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TRAVEL-TIME SENSITIVITY KERNELS FOR TIME-DISTANCE HELIOSEISMOLOGY

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ABSTRACT

The forward problem of time-distance helioseismology is to compute travel-time perturbations which result from perturbations to a solar model. We present a new and physically motivated general framework for calculations of the sensitivity of travel-times to small local perturbations. Our starting point is a description of the wavefield excited by distributed random sources. We employ the first Born approximation to model scattering from local inhomogeneities. We give a clear definition of travel-time perturbation, which allows a connection between observations and theory. In this framework travel-time sensitivity kernels depend explicitly on the details of the measurement procedure.

Key words: helioseismology: time-distance; waves: scattering.

1. INTRODUCTION

Time-distance helioseismology, as introduced by Duvall et al. (1993), has yielded numerous exciting insights into the interior of the Sun. This technique, which gives information about travel-times for wave packets moving between any two points on the solar surface, is an important complement to global mode helioseismology as it is able to probe subsurface structure and dynamics in three dimensions.

The forward problem of time-distance helioseismology is to compute travel-time perturbations due to perturbations to a solar model. An accurate solution to the forward problem is necessary for making quantitative inferences about the Sun from time-distance data. There have been a number of previous efforts to understand the effect of local perturbations on travel-times. Kosovichev (1996) used geometrical acoustics to describe the interaction of waves with sound-speed perturbations and flows. Bogdan (1997) argued that a finite-wavelength theory is needed. Jensen et al. (2000) and Birch & Kosovichev (2000) solved the linear forward problem for sound-speed perturbations, in the single-source approximation. Bogdan et al. (1998) used a normal mode approach to compute travel-time perturbations in a model sunspot. Woodard (1997) showed the effect of localized damping on travel-times.

Although the above mentioned efforts represent substantial progress, there is not yet a general procedure for relating actual travel-time measurements to perturbations to a solar model. This paper is an attempt to synthesize and extend the current knowledge into a flexible framework for the computation of the sensitivity of travel-times to local inhomogeneities. We start from a physical description of the wavefield, including wave excitation and damping. We incorporate the details of the measurement procedure. Two other key ingredients of our approach are the single scattering Born approximation and a clear observational definition of travel-time, both taken from the geophysics literature (e.g. Tong et al., 1998; Zhao & Jordan, 1998; Marquering et al., 1999).

2. DEFINITION OF TRAVEL TIMES

The fundamental data of modern helioseismology are high-resolution Doppler images of the Sun's surface (e.g. Scherrer et al., 1995). In general, the filtered line-of-sight projection of the velocity field can be written as

$$\phi = \mathcal{F}\left\{\hat{\boldsymbol{\ell}} \cdot \dot{\boldsymbol{\xi}}\right\} , \tag{1}$$

where $\boldsymbol{\xi}$ is the displacement vector and $\hat{\boldsymbol{\ell}}$ is a unit vector in the direction of the line of sight. Throughout this paper overdots denote time derivatives. The operator $\mathcal{F}$ describes the filter used in the data analysis which includes the time window, instrumental effects, and other filtering applied in the data analysis.

The basic computation in time-distance helioseismology is the temporal cross-correlation between the signal, ϕ, measured at two points, $\mathbf{1}$ and $\mathbf{2}$, on the solar surface,

$$C(\mathbf{1}, \mathbf{2}, t) = \int \mathrm{d}t' \; \phi(\mathbf{1}, t') \; \phi(\mathbf{2}, t' + t). \tag{2}$$

The cross-correlation is a solar seismogram; it provides information about travel-times, amplitudes and the shape of the wavepackets traveling between any two points on the solar surface. For acoustic waves, cross-correlations display several branches corresponding to multiple bounces off the surface in between **1** and **2**. For the rest of the paper we discuss only first-bounce travel-times in order to simplify the notation. The positive-time branch corresponds to waves moving from **1** to **2** and the negative-time branch represents waves moving in the opposite direction.

We define the travel-time perturbation for each branch to be the time lag that minimizes the difference between the measured cross-correlation, C, and a reference cross-correlation. This is analogous to the typical definition of travel-time perturbation used in the geophysics literature. The reference cross-correlation for measuring the travel-time perturbation from **1** to **2** we denote by $C_+^{\rm ref}$, which should be chosen by the observer to look like the first-bounce positive-time part of a cross-correlation. The other reference cross-correlation, denoted by $C_-^{\rm ref}$, is used for measuring the travel-time perturbation from **2** to **1** and should look like the first-bounce negative-time part of a cross-correlation. We measure the difference between the observed and reference cross-correlations by considering

$$X_\pm(\mathbf{1},\mathbf{2},t) = \int \mathrm{d}t' \; \left[C(\mathbf{1},\mathbf{2},t') - C_\pm^{\rm ref}(\mathbf{1},\mathbf{2},t' \mp t)\right]^2 . \quad (3)$$

Minimizing $X_\pm$ is equivalent to fitting $C_\pm^{\rm ref}$ to C in the least square sense, varying t only. Note that the time lag t appears with a minus sign in $C_+^{\rm ref}$ and a plus sign in $C_-^{\rm ref}$, so that for t positive $C_+^{\rm ref}$ is shifted to larger positive times while $C_-^{\rm ref}$ is shifted to more negative times.

According to our definition, the travel-time perturbations $\Delta\tau_+$ and $\Delta\tau_-$ are the time lags that minimize X_+ and X_-, and as a result $\dot{X}_\pm(\mathbf{1},\mathbf{2},\Delta\tau_\pm) = 0$. Linearizing $\dot{X}_\pm(\mathbf{1},\mathbf{2},t)$ about $t=0$ gives

$$\Delta\tau_\pm(\mathbf{1},\mathbf{2}) = \pm \frac{\int \mathrm{d}t \; \dot{C}_\pm^{\rm ref}(\mathbf{1},\mathbf{2},t)\; C(\mathbf{1},\mathbf{2},t)}{\int \mathrm{d}t \; \ddot{C}_\pm^{\rm ref}(\mathbf{1},\mathbf{2},t)\; C(\mathbf{1},\mathbf{2},t)} . \quad (4)$$

In order for this equation to be valid the linearization from which it was derived must be good, meaning $\Delta\tau_\pm$ must be small compared to the dominant period of the reference cross-correlation. Equation (4) is the fundamental equation from which the rest of the paper follows.

The definition of travel-time perturbations described here leaves observers free to measure without reference to a solar model. We note, however, that in order for a proper interpretation of measured travel-time perturbations to be made it is essential for the observer to report their choice of reference cross-correlations $C_\pm^{\rm ref}$ as well as the filter function $\mathcal{F}$. A solar model is only necessary for the next step, the interpretation of travel-time perturbations in terms of local perturbations to a solar model, to which we now turn.

3. INTERPRETATION OF TRAVEL TIMES

The first step in the interpretation of time-distance data is to relate measured $\Delta\tau_\pm$ to local perturbations to the solar model. Our approach is to compute the cross-correlation in a background solar model and then consider the perturbation to the cross-correlation, and thus travel-time, due to a local perturbation to the model.

We write the cross-correlation, C, as the cross-correlation in the background model, C^0, plus a first-order correction, δC,

$$C(\mathbf{1},\mathbf{2},t) = C^0(\mathbf{1},\mathbf{2},t) + \delta C(\mathbf{1},\mathbf{2},t). \quad (5)$$

We then write equation (4) in the form

$$\delta\tau_\pm(\mathbf{1},\mathbf{2}) = \int \mathrm{d}t \; W_\pm(\mathbf{1},\mathbf{2},t)\; \delta C(\mathbf{1},\mathbf{2},t). \quad (6)$$

The $\delta\tau_\pm$ are defined by

$$\delta\tau_\pm(\mathbf{1},\mathbf{2}) = \Delta\tau_\pm(\mathbf{1},\mathbf{2}) - \Delta\tau_\pm^0(\mathbf{1},\mathbf{2}) \quad (7)$$

with

$$\Delta\tau_\pm^0(\mathbf{1},\mathbf{2}) = \pm \frac{\int \mathrm{d}t \; \dot{C}_\pm^{\rm ref}(\mathbf{1},\mathbf{2},t)\; C^0(\mathbf{1},\mathbf{2},t)}{\int \mathrm{d}t \; \ddot{C}_\pm^{\rm ref}(\mathbf{1},\mathbf{2},t)\; C^0(\mathbf{1},\mathbf{2},t)}, \quad (8)$$

and the $W_\pm$ are given by

$$W_\pm(\mathbf{1},\mathbf{2},t) = \pm \frac{\dot{C}_\pm^{\rm ref}(\mathbf{1},\mathbf{2},t) \mp \Delta\tau_\pm^0(\mathbf{1},\mathbf{2})\; \ddot{C}_\pm^{\rm ref}(\mathbf{1},\mathbf{2},t)}{\int \mathrm{d}t \; \ddot{C}_\pm^{\rm ref}(\mathbf{1},\mathbf{2},t)\; C^0(\mathbf{1},\mathbf{2},t)} . \quad (9)$$

In order to obtain these equations we have linearized equation (4) in δC. The reference times, $\Delta\tau_\pm^0$, are the travel-time perturbations which would be measured if the Sun and the background model were identical. The measured travel-time perturbations, $\Delta\tau_\pm$, corrected by the reference times, $\Delta\tau_\pm^0$, we denote by $\delta\tau_\pm$. We emphasize that the $\delta\tau_\pm$ are proportional to δC, which is a first-order perturbation to the background solar model. The sensitivity of $\delta\tau_\pm$ to δC is given by $W_\pm$.

4. TEMPORAL CROSS-CORRELATION

In order to obtain the cross-correlation C^0 and the first-order perturbation δC, we need to compute the observable ϕ, which means we need the displacement vector $\boldsymbol{\xi}$. Linear oscillations are governed by an equation of the form

$$\boldsymbol{\mathcal{L}}\boldsymbol{\xi} = \boldsymbol{S}, \quad (10)$$

where $\boldsymbol{S}$ denotes the source of excitation for the waves. The linear operator $\boldsymbol{\mathcal{L}}$, acting on $\boldsymbol{\xi}$, encompasses all the physics of wave propagation in an inhomogeneous stratified medium permeated by flows and magnetic fields. Damping processes should also be accounted for in $\boldsymbol{\mathcal{L}}$. An explicit expression for the operator $\boldsymbol{\mathcal{L}}$ including steady flows is provided by

Lynden-Bell & Ostriker (1967). Bogdan (2000) includes magnetic field.

We now expand $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\xi}$ into zero- and first-order contributions, which refer to the background solar model and to the lowest order perturbation to that model:

$$\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{L}}^0 + \delta\boldsymbol{\mathcal{L}}, \quad (11)$$
$$\boldsymbol{\xi} = \boldsymbol{\xi}^0 + \delta\boldsymbol{\xi}. \quad (12)$$

The operator $\delta\boldsymbol{\mathcal{L}}$ depends on first-order quantities such as local perturbations in density, sound speed, and damping rate, as well as flows and magnetic field. Here we only consider time-independent perturbations, denoted by $\delta q_\alpha(\boldsymbol{r})$ for short, which are functions of position $\boldsymbol{r}$ in the solar interior. For the sake of simplicity, we ignore local perturbations to the source function.

To lowest order, equation (10) reduces to

$$\boldsymbol{\mathcal{L}}^0\boldsymbol{\xi}^0 = \boldsymbol{S}. \quad (13)$$

In order to solve this equation, we introduce a set of causal Green vectors $\mathbf{G}_j$ defined by

$$\boldsymbol{\mathcal{L}}^0\mathbf{G}_j(\boldsymbol{x}, t; \boldsymbol{s}, t_\mathrm{s}) = \hat{\mathbf{e}}_j(\boldsymbol{s})\ \delta_\mathrm{D}(\boldsymbol{x}-\boldsymbol{s})\ \delta_\mathrm{D}(t-t_\mathrm{s}), \quad (14)$$

where $\hat{\mathbf{e}}_j(\boldsymbol{s})$ are basis vectors at the point $\boldsymbol{s}$ and δ_D is the Dirac delta function. The vector $\mathbf{G}_j(\boldsymbol{x}, t; \boldsymbol{s}, t_\mathrm{s})$ is the displacement at $(\boldsymbol{x}, t)$ which results from a unit source in the $\hat{\mathbf{e}}_j$ direction at $(\boldsymbol{s}, t_\mathrm{s})$. Guided by equation (1), we define the zero-order Green functions for the observable ϕ:

$$\mathcal{G}_j(\boldsymbol{x}, t; \boldsymbol{s}, t_\mathrm{s}) = \mathcal{F}\left\{\hat{\boldsymbol{\ell}}(\boldsymbol{x}) \cdot \dot{\mathbf{G}}_j(\boldsymbol{x}, t; \boldsymbol{s}, t_\mathrm{s})\right\}. \quad (15)$$

In terms of $\mathcal{G}_j$, the unperturbed signal reads:

$$\phi^0(\boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{s}\, \mathrm{d}t_\mathrm{s}\ \mathcal{G}_j(\boldsymbol{x}, t; \boldsymbol{s}, t_\mathrm{s})\ S_j(\boldsymbol{s}, t_\mathrm{s}). \quad (16)$$

Sums are taken over repeated indices.

To the next order of approximation, equation (10) gives

$$\boldsymbol{\mathcal{L}}^0\delta\boldsymbol{\xi} = -\delta\boldsymbol{\mathcal{L}}\boldsymbol{\xi}^0. \quad (17)$$

This is the single-scattering Born approximation. We note that equation (17) is of the same form as equation (13): the term $-\delta\boldsymbol{\mathcal{L}}\boldsymbol{\xi}^0$ appears as a source for the scattered wave displacement $\delta\boldsymbol{\xi}$. The first-order Born approximation has been shown to work for small perturbations (e.g. Hung et al., 2000; Birch & Kosovichev, 2001). Employing the first Born approximation gives the first-order correction to the signal:

$$\begin{aligned}\delta\phi(\boldsymbol{x}, t) \quad = \quad & -\int \mathrm{d}\boldsymbol{r}\, \mathrm{d}t'\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}t_\mathrm{s}\ \mathcal{G}_j(\boldsymbol{x}, t; \boldsymbol{r}, t') \\ & \{\delta\boldsymbol{\mathcal{L}}\mathbf{G}_k(\boldsymbol{r}, t'; \boldsymbol{s}, t_\mathrm{s})\}_j\ S_k(\boldsymbol{s}, t_\mathrm{s}). \quad (18)\end{aligned}$$

We recall that $\delta\boldsymbol{\mathcal{L}}$ contains the first-order perturbations to the solar model, δq_α. Since we now have ϕ^0 and $\delta\phi$ we can next compute the zero- and first-order cross-correlations, C^0 and δC.

The source of solar oscillations is turbulent convection near the solar surface. As a result, the source function $\boldsymbol{S}$ is a realization of a random process and thus the signal ϕ and the cross-correlation C are stochastic as well. In order to interpret the measured cross-correlation we assume that it represents an expectation value. We do not try to interpret the difference between the cross-correlation and its expected value as it depends on the particular realization of the source function. Under the assumptions of the Ergodic theorem (see e.g. Yaglom, 1962) the cross-correlation tends to its expectation value as the observation time interval increases. In the rest of this paper, cross-correlations stand for their expectation values.

From equation (2) we obtain the zero-order cross-correlation,

$$C^0(\mathbf{1}, \mathbf{2}, t) = \int \mathrm{d}t'\ \mathrm{E}[\phi^0(\mathbf{1}, t')\ \phi^0(\mathbf{2}, t'+t)], \quad (19)$$

where $\mathrm{E}[\cdot]$ denotes the expectation value of the expression in square brackets. Using the expression for ϕ^0 given by equation (16) we find that C^0 depends on

$$M_{jk}(\boldsymbol{s}, t_\mathrm{s}; \boldsymbol{s}', t'_\mathrm{s}) = \mathrm{E}[S_j(\boldsymbol{s}, t_\mathrm{s})\ S_k(\boldsymbol{s}', t'_\mathrm{s})]. \quad (20)$$

The matrix $\mathbf{M}$ gives the correlation between any two components of $\boldsymbol{S}$, measured at two possibly different positions. With the assumptions of stationarity in time and homogeneity and isotropy in the horizontal direction, $\mathbf{M}$ only depends on the time difference $t_\mathrm{s} - t'_\mathrm{s}$, the horizontal distance between $\boldsymbol{s}$ and $\boldsymbol{s}'$, and their depths. Further assumptions could be made in order to simplify the computation of equation (19). We may assume, for example, that the sources are spatially uncorrelated or are located only at a particular depth.

We now perturb equation (2) and take the expectation value to obtain

$$\begin{aligned}\delta C(\mathbf{1}, \mathbf{2}, t) \quad = \quad & \int \mathrm{d}t'\ \mathrm{E}\left[\phi^0(\mathbf{1}, t')\ \delta\phi(\mathbf{2}, t'+t)\right. \\ & \left. +\phi^0(\mathbf{2}, t'+t)\ \delta\phi(\mathbf{1}, t')\right]. \quad (21)\end{aligned}$$

Using the expressions for ϕ^0 and $\delta\phi$ given by equations (16) and (18) we can express δC as the spatial integral of a function $\mathcal{C}$ that depends on the point of scattering, $\boldsymbol{r}$,

$$\delta C(\mathbf{1}, \mathbf{2}, t) = \int \mathrm{d}\boldsymbol{r}\ \mathcal{C}(\mathbf{1}, \mathbf{2}, t; \boldsymbol{r}). \quad (22)$$

We have explained how to obtain C^0 and δC from an assumed solar model consisting of a background model ($\boldsymbol{\mathcal{L}}^0$) and small perturbations ($\delta\boldsymbol{\mathcal{L}}$).

5. TRAVEL-TIME SENSITIVITY KERNELS

In section 3 we showed how to relate $\delta\tau_\pm$ to δC. The expression for δC that we wrote in the previous section enables us to express $\delta\tau_\pm$ as an integral over the scattering point $\boldsymbol{r}$:

$$\delta\tau_\pm(\mathbf{1},\mathbf{2}) = \int \mathrm{d}\boldsymbol{r}\ \mathcal{T}_\pm(\mathbf{1},\mathbf{2};\boldsymbol{r}), \tag{23}$$

where

$$\mathcal{T}_\pm(\mathbf{1},\mathbf{2};\boldsymbol{r}) = \int \mathrm{d}t\ W_\pm(\mathbf{1},\mathbf{2},t)\ \mathcal{C}(\mathbf{1},\mathbf{2},t;\boldsymbol{r}). \tag{24}$$

The functions $\mathcal{T}_\pm$ describe the local perturbation to the travel-time data and are linear in δq_α through $\mathcal{C}$.

For any choice of $\delta\boldsymbol{\mathcal{L}}$, travel-time kernels $K^\alpha_\pm$ can be derived from $\mathcal{T}_\pm$ such that

$$\delta\tau_\pm(\mathbf{1},\mathbf{2}) = \sum_\alpha \int \mathrm{d}\boldsymbol{r}\ K^\alpha_\pm(\mathbf{1},\mathbf{2};\boldsymbol{r})\ \frac{\delta q_\alpha}{q_\alpha}(\boldsymbol{r}). \tag{25}$$

Integration by parts may be necessary. By definition, $K^\alpha_\pm$ represent the local sensitivity of $\delta\tau_\pm$ to $\delta q_\alpha/q_\alpha$. In the previous expression, the sum over the index α is a sum over all relevant perturbations. Note that extra terms should be added to account for possible perturbations $\delta\mathbf{M}$ in the source function.

6. DISCUSSION

We now have a recipe for solving the linear forward problem, i.e. computing travel-time sensitivity kernels. This recipe is based on a physical description of the observed wavefield. The kernels give the dependence of travel-time perturbations on perturbations to a solar model and they take account of the details of the measurement procedure. The sensitivity kernels depend on the background solar model, on the filtering and fitting of the data, and on position on the solar disk.

The most significant obstacle to the computation of accurate travel-time kernels is our lack of a detailed understanding of turbulent convection. The excitation and damping of solar oscillations is due to convection and is thus extremely difficult to account for in the background model: approximations must be introduced. An important constraint on the zero-order solar model is that it must produce a k-ω diagram compatible with observations. A further complication introduced by turbulence is that, in principle, it demands a theory for wave propagation through random media, i.e. a treatment of perturbations that vary on short temporal and spatial scales.

There are a number of less fundamental issues relating to the interpretation of travel-times. We emphasize that the filter $\mathcal{F}$ includes the point spread function of the instrument, which is not always well known. It is unclear how an inaccurate estimate of the point spread function affects the interpretation of travel-time measurements. A straightforward issue is that cross-correlations are typically averaged over annuli or sectors of annuli; this can easily be accounted for by averaging the point-to-point kernels described in this paper.

Despite all of the aforementioned difficulties, the approach we have described here is feasible. Gizon et al. (2000) have shown the whole procedure described here to work on a 2D problem using surface waves. We have computed examples of sensitivity kernels for various perturbations. Papers describing these examples are in preparation.

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