

MEAN-FIELD VIEW ON GEODYNAMO MODELS

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A comparison is made between direct numerical simulations of magnetohydrodynamic processes in a rotating spherical shell and their mean-field description. The mean fields are defined by azimuthal averaging. The coefficients that occur in the traditional representation of the mean electromotive force considering spatial derivatives of the mean magnetic field up to the first order are calculated by two different methods, with the fluid velocity taken from the direct numerical simulations. While the first method does not use intrinsic approximations, the second one is based on the first-order smoothing approximation. There is satisfying agreement of the results of the both methods for sufficiently slow fluid motions. For the investigated example of rotating magnetoconvection, the mean magnetic field derived from the direct numerical simulation is well reproduced on the mean-field level. For a quasi-steady geodynamo model a discrepancy occurs, which is probably a consequence of the neglect of higher-order derivatives of the mean magnetic field in the mean electromotive force. At higher excitations, geodynamo models of the same type show highly time-dependent fluid motions and magnetic fields. The coefficients determining the mean electromotive force fluctuate then considerably in space and time, but on the average their profiles resemble those of their counterparts in the quasi-steady case.

Introduction. The mean-field concept has proved to be a useful tool for the investigation of magnetohydrodynamic, in particular dynamo processes with complex fluid motions. Within this concept, mean fields are defined by a proper averaging of the original fields. We denote the mean fields by overbars, e.g., the mean magnetic field and the mean fluid velocity by $\overline{\mathbf{B}}$ and $\overline{\mathbf{U}}$, and the deviations of the original fields \mathbf{B} and \mathbf{U} from these mean fields by \mathbf{b} and \mathbf{u} . The mean electromagnetic fields are governed by equations, which differ formally from the Maxwell's and the completing constitutive equations for the original fields, or from the corresponding induction equation, only in one point. In the mean-field versions of the Ohm's law and of the induction equation an additional electromotive force, \mathcal{E} , occurs, which is defined by $\mathcal{E} = \overline{\mathbf{u} \times \mathbf{b}}$. It may be considered as a functional of \mathbf{u} , $\overline{\mathbf{U}}$ and $\overline{\mathbf{B}}$. Under some simplifying assumptions, the representation

$$\mathcal{E}_i = a_{ij} \overline{B}_j + b_{ijk} \nabla_k \overline{B}_j \quad (1)$$

can be justified. We refer here to Cartesian coordinates and use the summation convention. The coefficients a_{ij} and b_{ijk} are determined by \mathbf{u} and $\overline{\mathbf{U}}$ and can depend on $\overline{\mathbf{B}}$ only via these quantities. A crucial condition for the validity of relation (1) is a sufficiently weak variation of $\overline{\mathbf{B}}$ in space and time.

Although by far not generally justified, the simple relation (1) for \mathcal{E} has been used in almost all mean-field models of magnetohydrodynamic processes, in particular in mean-field dynamo models. In many cases now direct numerical simulations are available. So the possibility opens up to calculate the tensors a_{ij} and b_{ijk} with the velocity field \mathbf{u} taken from these simulations. In this paper, we

deal with examples of magnetoconvection and geodynamo models as investigated by Olson *et al.* (1999) and by Christensen *et al.* (2001). We compare, with a view to the applicability of relation (1), the mean magnetic field resulting from mean-field models using the so determined a_{ij} and b_{ijk} with that derived immediately from the numerical simulations.

Some of the relevant ideas have already been presented by Schinnerer *et al.* (2005) and are discussed here further. More details can be found in Schinnerer (2005).

1. The examples considered. In both cases, magnetoconvection and geodynamo, a rotating spherical shell of an electrically conducting fluid is considered, in which the fluid velocity \mathbf{U} , the magnetic field \mathbf{B} and the deviation θ of the temperature from the temperature T_0 in a reference state is governed by

$$\begin{aligned} \partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla) \mathbf{U} &= -(1/\varrho) \nabla P + \nu \nabla^2 \mathbf{U} - 2\boldsymbol{\Omega} \times \mathbf{U} \\ &\quad + (1/\mu\varrho) (\nabla \times \mathbf{B}) \times \mathbf{B} - \alpha_T \mathbf{g} \theta \\ \partial_t \mathbf{B} - \nabla \times (\mathbf{U} \times \mathbf{B}) - \eta \nabla^2 \mathbf{B} &= \mathbf{0} \\ \partial_t \theta + \mathbf{U} \cdot \nabla \theta - \kappa \Delta \theta &= -\mathbf{U} \cdot \nabla T_0 \\ \nabla \cdot \mathbf{U} = \nabla \cdot \mathbf{B} &= 0. \end{aligned} \tag{2}$$

The fluid-dynamic equations have to be understood as Boussinesq approximation. As usual, ϱ is the mass density of the fluid, μ is its magnetic permeability assumed to be equal to that of free space; ν , η and κ are the kinematic viscosity, magnetic diffusivity and thermal conductivity, $\boldsymbol{\Omega}$ is the angular velocity responsible for the Coriolis force, α_T is the thermal volume expansion coefficient, and \mathbf{g} denotes the gravitational acceleration.

For the fluid velocity \mathbf{U} , no-slip conditions are posed at the boundaries, which are considered as surfaces of rigid bodies. Both parts of the surroundings of the spherical shell are assumed electrically non-conducting so that the magnetic field \mathbf{B} continues as a potential field in these spaces. In the magnetoconvection case, an imposed toroidal magnetic field is assumed. The temperature T_0 is assumed to be constant on each of the boundaries, and θ to vanish there.

Equations (2) can be written in a non-dimensional form, which contains only four non-dimensional parameters, that is, the Ekman number E , a modified Rayleigh number Ra , the Prandtl number Pr and the magnetic Prandtl number Pm ,

$$E = \nu/\Omega D^2, \quad Ra = \alpha_T g \Delta T D / \nu \Omega, \quad Pr = \nu/\kappa, \quad Pm = \nu/\eta. \tag{3}$$

Here D denotes the thickness of the spherical shell and ΔT is the difference of the temperatures at the inner and the outer boundaries. The typical magnitude B_0 of the imposed toroidal magnetic field can be expressed by the Elsasser number Λ ,

$$\Lambda = B_0^2 / \varrho \mu \eta \Omega. \tag{4}$$

In all simulations considered in the following, $D = 0.65r_0$ is assumed, where r_0 is the radius of the outer boundary. In order to characterize the results of the simulations, we use in particular the magnetic Reynolds number $Rm = uD/\eta$ with u interpreted as a r.m.s. value of \mathbf{u} .

For the numerical solution of the above equations a code is used, which was originally designed by Glatzmaier (1984) and later modified by Christensen *et al.* (1999).

2. Mean-field theory. We follow here the lines of mean-field electrodynamics as presented, e.g., by Krause *et al.* (1980) or Rädler (2000).

2.1. The general concept. When applying the mean-field concept, we focus attention on the induction equation only. We refer here to a spherical coordinate system (r, ϑ, φ) , the polar axis of which coincides with the rotation axis. For a scalar field F , the mean field \overline{F} is defined as the average of F over all values of the azimuthal coordinate φ . In the case of vector and tensor fields, the mean fields are defined by averaging the components of these fields with respect to the chosen coordinate system in this way. With this definition, the Reynolds averaging rules apply exactly. Of course, all mean fields are axisymmetric about the polar axis of the coordinate system.

Subjecting the induction equation given in (2) to averaging, we obtain

$$\partial_t \overline{\mathbf{B}} - \nabla \times (\overline{\mathbf{U}} \times \overline{\mathbf{B}} + \boldsymbol{\mathcal{E}}) - \eta \nabla^2 \overline{\mathbf{B}} = \mathbf{0}, \quad \nabla \cdot \overline{\mathbf{B}} = 0, \quad (5)$$

with the crucial electromotive force

$$\boldsymbol{\mathcal{E}} = \overline{\mathbf{u} \times \mathbf{b}} \quad (6)$$

mentioned above. If \mathbf{u} is given, the calculation of $\boldsymbol{\mathcal{E}}$ further requires the knowledge of \mathbf{b} . From the above equations we derive

$$\begin{aligned} \partial_t \mathbf{b} - \nabla \times (\overline{\mathbf{U}} \times \mathbf{b} + \mathbf{G}) - \eta \nabla^2 \mathbf{b} &= \nabla \times (\mathbf{u} \times \overline{\mathbf{B}}) \\ \mathbf{G} &= \mathbf{u} \times \mathbf{b} - \overline{\mathbf{u} \times \mathbf{b}}, \quad \nabla \cdot \mathbf{b} = 0. \end{aligned} \quad (7)$$

On this basis, we conclude that $\boldsymbol{\mathcal{E}}$ is a functional of \mathbf{u} , $\overline{\mathbf{U}}$ and $\overline{\mathbf{B}}$, which is linear in $\overline{\mathbf{B}}$. Cancelling \mathbf{G} in the first line of (7) leads to the often used “first-order smoothing” approximation.

In some of the examples envisaged the dominant feature of the flow pattern are convection columns, showing no other variation in time than some azimuthal drift. We may change to a rotating frame of reference, in which they are stationary. In simple cases, in addition to \mathbf{u} also \mathbf{b} , \mathbf{U} , $\overline{\mathbf{B}}$ and $\boldsymbol{\mathcal{E}}$ are then steady. The result for $\boldsymbol{\mathcal{E}}$ obtained in this rotating frame applies in any case to the original frame, too.

In view of the examples considered, we may assume that \mathbf{b} vanishes if $\overline{\mathbf{B}}$ does so. Then $\boldsymbol{\mathcal{E}}$ must not only be linear in the sense explained above, but also homogeneous in $\overline{\mathbf{B}}$. The definition of $\boldsymbol{\mathcal{E}}$ at a given point of the (r, ϑ) plane at a given time requires the knowledge of the components of $\overline{\mathbf{B}}$ in some surroundings of this point at this time and in some past. For the sake of simplicity, it is further assumed that their variation in this surroundings is sufficiently weak so that they can be represented there by their values and their first derivatives at this point, and that any time variation is negligible. While the assumption concerning the homogeneity of $\boldsymbol{\mathcal{E}}$ in $\overline{\mathbf{B}}$ is here well satisfied, the latter one remains to be checked in all applications. Both together enable us to write

$$\boldsymbol{\mathcal{E}}_\kappa = \tilde{a}_{\kappa\lambda} \overline{B}_\lambda + \tilde{b}_{\kappa\lambda r} \frac{\partial \overline{B}_\lambda}{\partial r} + \tilde{b}_{\kappa\lambda\vartheta} \frac{1}{r} \frac{\partial \overline{B}_\lambda}{\partial \vartheta}. \quad (8)$$

We refer here again to the spherical coordinate system introduced above. The coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$, called “mean-field coefficients” in the following, are determined by \mathbf{u} and $\overline{\mathbf{U}}$ and can depend only via these quantities on $\overline{\mathbf{B}}$. They depend, of course, on r and ϑ and, in case of non-stationary flows, also on time. The indices κ and λ stand for r , ϑ or φ , and again the summation convention is adopted. Note that $\boldsymbol{\mathcal{E}}$ is here, with $\overline{\mathbf{B}}$ being axisymmetric, determined by 27 independent coefficients.

2.2. *Definition of the mean-field coefficients.* Two methods have been used for the calculation of the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$ on the basis of the numerical simulations addressed above.

Method (i) is based on equation (7) for \mathbf{b} . It is solved numerically with \mathbf{u} and $\overline{\mathbf{U}}$ taken from the numerical simulations mentioned, but employing nine properly chosen steady “test fields” $\overline{\mathbf{B}} = \overline{\mathbf{B}}^{(\nu)}$, $\nu = 1, \dots, 9$. Note that the velocities are treated as independent of the test fields and are therefore the same in all 9 cases. One criterion for the choice of the test fields is that higher than first-order derivatives of their components with respect to r and ϑ are equal to zero or at least as small as possible. With the results for \mathbf{b} obtained in this way, $\mathcal{E} = \mathcal{E}^{(\nu)}$ is calculated for each ν . Writing then down equations (8) with $\mathcal{E}_\kappa = \mathcal{E}_\kappa^{(\nu)}$ and $\overline{B}_\lambda = \overline{B}_\lambda^{(\nu)}$ for $\nu = 1, \dots, 9$ for any given r and ϑ , we arrive at three sets of nine linear algebraic equations for the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$. Solving these equations, we can define all 27 coefficients for all r and ϑ .

Method (ii) is applicable to steady situations only, ignores any mean fluid motion and uses the first-order smoothing approximation. The steady version of equation (7) for \mathbf{b} with $\overline{\mathbf{U}} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}$ can be solved analytically for arbitrary \mathbf{u} and $\overline{\mathbf{B}}$. On this basis, \mathcal{E} and so the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$ can be defined for arbitrary \mathbf{u} in the usual way and later specified by choosing \mathbf{u} according to the numerical simulations mentioned.

We point out that \mathbf{u} needed to determine the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$ in both methods was taken from simulations with non-zero $\overline{\mathbf{B}}$. That is, the resulting coefficients are already subject to a magnetic quenching corresponding to this $\overline{\mathbf{B}}$.

2.3. *A coordinate-independent representation of the mean electromotive force.*

Relation (1) for \mathcal{E} can be understood as establishing a coordinate-independent relationship between \mathcal{E} , $\overline{\mathbf{B}}$ and $\nabla\overline{\mathbf{B}}$ by representing it in a Cartesian coordinate system. In that sense, \mathcal{E}_i and \overline{B}_j have to be understood as vectors, a_{ij} , b_{ijk} and $\nabla_j\overline{B}_k$ as tensors, all with the usual properties under coordinate transformations. By contrast, relation (8) is from the very beginning a specific one, which applies only in the chosen spherical coordinate system, and the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$ should not be interpreted as tensor components.

The coordinate-independent relationship between \mathcal{E} , $\overline{\mathbf{B}}$ and $\nabla\overline{\mathbf{B}}$, expressed above in the form (1), is equivalent to

$$\begin{aligned} \mathcal{E} = & -\boldsymbol{\alpha} \cdot \overline{\mathbf{B}} - \boldsymbol{\gamma} \times \overline{\mathbf{B}} \\ & -\boldsymbol{\beta} \cdot (\nabla \times \overline{\mathbf{B}}) - \boldsymbol{\delta} \times (\nabla \times \overline{\mathbf{B}}) - \boldsymbol{\kappa} \cdot (\nabla\overline{\mathbf{B}})^{(s)}; \end{aligned} \quad (9)$$

see, e.g., Rädler (1980, 2000). Here $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are symmetric second-rank tensors, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are vectors, $\boldsymbol{\kappa}$ is a third-rank tensor with some symmetries, all determined by \mathbf{u} and $\overline{\mathbf{U}}$ only, and $(\nabla\overline{\mathbf{B}})^{(s)}$ is the symmetric part of the gradient tensor of $\overline{\mathbf{B}}$, that is, when referring again to a Cartesian coordinate system, $(\nabla\overline{\mathbf{B}})_{ij}^{(s)} = \frac{1}{2}(\nabla_j\overline{B}_i + \nabla_i\overline{B}_j)$. We extend the notation “mean-field coefficients” also to the components of $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$. The $\boldsymbol{\alpha}$ term in (9) describes in general an anisotropic α -effect, and the $\boldsymbol{\gamma}$ term an advection of the mean magnetic field like that by a mean motion of the fluid. The $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ terms can be interpreted in the sense of an anisotropic electric mean-field conductivity and the $\boldsymbol{\kappa}$ term covers several other influences on the mean fields.

Like the number of the components of a_{ij} and b_{ijk} in (1), that of the components of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ in (9) is 36, too. If we, however, specify (9) to our spherical coordinate system and consider the axisymmetry of $\overline{\mathbf{B}}$, this number reduces to 27, just in agreement with the number of the coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and

$\tilde{b}_{\kappa\lambda\vartheta}$. We may therefore, without changing \mathcal{E} , choose nine coefficients of α , β , γ , δ and κ arbitrarily, e.g., put them equal to zero. The remaining 27 components of α , β , γ , δ and κ are then uniquely determined by the 27 coefficients $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$. In that sense we express in the following all results originally obtained for $\tilde{a}_{\kappa\lambda}$, $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$ in terms of α , β , γ , δ and κ . We stress that these last quantities are chosen with some arbitrariness, which, however, has no any influence on \mathcal{E} .

We also point out that the components of $\nabla \times \overline{\mathbf{B}}$ and $(\nabla \overline{\mathbf{B}})^{(s)}$ with respect to the spherical coordinate system contain the $\overline{\mathbf{B}}_{\kappa}$ not only in the form of their derivatives, but also without derivatives. Therefore, α and γ depend not only on $\tilde{a}_{\kappa\lambda}$, but also on $\tilde{b}_{\kappa\lambda r}$ and $\tilde{b}_{\kappa\lambda\vartheta}$. With our special choice of α , β , γ , δ and κ , we have, e.g., $\alpha_{rr} = -(\tilde{a}_{rr} - \tilde{b}_{r\vartheta\vartheta}/r)$, $\alpha_{\vartheta\vartheta} = -(\tilde{a}_{\vartheta\vartheta} + \tilde{b}_{\vartheta r\vartheta}/r)$ and $\alpha_{\varphi\varphi} = -\tilde{a}_{\varphi\varphi}$.

3. Magnetoconvection. We consider here a simulation by Olson *et al.* (1999) with $E = 10^{-3}$, $Ra = 94$, $Pr = Pm = 1$ and an imposed toroidal magnetic field corresponding to $\Lambda = 1$. In this case convection columns occur, which drift in the azimuthal direction. In a properly defined frame of reference the fluid motion as well as the magnetic field are steady. The intensity of the motion is characterized by $Rm \approx 12$. A flow pattern is shown in Fig. 1.

The results for α , β , γ , δ and κ obtained by the two methods explained above, (i) and (ii), do not completely coincide. This was to be expected since method (ii) is based on the assumption $\overline{\mathbf{U}} = \mathbf{0}$ and first-order smoothing. In the steady case considered here the latter is surely justified for $Rm' \ll 1$ (a sufficient condition), where $Rm' = ul/\eta$, with u being again the r.m.s. value of \mathbf{u} and l a characteristic length of the \mathbf{u} -field. It seems reasonable to assume that l is not much smaller than D , that is, Rm' is not much smaller than Rm . We consider of course the results for α , β , γ , δ and κ obtained by method (i) as most reliable. A part of them is represented in Figs. 2 and 3. The results obtained by method (ii) fairly agree with them as far as the profiles of these quantities with respect to r and ϑ are concerned, but overestimate their magnitudes typically by a few per cent. When calculating α , β , γ , δ and κ for the given \mathbf{u} , we may scale down Rm by a proper reduction of Pm . For values of Rm up to about 10 the results of the two methods are in satisfying agreement; see also Fig. 5 below.

Inspecting Figs. 2 and 3 and recalling that $Pm = 1$, we see that the components of α and γ reach values of several η/D . The dominating component of α is $\alpha_{\varphi\varphi}$, which is crucial for the generation of a poloidal from a toroidal mean magnetic field, followed by α_{rr} and $\alpha_{\vartheta\vartheta}$. The effect of γ consists in the expulsion of the mean magnetic flux from the main convection region. The components of β ,

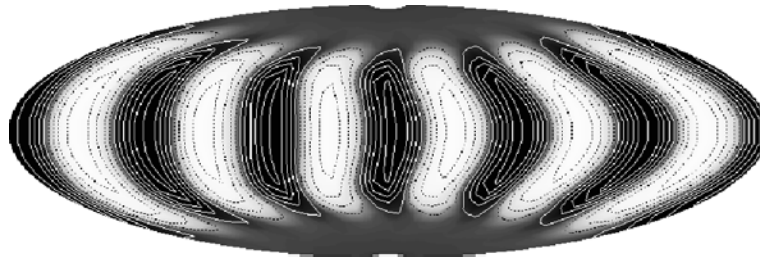


Fig. 1. The radial velocity in the magnetoconvection case at $r = 0.59 r_0$, normalized with its maximum given by $U_r = 16.98 \nu/D$. In the grey scale coding, white and black correspond to -1 and $+1$, respectively, and the contour lines to ± 0.1 , ± 0.3 , ± 0.5 , ± 0.7 , ± 0.9 .

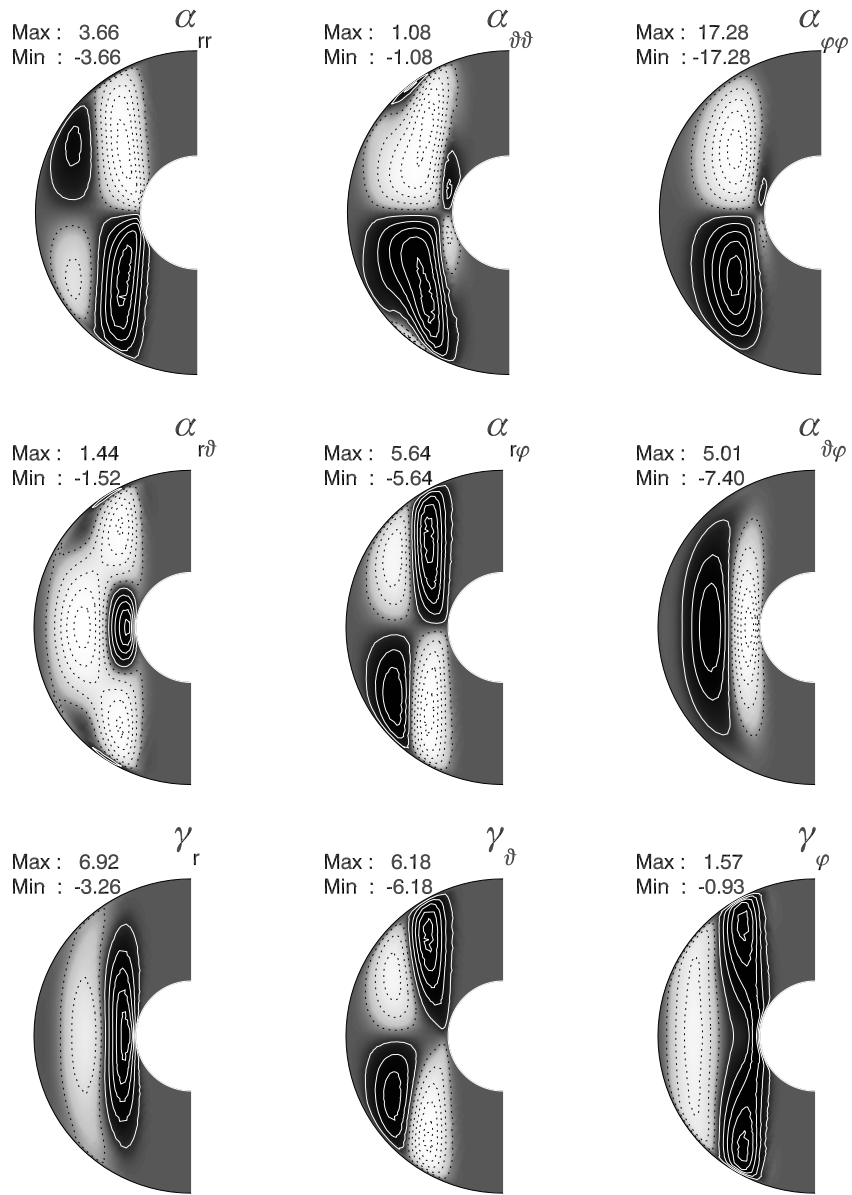


Fig. 2. Components of the (symmetric) α tensor and the γ vector in a meridional plane in the magnetoconvection case determined by method (i) in units of ν/D . For each component the grey scale (white – negative, black – positive values) is separately adjusted to its maximum modulus. Note that there is a sign error in the presentation of γ in Fig. 2 of Schirner *et al.* (2005).

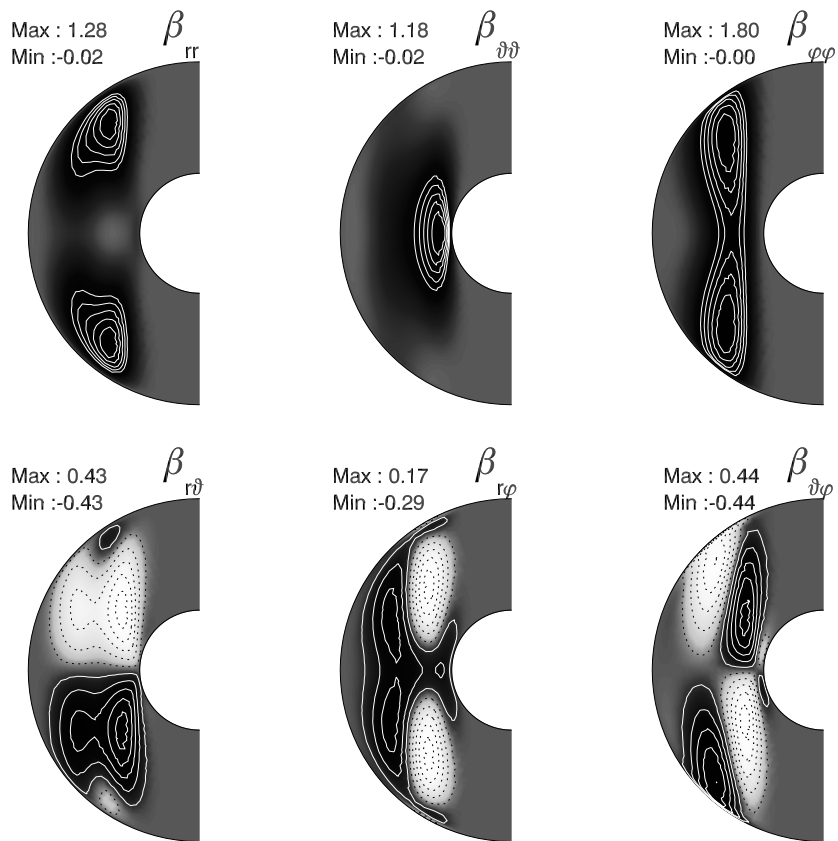


Fig. 3. Components of the (symmetric) β tensor in a meridional plane in the magneto-convection case determined by method (i) in units of ν . Grey scales as in Fig. 2.

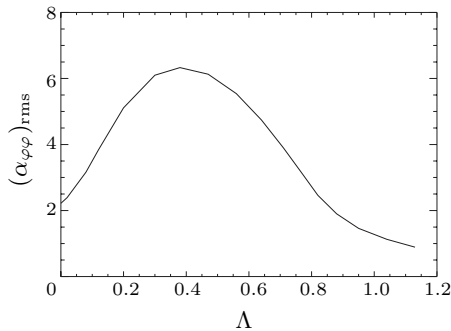


Fig. 4. The quantity $(\alpha_{\varphi\varphi})_{rms}$ in the magnetoconvection case in ν/D units, in dependence on Λ .

in particular the diagonal ones, take values up to about η . That is, the mean-field diffusivity, and the mean-field conductivity, differ clearly from the original ones.

A mean-field model of magnetoconvection, using (9) and our results for α , β , γ , δ and κ , reproduces very well the $\overline{\mathbf{B}}$ -field obtained from the direct numerical simulations.

We have also investigated the quantity $\delta\mathcal{E}$ defined by

$$\delta\mathcal{E} = (\mathcal{E}^{\text{DNS}} - \mathcal{E}^{\text{MF1}})/\mathcal{E}_{rms}^{\text{DNS}}, \quad (10)$$

where \mathcal{E}^{DNS} corresponds to the quantity \mathcal{E} immediately extracted from the direct numerical simulation and \mathcal{E}^{MF1} corresponds to this quantity determined according to (9) (considering no higher than first-order derivatives of $\overline{\mathbf{B}}$) with α , β , γ , δ and κ as obtained by the above-described calculations, and $\overline{\mathbf{B}}$ corresponding to the direct numerical simulations (or, what here is the same, to the mean-field model). $\mathcal{E}_{rms}^{\text{DNS}}$ is the r.m.s. value of \mathcal{E}^{DNS} . Slightly deviating from the standard definition of the r.m.s. value used in the case of u_{rms} , with averaging over the volume of all fluid shell, here the average is taken over the part of the meridional plane, which corresponds to the fluid layer. With the exception of few small areas of this plane, $|\delta\mathcal{E}|$ is much smaller than unity. This indicates that the representation (9) is indeed sufficient for the purposes of the example considered.

As explained above, the results reported here imply some magnetic quenching. In this respect, the results obtained with various $\overline{\mathbf{B}}$, i.e., various Λ , are of interest. Fig. 4 shows $(\alpha_{\varphi\varphi})_{rms}$, the r.m.s. value of $\alpha_{\varphi\varphi}$ defined as in the case of $\mathcal{E}_{rms}^{\text{DNS}}$, as a function of Λ . The increase of α in the presence of a weak magnetic field, that is, for small Λ , is due to an increasing vigour of convection by the relaxation of the geostrophic constraint. A strong magnetic field, however, inhibits convection and reduces $\alpha_{\varphi\varphi}$. Like $\alpha_{\varphi\varphi}$, the other components of α are also quenched. The quenching is, however, not the same for different components what leads to varying amplitude relations among these components for the varying strength of the mean magnetic field. In addition to the α -quenching, e.g., a β -quenching also takes place.

4. A quasi-steady geodynamo model. We consider now the case with $E = 10^{-3}$, $Ra = 100$, $Pr = 1$ and $Pm = 5$, in which the numerical simulations by Christensen *et al.* (2001) indeed show a dynamo. Again the fluid motion and the magnetic field in a properly chosen rotating frame of reference are steady, and the intensity of the motion can now be characterized by $Rm \approx 40$.

In this case, there is a clear difference in the results for α , β , γ , δ and κ obtained by the two methods. To give an example for that, we consider again the

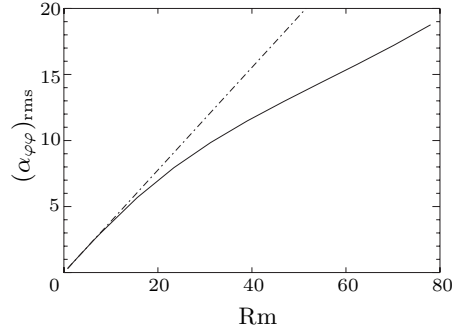


Fig. 5. The quantity $(\alpha_{\varphi\varphi})_{\text{rms}}$ in the quasi-steady dynamo case in ν/D units, with $\alpha_{\varphi\varphi}$ determined by methods (i) and (ii) (solid and dashed lines, respectively) in dependence on Rm .

quantity $(\alpha_{\varphi\varphi})_{\text{rms}}$ defined above. Fig. 5 shows the dependence of this quantity on Rm , which is again varied by varying Pm . In this case, the first-order smoothing approximation provides us with reasonable results only up to values of Rm of about 10, but in general it overestimates $\alpha_{\varphi\varphi}$.

An attempt has been made to reproduce the quasi-steady dynamo observed in the direct numerical simulations by a mean-field model using the representation (9) of \mathcal{E} with the calculated α , β , γ , δ and κ . The results were not completely satisfying. The mean-field model with the most reliable choice of these quantities, that is, according to method (i), proved to be slightly subcritical. As Fig. 6 shows, however, the steady mean magnetic field extracted from the direct numerical simulations is geometrically rather similar to the slowly decaying one of the mean-field model.

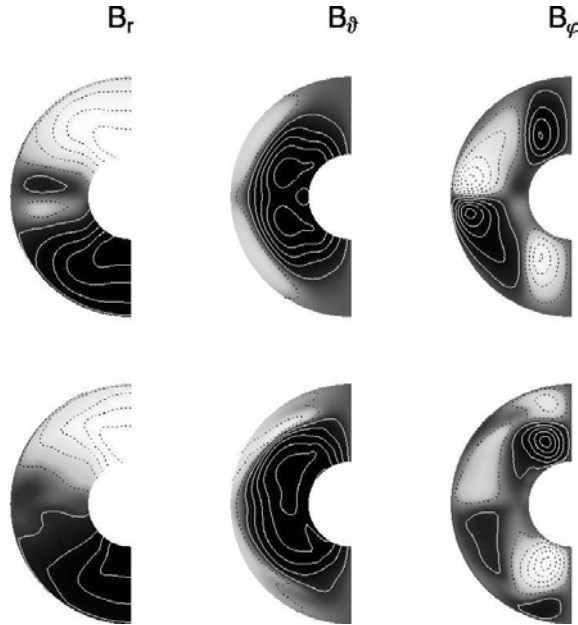


Fig. 6. The mean magnetic field components from a direct numerical dynamo simulation (upper panel) and from the corresponding mean-field model (lower panel). Grey scales as in Fig. 2.

The quantity $\delta\mathcal{E}$ turns out to be larger than in the case of magnetoconvection by a factor of the order of 10. This seems to indicate that the representation (9) no longer describes the real \mathcal{E} reasonably. The neglect of higher than first-order derivatives of $\bar{\mathbf{B}}$ is no longer justified. This statement is in some agreement with findings by Avalos *et al.* (2006).

5. Time-dependent geodynamo models. At higher excitations, geodynamo models of the type considered so far show highly time-dependent motions and magnetic fields. Let us add two examples of that kind.

The first one is defined by $E = 10^{-4}$, $Ra = 334$, $Pr = 1$ and $Pm = 2$. The numerical simulation by Olson *et al.* (1999) indeed shows a highly time-dependent but still strongly columnar convection. Compared to the quasi-steady case considered before, Rm has roughly doubled, $Rm \approx 88$. When applying the mean-field concept to this situation, a generalization of relation (8) for \mathcal{E}_κ might be necessary so that the time derivatives of \bar{B}_λ are also included. Nevertheless, method (i), which was established on the basis of (8), provides us with correct results for the mean-field coefficients considered so far if steady test fields are used. We have defined them in this way. Their time-averaged profiles are similar to those in the quasi-steady case what suggests similar dynamo processes. The amplitudes of the coefficients, however, vary considerably with time, on the scale of the convective turnover time, with fluctuations as large as the mean values. While the dominating coefficients fluctuate somewhat moderately, the variation of the less significant coefficients is even stronger. Fig. 7 displays exemplarily the profile and the time-variability of $\alpha_{\varphi\varphi}$.

In the last example we have $E = 3 \cdot 10^{-4}$, $Ra = 990$, $Pr = 1/3$, $Pm = 1$ and, as a result, $Rm = 350$. As shown by Kutzner and Christensen (2002), it belongs to the dynamos with a reversing dipole field. The convection is no longer columnar and shows no longer equatorial symmetry. Again the mean-field coefficients have been defined according to method (i) with steady test fields. At any given time they vary, compared to the previous example, on much smaller spatial scales and do not exhibit equatorial symmetries. If averaged over sufficiently long times, however, the coefficients show profiles similar to those of their steady counterparts, see

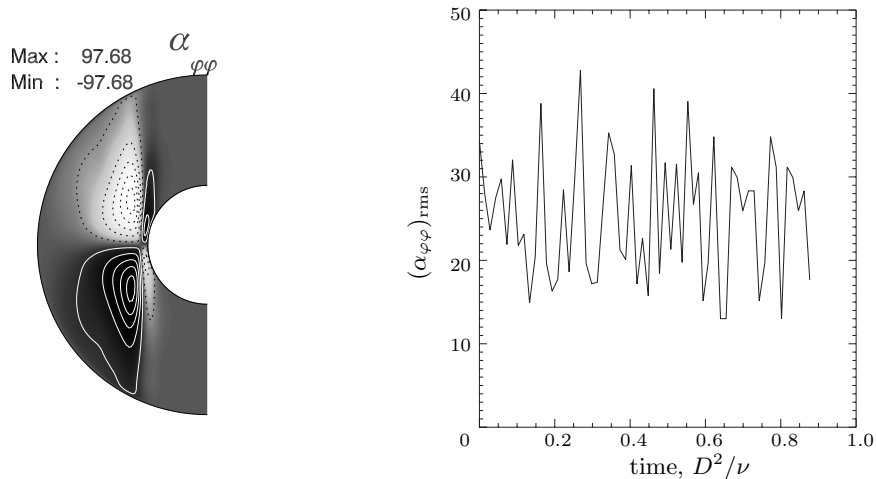


Fig. 7. Time-averaged $\alpha_{\varphi\varphi}$ (left) and a time series of $(\alpha_{\varphi\varphi})_{rms}$ (right), both in units of ν/D , in the case of a dynamo with time-dependent but columnar convection. Grey scale as in Fig. 2.

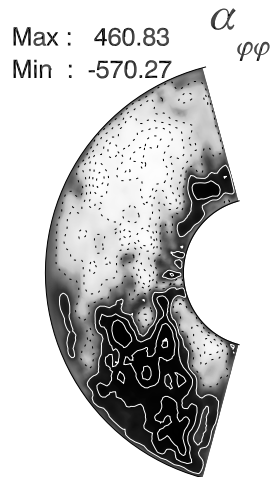


Fig. 8. Time-averaged $\alpha_{\varphi\varphi}$, in units of ν/D , in the case of a dynamo with time-dependent convection showing no longer columnar structure and equatorial symmetry. Grey scale as in Fig. 2.

Fig. 8. The relative amplitude fluctuations of the mean-field coefficients are by a factor of the order of 10 larger than in the previous example and the fluctuations exceed the mean values by far. Such strong fluctuations may be the cause of polarity reversals, see Hoyng *et al.* (2001).

6. Summary. The first two examples considered in this paper, that is, the cases of rotating magnetoconvection and of a quasi-steady geodynamo model, lead us to limits of the applicability of two simplifications frequently used in the mean-field theory. Although in these examples the validity of the first-order smoothing approximation has proved not to be rigorously restricted to Rm much smaller than unity, it has become clear that this approximation does not work well with Rm exceeding 10. In the second example, in addition to the traditional representation of the mean electromotive force considering no higher than first-order spatial derivatives of the mean magnetic field seems to be no longer justified. Nonetheless, the results derived from our mean-field models match the azimuthal averages extracted from the direct numerical simulations surprisingly well.

In two more examples, geodynamo models of the same type but at higher excitations are considered, which exhibit highly time-dependent motions and magnetic fields. The coefficients, determining the mean electromotive force, fluctuate then considerably in space and time, but on the average their profiles resemble those of their steady counterparts.

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