5\phi + \mathcal{A} - 3\Sigma + \frac{3\mathcal{E}}{2f}\left(\Sigma - \frac{2}{3}\theta + \phi\right)\right]\mathcal{L}_n - \tilde{A} - \tilde{B}\right\}\Psi_- = S_-. \quad (46)$$

Since the differential operator in (45)–(46) is completely real and the dependent variables, $\Psi_\pm$, are complex, there are in fact four real decoupled equations which can be readily found by taking the real and imaginary parts of (45), (46) separately.

It is now of interest to see how these relate to the original GEM 2-vectors and other 1+1+2 quantities. We expand the 2-tensors describing the shear of the 2/3-sheets using the arbitrary 2-tensor harmonics,

$$\Sigma_{\mu\nu} = \Sigma_T Q_{\mu\nu} + \bar{\Sigma}_T \bar{Q}_{\mu\nu} \qquad \text{and} \qquad \zeta_{\mu\nu} = \zeta_T Q_{\mu\nu} + \bar{\zeta}_T \bar{Q}_{\mu\nu}.$$

Then, by using the definitions (10), (11), (30) and (44), the four quantities that each decouple are

$$\Re[\Psi_+] = \tilde{f}\left[\left(\mathcal{E}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\zeta_{\rm T}\right) - \left(\bar{\mathcal{H}}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\Sigma_{\rm T}\right)\right], \quad (47)$$
$$\Re[\Psi_-] = \tilde{f}\left[\left(\mathcal{E}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\zeta_{\rm T}\right) + \left(\bar{\mathcal{H}}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\Sigma_{\rm T}\right)\right], \quad (48)$$
$$\Im[\Psi_+] = \tilde{f}\left[\left(\mathcal{H}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\bar{\Sigma}_{\rm T}\right) + \left(\bar{\mathcal{E}}_{\rm v} - \tfrac{3}{2}\mathcal{E}r\bar{\zeta}_{\rm T}\right)\right], \quad (49)$$
$$\Im[\Psi_-] = \tilde{f}\left[\left(\mathcal{H}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\bar{\Sigma}_{\rm T}\right) - \left(\bar{\mathcal{E}}_{\rm v} - \tfrac{3}{2}\mathcal{E}r\bar{\zeta}_{\rm T}\right)\right], \quad (50)$$

where $\tilde{f} := f - 3K - 3\mathcal{E}$. Moreover, the four specific combinations may be separated into their polar and axial parts according to

$$\text{polar:} \quad \tilde{f}\left\{\left(\mathcal{E}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\zeta_{\rm T}\right) - \left(\bar{\mathcal{H}}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\Sigma_{\rm T}\right), \left(\mathcal{E}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\zeta_{\rm T}\right) + \left(\bar{\mathcal{H}}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\Sigma_{\rm T}\right)\right\}, \quad (51)$$
$$\text{axial:} \quad \tilde{f}\left\{\left(\mathcal{H}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\bar{\Sigma}_{\rm T}\right) + \left(\bar{\mathcal{E}}_{\rm v} - \tfrac{3}{2}\mathcal{E}r\bar{\zeta}_{\rm T}\right), \left(\mathcal{H}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\bar{\Sigma}_{\rm T}\right) - \left(\bar{\mathcal{E}}_{\rm v} - \tfrac{3}{2}\mathcal{E}r\bar{\zeta}_{\rm T}\right)\right\}. \quad (52)$$

Finally, it is also clear that if one were to integrate the four decoupled equations (a non-trivial task) then linear combinations of the solutions would reveal each of $\left(\mathcal{E}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\zeta_{\rm T}\right)$, $\left(\bar{\mathcal{H}}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\Sigma_{\rm T}\right)$, $\left(\mathcal{H}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\bar{\Sigma}_{\rm T}\right)$ and $\left(\bar{\mathcal{E}}_{\rm v} - \tfrac{3}{2}\mathcal{E}r\bar{\zeta}_{\rm T}\right)$.

As a final comment, we note that one of these latter terms is also found in the application presented by [2]. Therein, they analyzed a Schwarzschild spacetime for which the background consists of only three non-vanishing scalars $(\mathcal{A}, \phi, \mathcal{E})$ and were able to demonstrate that the Zerilli variable, $\mathcal{Z} = \frac{1}{3}\phi\mathcal{E}^{-1}[(\ell-1)(\ell+2) - 3\mathcal{E}r^2]^{-1}\left(\bar{\mathcal{H}}_{\rm v} + \tfrac{3}{2}\mathcal{E}r\Sigma_{\rm T}\right)$ where ℓ is the spherical harmonic degree, satisfies a decoupled Zerilli [21] equation. It is currently an open problem to generalize this for arbitrary LRS class II spacetimes, and if possible, it would be an excellent supplement to the results demonstrated here.

5. Decoupling Ξ_μ and its harmonic amplitudes

In order to decouple Ξ_μ, we take the Lie derivative of (25) with respect to u^μ and after much manipulation, we find

$$\left[\left(\mathcal{L}_u - 4\Sigma + \tfrac{11}{3}\theta\right)\mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + 5\phi)\mathcal{L}_n - W\right]\Xi_{\bar{\mu}} = \mathcal{M}_\mu, \quad (53)$$

where the potential and energy–momentum source are

$$W := \delta^2 + 19K + 12\mathcal{E} - 10\Lambda, \quad (54)$$

$$\mathcal{M}_\mu := \left(\mathcal{L}_u - \tfrac{3}{2}\Sigma + 2\theta\right)T_\mu - \mathrm{i}\epsilon_\mu{}^\alpha\left(\mathcal{L}_n + \mathcal{A} + \tfrac{5}{2}\phi\right)T_\alpha + 3\mathcal{E}\left(\Sigma - \tfrac{2}{3}\theta\right)\delta^\alpha\mathcal{F}_{\mu\alpha} + \mathrm{i}3\mathcal{E}\phi\epsilon_\mu{}^\alpha\delta^\beta\mathcal{F}_{\alpha\beta} + (\delta^2 - K - 3\mathcal{E})(\check{\mathcal{G}}_\mu - \mathrm{i}\epsilon_\mu{}^\alpha\check{\mathcal{F}}_\alpha). \tag{55}$$

This clearly demonstrates the decoupling of Ξ_μ from the remaining first-order quantities. In addition, we are able to push the decoupling even further as the differential operator in (53) is real and thus it is also possible to consider the real and imaginary parts of Ξ_μ separately,

$$\Re[\Xi_\mu] := (\delta^2 + K)X_\mu - 3\mathcal{E}\delta^\alpha\left[\phi\zeta_{\mu\alpha} + \left(\Sigma - \tfrac{2}{3}\theta\right)\Sigma_{\mu\alpha} + \Pi_{\mu\alpha}\right], \tag{56}$$

$$\Im[\Xi_\mu] := (\delta^2 + K)\delta_\mu\mathcal{H}, \tag{57}$$

which demonstrates the decoupling between the 1+1+2 GEM scalars. The gravito-magnetic scalar is entirely contained in the decoupled quantity (57). The gravito-electric scalar is contained entirely in (56) and also requires an additional 2-divergence term involving the 2/3-sheet shears and the anisotropic stress.

5.1. Decoupling harmonic amplitudes of Ξ_μ

The dependent variable Ξ_μ and the energy–momentum source $\mathcal{M}_\mu$ are expanded into 2-vector harmonics according to

$$\Xi_\mu = \Xi_{\mathrm{v}}Q_\mu + \bar{\Xi}_{\mathrm{v}}\bar{Q}_\mu \qquad \text{and} \qquad \mathcal{M}_\mu = \mathcal{M}_{\mathrm{v}}Q_\mu + \bar{\mathcal{M}}_{\mathrm{v}}\bar{Q}_\mu. \tag{58}$$

Therefore, (53) naturally decouples into

$$\left[\left(\mathcal{L}_u - 5\Sigma + \tfrac{13}{3}\theta\right)\mathcal{L}_u - (\mathcal{L}_n + 6\phi + \mathcal{A})\mathcal{L}_n - \tilde{W}\right]\Xi_{\mathrm{v}} = \mathcal{M}_{\mathrm{v}}, \tag{59}$$

$$\left[\left(\mathcal{L}_u - 5\Sigma + \tfrac{13}{3}\theta\right)\mathcal{L}_u - (\mathcal{L}_n + 6\phi + \mathcal{A})\mathcal{L}_n - \tilde{W}\right]\bar{\Xi}_{\mathrm{v}} = \bar{\mathcal{M}}_{\mathrm{v}}, \tag{60}$$

and the new potential has been defined

$$\tilde{W} := -\frac{k^2}{r^2} + 30K + 21\mathcal{E} - \frac{42}{3}\Lambda. \tag{61}$$

Now since the differential operators in (59), (60) are purely real (they are also identical), the real and imaginary parts may again be considered separately. It is of interest to see how the amplitudes of Ξ_μ are related back to the amplitudes of the GEM scalars and other 1+1+2 quantities. The 2-gradient of the gravito-electric scalar is expanded in terms of 2-vector harmonics and the gravito-magnetic scalar in terms of scalar harmonics according to

$$X_\mu := X_{\mathrm{v}}Q_\mu + \bar{X}_{\mathrm{v}}\bar{Q}_\mu \qquad \text{and} \qquad \mathcal{H} := \mathcal{H}_{\mathrm{s}}Q. \tag{62}$$

Here, Q is the scalar harmonic function defined in paper I (this definition was initially given in [2, 7]). Then, by using the definitions (56) and (57), it follows that

$$\Re[\Xi_V] = \tilde{f}\left\{X_V - \frac{3}{2}\mathcal{E}r\left[\phi\zeta_{\mathrm{T}} + \left(\Sigma - \frac{2}{3}\theta\right)\Sigma_{\mathrm{T}} + \Pi_{\mathrm{T}}\right]\right\}, \tag{63}$$

$$\Re[\bar{\Xi}_V] = \tilde{f}\left\{\bar{X}_V + \frac{3}{2}\mathcal{E}r\left[\phi\bar{\zeta}_{\mathrm{T}} + \left(\Sigma - \frac{2}{3}\theta\right)\bar{\Sigma}_{\mathrm{T}} + \bar{\Pi}_{\mathrm{T}}\right]\right\}, \tag{64}$$

$$\Im[\Xi_{\mathrm{v}}] = \frac{\tilde{f}}{r}\mathcal{H}_{\mathrm{s}}, \tag{65}$$

$$\Im[\bar{\Xi}_{\mathrm{v}}] = 0, \tag{66}$$

where the anisotropic stress 2-tensor has also been expanded according to $\Pi_{\mu\nu} = \Pi_{\mathrm{T}} Q_{\mu\nu} + \bar{\Pi}_{\mathrm{T}} \bar{Q}_{\mu\nu}$. Therefore, there are actually only three non-trivial equations here. One of them governs the gravito-magnetic scalar harmonic amplitude, $\mathcal{H}_{\mathrm{s}}$, whereas the remaining two govern the 2-gradient of the gravito-electric scalar combined with the 2-divergence of the 2-tensors for the shear of the 2/3-sheets and the anisotropic stress.

The factors, $\tilde{f}$ and r, that arise from the harmonic expansions can be differentiated and factorized if desired. For example, the decoupled equation governing $\mathcal{H}_{\mathrm{s}}$ (i.e. taking the imaginary part of (59)) becomes

$$\frac{\tilde{f}}{r}\left[\left(\mathcal{L}_u - 2\Sigma + \frac{7}{3}\theta\right)\mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + 3\phi)\mathcal{L}_n - V_{\mathcal{H}}\right]\mathcal{H}_{\mathrm{s}} = \Im[\mathcal{M}_V], \tag{67}$$

where

$$V_{\mathcal{H}} := -\frac{k^2}{r^2} + \frac{3}{2}\phi^2 - 6\mathcal{E} - \frac{3}{2}\left(\Sigma - \frac{2}{3}\theta\right)^2 - \frac{10}{3}\Lambda. \tag{68}$$

By inspecting the first-order constraints from paper I, the first-order gravito-magnetic scalar is directly related to first-order twisting effects of the 2-sheets and first-order vorticity effects of the 3-sheets.

Summarily, the three quantities which each decouple are categorized into polar and axial perturbations according to

$$\text{decoupled polar perturbations:} \quad \left\{X_V - \tfrac{3}{2}\mathcal{E}r\left[\phi\zeta_{\mathrm{T}} + \left(\Sigma - \tfrac{2}{3}\theta\right)\Sigma_{\mathrm{T}} + \Pi_{\mathrm{T}}\right]\right\}, \tag{69}$$

$$\text{decoupled axial perturbations:} \quad \left\{\bar{X}_V + \tfrac{3}{2}\mathcal{E}r\left[\phi\bar{\zeta}_{\mathrm{T}} + \left(\Sigma - \tfrac{2}{3}\theta\right)\bar{\Sigma}_{\mathrm{T}} + \bar{\Pi}_{\mathrm{T}}\right], \mathcal{H}_{\mathrm{s}}\right\}. \tag{70}$$

Finally, we comment on how these results are related to previous work throughout the literature. In the particular Schwarzschild spacetime example, a RW-type equation that governs what is referred to as a RW 2-tensor, $W_{\mu\nu}$, was derived by [2]. Thus in this case, the real part of Ξ_μ simplifies and is related to the 2-divergence of the RW tensor according to $\Re[\Xi_\mu] = -6\mathcal{E}/r^2\delta^\alpha W_{\mu\alpha}$. Furthermore, by once again reducing to a Schwarzschild background, equation (67) corresponds precisely to that presented in [28] using the Newman–Penrose (NP) formalism [29].[2] This is achieved by expressing the 1+1+2 frame vectors in terms of the NP null vectors as indicated in [10], and consequently, the 1+1+2 gravito-magnetic harmonic scalar can be written in terms of the zero spin-weighted NP scalar. Finally, after transforming to the famous tortoise coordinate and introducing a scaling factor of r^3, a precise correspondence is achieved.

6. Summary

We have provided a comprehensive decoupling analysis of the first-order 1+1+2 complex GEM system. The first paper decoupled the first four quantities and set the foundation for future work. This paper delivered the final seven decoupled quantities. Summarily, the 11 decoupled quantities are categorized according to

$$\text{decoupled polar perturbations:} \left\{(\mathcal{E}_{\mathrm{T}} + \bar{\mathcal{H}}_{\mathrm{T}}), (\mathcal{E}_{\mathrm{T}} - \bar{\mathcal{H}}_{\mathrm{T}}),\right.$$
$$\tilde{f}\left[\left(\mathcal{E}_{\mathrm{v}} + \frac{3}{2}\mathcal{E}r\zeta_{\mathrm{T}}\right) - \left(\bar{\mathcal{H}}_{\mathrm{v}} + \frac{3}{2}\mathcal{E}r\Sigma_{\mathrm{T}}\right)\right], \tilde{f}\left[\left(\mathcal{E}_{\mathrm{v}} + \frac{3}{2}\mathcal{E}r\zeta_{\mathrm{T}}\right) + \left(\bar{\mathcal{H}}_{\mathrm{v}} + \frac{3}{2}\mathcal{E}r\Sigma_{\mathrm{T}}\right)\right],$$
$$\left.\tilde{f}\left[X_{\mathrm{v}} - \frac{3}{2}\mathcal{E}r\left[\phi\zeta_{\mathrm{T}} + \left(\Sigma - \frac{2}{3}\theta\right)\Sigma_{\mathrm{T}} + \Pi_{\mathrm{T}}\right]\right]\right\} \tag{71}$$

[2] In the Schwarzschild case, (67) from this paper may be transformed and scaled to achieve equation (56*a*) from [28].

decoupled axial perturbations: $\Big\{(\mathcal{H}_{\rm T}+\bar{\mathcal{E}}_{\rm T}),\,(\mathcal{H}_{\rm T}-\bar{\mathcal{E}}_{\rm T}),$

$$\tilde{f}\left[\left(\mathcal{H}_{\rm v}+\frac{3}{2}\mathcal{E}r\bar{\Sigma}_{\rm T}\right)+\left(\bar{\mathcal{E}}_{\rm v}-\frac{3}{2}\mathcal{E}r\bar{\zeta}_{\rm T}\right)\right],\ \tilde{f}\left[\left(\mathcal{H}_{\rm v}+\frac{3}{2}\mathcal{E}r\bar{\Sigma}_{\rm T}\right)-\left(\bar{\mathcal{E}}_{\rm v}-\frac{3}{2}\mathcal{E}r\bar{\zeta}_{\rm T}\right)\right],$$

$$\tilde{f}\left[\bar{X}_{\rm v}+\frac{3}{2}\mathcal{E}r\left[\phi\bar{\zeta}_{\rm T}+\left(\Sigma-\frac{2}{3}\theta\right)\bar{\Sigma}_{\rm T}+\bar{\Pi}_{\rm T}\right]\right],\ \frac{\tilde{f}}{r}\mathcal{H}_{\rm s}\Big\}. \tag{72}$$

From the three 1+1+2 tensors decoupled in this series, the RW equation governing Ξ_μ is the most straightforward to solve. This is primarily because of its simple form (as also noted by [2]) and also because it is a well-studied equation from a numerical point of view, especially in the case of vacuum perturbations. Once the RW equations have been solved, those results may then be used to calculate the remaining perturbation quantities. From a more astrophysical point of view, in the case of energy momentum perturbations, it is possible for the complexity of the first-order energy–momentum sources to have a strong influence on how the equations are solved. One should consider the RW equation first, but if this proves too difficult then it is possible for one of the other equations to take primary focus.

Thus far we have established a direct connection between the RW 2-vector, Ξ_μ, and the 2-divergence of the RW 2-tensor, $\delta^\alpha W_{\mu\alpha}$, from [2]. In a forthcoming paper we will choose new variables yet again, and consequently, demonstrate how to avoid the '2-divergence' part thereby revealing a relationship between the full RW 2-tensor and some newly defined quantities.

There are two remaining primary areas of research to be undertaken in this series. One is to present the various options that arise for solving the decoupled 1+1+2 GEM system and it will be shown that there are only two dynamical quantities. Furthermore, it will also be demonstrated how the solution of the 1+1+2 GEM system aides in subsequently solving the 1+1+2 Ricci identities. The other area of research will embark on astrophysical applications that demonstrate all the features presented. The complexity of the equations will most likely require numerical analysis, and since the background spacetime is LRS class II, the applications may be posed as an initial-value problem.

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