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Mean-field view on rotating magnetoconvection and a geodynamo model

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Abstract. A comparison is made between direct numerical simulations of magnetohydrodynamic processes in a rotating spherical shell and their mean–field description. The mean fields are defined by azimuthal averaging. The coefficients that occur in the traditional representation of the mean electromotive force considering derivatives of the mean magnetic field up to the first order are calculated with the fluid velocity taken from the direct numerical simulations by two different methods. While the first one does not use specific approximations, the second one is based on the first–order smoothing approximation. There is satisfying agreement of the results of both methods for sufficiently slow fluid motions. For the investigated example of rotating magnetoconvection the mean magnetic field derived from the direct numerical simulation is well reproduced on the mean–field level. For the simple geodynamo model a discrepancy occurs, which is probably a consequence of the neglect of higher–order derivatives of the mean magnetic field in the mean electromotive force.

Key words: mean–field electrodynamics – magnetoconvection – geodynamo

1. Introduction

The mean–field concept has proved to be a useful tool for the investigation of magnetohydrodynamic, in particular dynamo processes with complex fluid motions. Within this concept mean fields are defined by a proper averaging of the original fields. As usual we denote mean fields by overbars, e.g., the mean magnetic field and the mean fluid velocity by $\overline{\boldsymbol{B}}$ and $\overline{\boldsymbol{U}}$, and the deviations of the original fields $\boldsymbol{B}$ and $\boldsymbol{U}$ from these mean fields by $\boldsymbol{b}$ and $\boldsymbol{u}$. The mean electromagnetic fields are governed by equations which differ formally from Maxwell's and the completing constitutive equations for the original fields, or from the corresponding induction equation, only in one point. In the mean–field versions of Ohm's law and of the induction equation an additional electromotive force, $\boldsymbol{\mathcal{E}}$, occurs which is defined by $\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$. It may be considered as a functional of $\boldsymbol{u}$, $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{B}}$. If some simplifying assumptions are adopted, the representation

$$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\nabla_k\overline{B}_j \quad (1)$$

can be justified. We refer here to Cartesian coordinates and use the summation convention. The coefficients a_{ij} and b_{ijk} are determined by $\boldsymbol{u}$ and $\overline{\boldsymbol{U}}$ and can depend on $\overline{\boldsymbol{B}}$ only via these quantities. A crucial condition for the validity of relation (1) is a sufficiently small variation of $\overline{\boldsymbol{B}}$ in space and time.

Although by far not generally justified the simple relation (1) for $\boldsymbol{\mathcal{E}}$ has been used in almost all mean–field models of magnetohydrodynamic phenomena, in particular in mean–field dynamo models. In a few cases of such phenomena now direct numerical simulations are available. So the possibility opens up to calculate the tensors a_{ij} and b_{ijk} with the field $\boldsymbol{u}$ taken from these simulations. In this paper we deal with an example of magnetoconvection as investigated by Olsen et al. (1999) and a geodynamo model by Christensen et al. (2001). In both cases we compare, with a view to the applicability of relation (1), the mean magnetic field resulting from mean–field models using the so determined a_{ij} and b_{ijk} with that derived immediately from the numerical simulations.

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2. The examples considered

In both cases, magnetoconvection and geodynamo, a rotating spherical shell of electrically conducting fluid is considered in which the fluid velocity $\boldsymbol{U}$, the magnetic field $\boldsymbol{B}$ and the deviation θ of the temperature from the temperature T_0 in a reference state is governed by

$$\begin{aligned}
&\partial_t \boldsymbol{U} + (\boldsymbol{U}\cdot\boldsymbol{\nabla})\boldsymbol{U} = -(1/\varrho)\,\boldsymbol{\nabla}P + \nu\boldsymbol{\nabla}^2\boldsymbol{U} - 2\boldsymbol{\Omega}\times\boldsymbol{U}\\
&\qquad\qquad + (1/\mu\varrho)\,(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} - \alpha_T\boldsymbol{g}\,\theta\\
&\partial_t \boldsymbol{B} - \boldsymbol{\nabla}\times(\boldsymbol{U}\times\boldsymbol{B}) - \eta\boldsymbol{\nabla}^2\boldsymbol{B} = \boldsymbol{0} \qquad (2)\\
&\partial_t\,\theta + \boldsymbol{U}\cdot\boldsymbol{\nabla}\theta - \kappa\Delta\theta = -\,\boldsymbol{U}\cdot\boldsymbol{\nabla}T_0\\
&\boldsymbol{\nabla}\cdot\boldsymbol{U} = \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\,.
\end{aligned}$$

The fluiddynamic equations have to be understood as Boussinesq approximation. As usual, ϱ is the mass density of the fluid, μ its magnetic permeability, assumed to be equal to that of free space; ν, η and κ are kinematic viscosity, magnetic diffusivity and thermal conductivity, $\boldsymbol{\Omega}$ is the angular velocity responsible for the Coriolis force, α_T the thermal volume expansion coefficient and $\boldsymbol{g}$ the gravitational acceleration.

For the fluid velocity $\boldsymbol{U}$, no–slip conditions are posed at the boundaries, which are in this respect considered as rigid bodies. All surroundings of the spherical shell are considered as electrically non-conducting so that the magnetic field $\boldsymbol{B}$ continues as a potential field in both parts of the outer space. In the magnetoconvection case an imposed toroidal magnetic field is assumed resulting from electric currents due to sources or sinks on the boundaries. The temperature T_0 is assumed to be constant on each of the boundaries, and θ to vanish there.

The equations (2) can be written in a non–dimensional form which contains only four non–dimensional parameters, that is, the Ekman number E, a modified Rayleigh number Ra, the Prandtl number Pr and the magnetic Prandtl number Pm,

$$\begin{aligned}
&E\ = \nu/\Omega D^2\,, \quad Ra = \alpha_T g\Delta T D/\nu\Omega\\
&Pr = \nu/\kappa\,, \quad Pm = \nu/\eta\,. \qquad (3)
\end{aligned}$$

Here D means the thickness of the spherical shell and ΔT the difference of temperatures at the inner and the outer boundary. The typical magnitude B_0 of the imposed toroidal magnetic field can be expressed by the Elsasser number Λ,

$$\Lambda = B_0^2/\varrho\mu\eta\Omega\,. \qquad (4)$$

In all simulations considered in the following, $D = 0.65r_0$ is assumed, where r_0 is the radius of the outer boundary. In order to characterize the results of the simulations we use in particular the magnetic Reynolds number $Rm = uD/\eta$ with u interpreted as r.m.s. value of $\boldsymbol{u}$.

For the numerical solution of the above equations a code is used which was originally designed by Glatzmaier (1984) and later modified by Christensen et al. (1999).

3. The mean–field concept

When applying the mean–field concept we focus attention on the induction equation only. We refer here to a spherical coordinate system (r, ϑ, φ) the polar axis of which coincides with the rotation axis occurring in the above examples. In order to define a mean vector field, we average its components with respect to the spherical coordinate system over all values of the azimuthal coordinate φ. For example, $\overline{\boldsymbol{B}} = \overline{B}_r(r,\vartheta)\,\boldsymbol{e}_r + \overline{B}_\vartheta(r,\vartheta)\,\boldsymbol{e}_\vartheta + \overline{B}_\varphi(r,\vartheta)\,\boldsymbol{e}_\varphi$, where $\overline{B}_r(r,\vartheta)$, $\overline{B}_\vartheta(r,\vartheta)$ and $\overline{B}_\varphi(r,\vartheta)$ are the averages of $B_r(r,\vartheta,\varphi)$, $B_\vartheta(r,\vartheta,\varphi)$ and $B_\varphi(r,\vartheta,\varphi)$. With this definition of mean fields the Reynolds averaging rules apply exactly. Of course, all mean fields are axisymmetric about the polar axis of the coordinate system chosen.

Subjecting the induction equation given in (2) to averaging, we obtain

$$\partial_t\overline{\boldsymbol{B}} - \boldsymbol{\nabla}\times(\overline{\boldsymbol{U}}\times\overline{\boldsymbol{B}} + \boldsymbol{\mathcal{E}}) - \eta\boldsymbol{\nabla}^2\overline{\boldsymbol{B}} = \boldsymbol{0}\,,\ \boldsymbol{\nabla}\cdot\overline{\boldsymbol{B}} = 0\,, \qquad (5)$$

with the crucial electromotive force

$$\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u}\times\boldsymbol{b}} \qquad (6)$$

mentioned above.

If $\boldsymbol{u}$ is given, the calculation of $\boldsymbol{\mathcal{E}}$ further requires the knowledge of $\boldsymbol{b}$. From the above equations we may derive

$$\begin{aligned}
&\partial_t\,\boldsymbol{b} - \boldsymbol{\nabla}\times(\overline{\boldsymbol{U}}\times\boldsymbol{b} + \boldsymbol{G}) - \eta\boldsymbol{\nabla}^2\boldsymbol{b} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\overline{\boldsymbol{B}})\\
&\qquad\qquad \boldsymbol{G} = \boldsymbol{u}\times\boldsymbol{b} - \overline{\boldsymbol{u}\times\boldsymbol{b}}\,, \quad \boldsymbol{\nabla}\cdot\boldsymbol{b} = 0\,. \qquad (7)
\end{aligned}$$

On this basis we may conclude that $\boldsymbol{\mathcal{E}}$ is a functional of $\boldsymbol{u}$, $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{B}}$, which is linear in $\overline{\boldsymbol{B}}$. Cancelling $\boldsymbol{G}$ in the first line of (7) leads to the often used "first–order smoothing" approximation.

In view of the examples envisaged we introduce some simplifications. Firstly we assume that $\boldsymbol{b}$ vanishes if $\overline{\boldsymbol{B}}$ does so. Then $\boldsymbol{\mathcal{E}}$ must be not only linear but also homogeneous in $\overline{\boldsymbol{B}}$. Secondly we use the fact that in both these examples the configurations of $\boldsymbol{U}$, $\boldsymbol{B}$ and θ rotate like a rigid body. This allows us to change to a rotating frame of reference in which they are steady. Then $\overline{\boldsymbol{B}}$ and $\boldsymbol{\mathcal{E}}$ are steady, too. The result for $\boldsymbol{\mathcal{E}}$ obtained in this rotating frame applies also in the original frame. In addition to these two simplifications, which are well justified for the examples under discussion, we introduce a third one, which must be considered as an assumption to be checked. The determination of $\boldsymbol{\mathcal{E}}$ in a given point of the (r,ϑ) plane requires the knowledge of the components of $\overline{\boldsymbol{B}}$ in some surroundings of this point. It is assumed that their variation in this surroundings is sufficiently weak so that they can be represented there by their values and their first derivatives in this point. These three simplifications enable us to write

$$\mathcal{E}_\kappa = \tilde{a}_{\kappa\lambda}\overline{B}_\lambda + \tilde{b}_{\kappa\lambda r}\frac{\partial\overline{B}_\lambda}{\partial r} + \tilde{b}_{\kappa\lambda\vartheta}\frac{1}{r}\frac{\partial\overline{B}_\lambda}{\partial\vartheta}\,. \qquad (8)$$

We refer here again to the spherical coordinate system introduced above. The coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$ are determined by $\boldsymbol{u}$ and $\overline{\boldsymbol{U}}$ and can depend only via these quantities on $\overline{\boldsymbol{B}}$. They depend, of course, on r and ϑ. The indices κ and λ stand for r, ϑ or φ, and again the summation convention is adopted. Note that $\boldsymbol{\mathcal{E}}$ is here, with $\overline{\boldsymbol{B}}$ being axisymmetric, determined by 27 independent coefficients.

Two methods have been used for the calculation of the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$ on the basis of the numerical simulations addressed in Sect. 2.

Method (i) is based on eq. (7) for $\boldsymbol{b}$, specified to the steady case. This equation is solved numerically with $\boldsymbol{u}$ and $\overline{\boldsymbol{U}}$ taken from the numerical simulations mentioned in Sect. 2, but employing nine properly chosen "test fields" $\overline{\boldsymbol{B}} = \overline{\boldsymbol{B}}^{(\nu)}$,

$\nu = 1, \ldots, 9$. Note, that the velocities are treated as independent of the test fields and are therefore the same in all 9 cases. One criterion for the choice of the test fields is that higher than first–order derivatives of their components with respect to r and ϑ are equal to zero or at least as small as possible. With the results for $\boldsymbol{b}$ obtained in this way, $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(\nu)}$ is calculated for each ν. Writing then down the eqs. (8) with $\mathcal{E}_\kappa = \mathcal{E}_\kappa^{(\nu)}$ and $\overline{B}_\lambda = \overline{B}_\lambda^{(\nu)}$ for $\nu = 1, \ldots, 9$, with any given r and ϑ, we arrive at three sets of nine linear algebraic equations for the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$. Solving these equations we can determine all 27 of these coefficients for the chosen r and ϑ.

Method (ii) ignores any mean fluid motion and uses the first–order smoothing approximation. The steady version of eq. (7) for $\boldsymbol{b}$ with $\overline{\boldsymbol{U}} = \boldsymbol{0}$ and $\boldsymbol{G} = \boldsymbol{0}$ can be solved analytically for arbitrary $\boldsymbol{u}$ and $\overline{\boldsymbol{B}}$. On this basis, $\boldsymbol{\mathcal{E}}$ and so the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$ can be determined for arbitrary $\boldsymbol{u}$ in the usual way, and later be specified by choosing $\boldsymbol{u}$ according to the numerical simulations mentioned.

We note that the $\boldsymbol{u}$ needed for the determination of the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$ were taken in both methods from simulations with non–zero $\overline{\boldsymbol{B}}$. That is, the resulting coefficients are already subject to a quenching corresponding to this $\overline{\boldsymbol{B}}$. In a further study they should be compared with those for the limit of vanishing $\overline{\boldsymbol{B}}$.

4. A more general representation of the mean electromotive force

Relation (1) for $\boldsymbol{\mathcal{E}}$ can be understood as establishing a coordinate–independent connection between the vectors $\boldsymbol{\mathcal{E}}$ and $\overline{\boldsymbol{B}}$ and the tensor $\boldsymbol{\nabla}\overline{\boldsymbol{B}}$ by this representation in a Cartesian coordinate system. In that sense a_{ij} and b_{ijk} have to be understood as tensors, too. By contrast, relation (8) is from the very beginning a specific one, which applies only in the chosen spherical coordinate system, and the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$ should not be interpreted as tensor components.

The coordinate–independent connection between $\boldsymbol{\mathcal{E}}$, $\overline{\boldsymbol{B}}$ and its derivatives $\boldsymbol{\nabla}\overline{\boldsymbol{B}}$ expressed above in the form (1) is equivalent to

$$\begin{aligned}\boldsymbol{\mathcal{E}} \;=\; &-\boldsymbol{\alpha}\cdot\overline{\boldsymbol{B}} - \boldsymbol{\gamma}\times\overline{\boldsymbol{B}} \\ &-\boldsymbol{\beta}\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) - \boldsymbol{\delta}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) - \boldsymbol{\kappa}\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(s)}\,; \quad (9)\end{aligned}$$

see, e.g., Rädler (1980). Here $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are symmetric second–rank tensors, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ vectors, $\boldsymbol{\kappa}$ is a third–rank tensor with some symmetries, all determined by $\boldsymbol{u}$ and $\overline{\boldsymbol{U}}$ only, and $(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(s)}$ is the symmetric part of the gradient tensor of $\overline{\boldsymbol{B}}$, that is, when referring again to a Cartesian coordinate system, $(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(s)}_{ij} = \frac{1}{2}(\nabla_j\overline{B}_i + \nabla_i\overline{B}_j)$. The $\boldsymbol{\alpha}$ term in (9) describes in general an anisotropic α–effect, the $\boldsymbol{\gamma}$ term an advection of the mean magnetic field like that by a mean motion of the fluid. The $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ terms can be interpreted in the sense of an anisotropic electrical mean–field conductivity and the $\boldsymbol{\kappa}$ term covers various other influences on the mean fields.

Like the number of the components of a_{ij} and b_{ijk} in (1) also that of the components of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ in (9) is 36. If we however specify (9) to our spherical coordinate system and consider the axisymmetry of $\overline{\boldsymbol{B}}$ this number reduces to 27, just in agreement with the number of the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$. We may therefore, without changing $\boldsymbol{\mathcal{E}}$, choose nine coefficients of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ arbitrarily, e.g., put them equal to zero. The remaining 27 components of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ are then uniquely determined by the 27 coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$. In that sense we express in the following all results originally obtained for the $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$ in terms of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$. We stress that these last quantities are chosen with some arbitrariness, which is, however, without any influence on $\boldsymbol{\mathcal{E}}$.

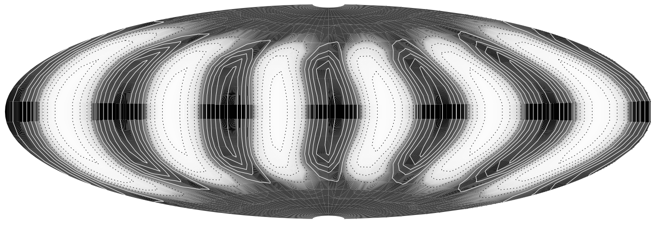

Fig. 1. The radial velocity in the magnetoconvection case at $r = 0.59\,r_0$, normalized with its maximum given by $U_r = 16.98\,\nu/D$. In the grey scale coding, white and black correspond to -1 and $+1$, respectively, and the contour lines to ±0.1, ±0.3, ±0.5, ±0.7, ±0.9.

5. Magnetoconvection

We consider here a simulation by Olsen et al. (1999) with $E = 10^{-3}$, $Ra = 94$, $Pr = Pm = 1$ and an imposed toroidal magnetic field corresponding to $\Lambda = 1$. In this case the intensity of the fluid motion is characterized by $Rm \approx 12$. A flow pattern is shown in Fig. 1.

The results for the $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ obtained by the two methods explained above, (i) and (ii), do not completely coincide. This was to be expected since method (ii) is based on first–order smoothing. In the steady case considered here it is surely justified for $Rm' \ll 1$ (a sufficient condition), where $Rm' = ul/\eta$, with u being again the r.m.s. value of $\boldsymbol{u}$ and l a characteristic length of the $\boldsymbol{u}$–field. It seems reasonable to assume that l is not much smaller than D, that is, Rm' not much smaller than Rm. We consider the results for $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ obtained with method (i) as most reliable. Those obtained with method (ii) fairly agree with them as far as the profiles of these quantities are concerned, but overestimate their magnitudes typically by a few per cent; see also Fig. 4 below. When calculating the $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ for the given $\boldsymbol{u}$ we may scale down Rm by a proper reduction of Pm. For $Rm \leq 1$ the results of the two methods come to a satisfying agreement. Fig. 2 shows results of method (i) for $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$, again with $Rm \approx 12$.

A mean–field model of magnetoconvection using (9) and our results for $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ reproduces very well the $\overline{\boldsymbol{B}}$–field obtained from the direct numerical simulations.

We have also investigated the quantity $\delta\boldsymbol{\mathcal{E}}$ defined by

$$\delta\boldsymbol{\mathcal{E}} = (\boldsymbol{\mathcal{E}}^{\rm DNS} - \boldsymbol{\mathcal{E}}^{\rm MF1})/\sqrt{\langle(\boldsymbol{\mathcal{E}}^{\rm DNS})^2\rangle}\;, \quad (10)$$

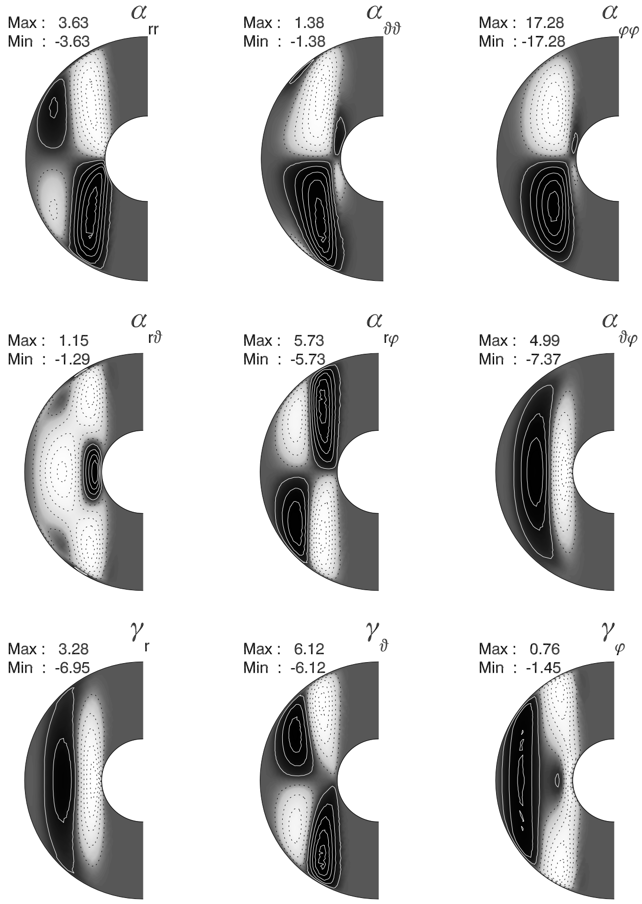

Fig. 2. Components of the symmetric α-tensor and the γ-vector in a meridional plane in the magnetoconvection case, determined by method (i), in units of ν/D. For each component the grey scale (white – negative, black – positive values) is separately adjusted to its maximum or, if having a larger modulus, to its minimum. Note the negative sign in the definition of $\boldsymbol{\alpha}$ in equation (9).

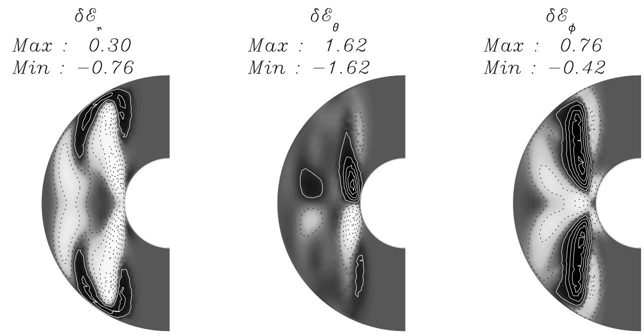

Fig. 3. The components of $\delta\boldsymbol{\mathcal{E}}$, defined by (10), in the magnetoconvection case. As in Fig. 2 the grey scale for each component is separately adjusted.

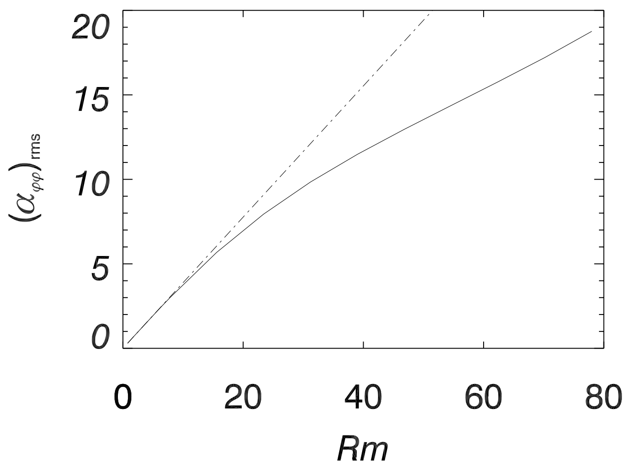

Fig. 4. The quantity $(\alpha_{\varphi\varphi})_{\mathrm{rms}}$ in the dynamo case, in units of ν/D, with $\alpha_{\varphi\varphi}$ determined by the methods (i) and (ii) (solid and dashed lines, respectively) in dependence on Rm.

where $\boldsymbol{\mathcal{E}}^{\mathrm{DNS}}$ corresponds to the quantity $\boldsymbol{\mathcal{E}}$ immediately extracted from the direct numerical simulation and $\boldsymbol{\mathcal{E}}^{\mathrm{MF1}}$ to this quantity determined according to (9) (considering no higher than first–order derivatives of $\overline{\boldsymbol{B}}$) with $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ as obtained by the above–described calculations and $\overline{\boldsymbol{B}}$ corresponding to the direct numerical simulations (or, what is here the same, to the mean-field model). $\langle\cdots\rangle$ means averaging over all r and ϑ of interest in the meridional plane. As Fig. 3 shows, $|\delta\boldsymbol{\mathcal{E}}|$ is, with the exception of a few small areas in the (r, ϑ) plane, much smaller than unity. This indicates that the representation (9) is indeed sufficient for the purposes of the example considered.

6. Geodynamo model

We consider now the case $E = 10^{-3}$, $Ra = 100$, $Pr = 1$, $Pm = 5$ and $\Lambda = 0$, in which the numerical simulations by Christensen et al. (2001) indeed show a dynamo. The intensity of the fluid motions can be characterized by $Rm \approx 40$.

In this case there is a clear difference in the results for $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ obtained by the two methods. To give an example for that, we consider the quantity $(\alpha_{\varphi\varphi})_{\mathrm{rms}}$, where the r.m.s. value is defined by averaging over all r and ϑ of interest in a meridional plane. Fig. 4 shows the dependence of this quantity on Rm, which is again varied by varying Pm.

Several attempts have been made to reproduce the quasi–steady dynamo observed in the direct numerical simulations by a mean–field model using the representation (9) of $\boldsymbol{\mathcal{E}}$ with the calculated $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$. The results were not completely satisfying. The mean–field model with the most reliable choice of these quantities, that is, according to method (i), proved to be slightly subcritical. As Fig. 5 shows, however, the steady mean magnetic field extracted from the direct numerical simulations is geometrically rather similar to the slowly decaying one of the mean–field model.

The quantity $\delta\boldsymbol{\mathcal{E}}$ turns out to be larger than in the case of magnetoconvection by a factor in the order of 10. This seems to indicate that the representation (9) no longer describes the real $\boldsymbol{\mathcal{E}}$ reasonably. The neglect of higher than first–order derivatives of $\overline{\boldsymbol{B}}$ is no longer justified. This statement is in some agreement with findings by Avalos et al. (2005).

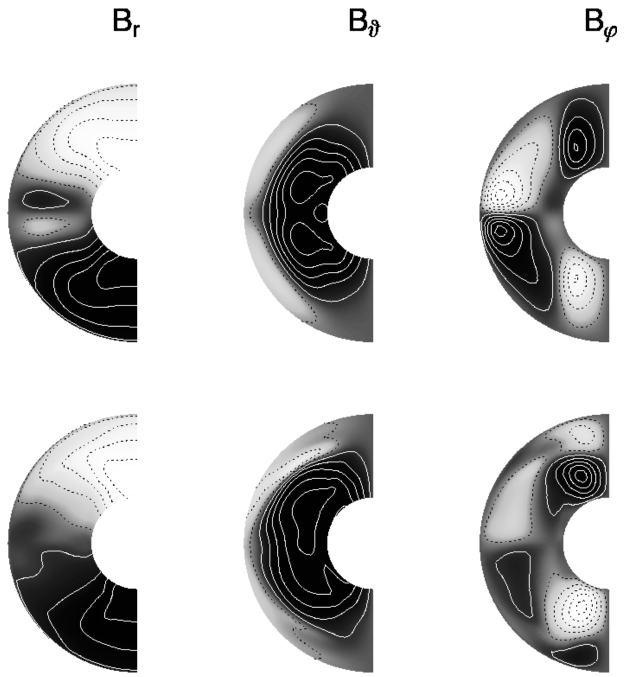

Fig. 5. The mean magnetic field components in the direct numerical dynamo simulation (upper panel) and in the corresponding mean–field model (lower panel).

7. Summary

The two examples considered in this paper lead us to limits of the applicability of two simplifications frequently used in mean–field theory: Although in these examples the validity of the first–order smoothing approximation has proved not to be rigourously restricted to Rm much smaller than unity, it became clear that this approximation does not work well with Rm exceeding the order of unity. In the second example, in addition the traditional representation of the mean electromotive force considering no higher than first–order derivatives of the mean magnetic field seems to be no longer justified. Nonetheless, the results derived from our mean-field models match the azimuthal averages extracted from the direct numerical simulations surprisingly well.

We want to stress that neither the neglect of higher than first-order derivatives in the expressions (1) or (8) for the mean electromotive force nor the restriction to steady velocities is intrinsic for the presented method (i). Corresponding extensions are planned for the future.

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