### Part I: Fundamentals on Time Series
- Classification
- Prob. Density Func.
- Auto-Correlation
- Power Spectral Density
- Cross-Correlation
- Applications
- Pre-Processing
- Sampling
- Trend Removal

### Part II: Fourier Series
- Definition
- Method
- Properties
- Convolution
- Correlations
- Leakage / Windowing
- Irregular Grid
- Noise Removal

### Part III: Wavelets
- Why Wavelet Transforms?
- Fundamentals: FT, STFT and Resolution Problems
- Multiresolution Analysis: CWT
- DWT

**Exercises**
A. Lagg – Spectral Analysis

Basic description of physical data

deterministic: described by explicit mathematical relation

\[ x(t) = X \cos\left(\sqrt{\frac{k}{t}} t \right) \]

non deterministic: no way to predict an exact value at a future instant of time
Classifications of deterministic data

- Deterministic
  - Periodic
    - Sinusoidal
    - Complex Periodic
  - Nonperiodic
    - Almost periodic
    - Transient
Sinusoidal data

\[ x(t) = X \sin(2\pi f_0 t + \Theta) \]
\[ T = \frac{1}{f_0} \]
Complex periodic data

\[ x(t) = x(t \pm nT) \quad n = 1, 2, 3, \ldots \]

\[ x(t) = \frac{a_0}{2} + \sum (a_n \cos 2\pi n f_1 t + b_n \sin 2\pi n f_1 t) \]

(T = fundamental period)
Almost periodic data

\[ x(t) = X_1 \sin(2t + \Theta_1) + X_2 \sin(3t + \Theta_2) + X_3 \sin(\sqrt{50}t + \Theta_3) \]

no highest common divisor \( \rightarrow \) infinitely long period \( T \)
Transient non-periodic data

all non-periodic data other than almost periodic data

\[ x(t) = \begin{cases} 
  A e^{-at} & t \geq 0 \\
  0 & t < 0 
\end{cases} \]

\[ x(t) = \begin{cases} 
  A e^{-at} \cos bt & t \geq 0 \\
  0 & t < 0 
\end{cases} \]

\[ x(t) = \begin{cases} 
  A & c \geq t \geq 0 \\
  0 & c < t < 0 
\end{cases} \]
Classification of random data

Random Data

Stationary

- Ergodic
- Nonergodic

Nonstationary

- Special classifications of nonstationarity
stationary / non stationary

collection of sample functions = ensemble

data can be (hypothetically) described by computing ensemble averages (averaging over multiple measurements / sample functions)

mean value (first moment):

$$\mu_x(t_1) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t_1)$$

autocorrelation function (joint moment):

$$R_x(t_1, t_1 + \tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t_1) x_k(t_1 + \tau)$$

stationary: $$\mu_x(t_1) = \mu_x, \quad R_x(t_1, t_1 + \tau) = R_x$$

weakly stationary: $$\mu_x(t_1) = \mu_x, \quad R_x(t_1, t_1 + \tau) = R_x(\tau)$$
Ergodic random process:
properties of a stationary random process
described by computing averages over only **one**
**single sample function** in the ensemble

mean value of k-th sample function:

$$\mu_x(k) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x_k(t) \, dt$$

autocorrelation function (joint moment):

$$R_x(\tau, k) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x_k(t) x_k(t+\tau) \, dt$$

**ergodic:**  $$\mu_x(k) = \mu_x, \quad R_x(\tau, k) = R_x(\tau)$$
Basic descriptive properties of random data

- mean square values
- probability density function
- autocorrelation functions
- power spectral density functions

(from now on: assume random data to be stationary and ergodic)
Mean square values

(mean values and variances)

describes general intensity of random data:

\[ \Psi_x^2 = \lim_{T \to \infty} \frac{1}{T} \int_0^T x^2(t) \, dt \]

rout mean square value:

\[ \Psi_{x}^{rms} = \sqrt{\Psi_x^2} \]

often convenient:

- **static component described by mean value:**
  \[ \mu_x = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) \, dt \]

- **dynamic component described by variance:**
  \[ \sigma_x^2 = \lim_{T \to \infty} \frac{1}{T} \int_0^T [x(t) - \mu_x]^2 \, dt \]

standard deviation:

\[ \sigma_x = \sqrt{\sigma_x^2} = \Psi_x^2 - \mu_x^2 \]
Probability density functions

describes the probability that the data will assume a value within some defined range at any instant of time

\[
\text{Prob}[x < x(t) \leq x + \Delta x] = \lim_{T \to \infty} \frac{T_x}{T}, \quad T_x = \sum_{i=1}^{k} \Delta t_i
\]

gives

\[
p(x) = \lim_{\Delta x \to 0} \frac{\text{Prob}[x < x(t) \leq x + \Delta x]}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[ \lim_{T \to \infty} \frac{T_x}{T} \right]
\]

gives

\[
P(x) = \text{Prob}[x(t) \leq x]
\]

\[
= \int_{-\infty}^{x} p(\xi) d\xi
\]

\[
x(t)
\]

\[
\Delta t_1 \quad \Delta t_2 \quad \Delta t_3 \quad \Delta t_4
\]

\[
x + \Delta x
\]

\[
x
\]

\[
0
\]

\[
t
\]
Illustration: probability density function

Sample time histories:
- Sine wave (a)
- Sine wave + random noise
- Narrow-band random noise
- Wide-band random noise

All 4 cases: mean value $\mu_x = 0$
Illustration: probability density function

\[ x(t) \]

\[ x(t) \]

\[ x(t) \]

\[ x(t) \]

(a)

(b)

(c)

(d)

\[ p(x) \]

\[ p(x) \]

\[ p(x) \]

\[ p(x) \]

(a)

(b)

(c)

(d)

probability density function
Autocorrelation functions

describes the general dependence of the data values at one time on the values at another time.

\[ R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} x(t)x(t+\tau) \, dt \]

\[ \mu_x = \sqrt{R_x(\infty)} \quad \Psi_x^2 = R_x(0) \quad \text{(not for special cases like sine waves)} \]
Illustration: ACF

autocorrelation functions (autocorrelogram)
Illustrations

autocorrelation function of a rectangular pulse

\[ x(t) x(t - \tau) \]

autocorrelation function

\[ R_f(\tau) = \begin{cases} A^2 \theta (1 - \frac{|\tau|}{\theta}) & |\tau| < \theta \\ 0 & |\tau| \geq \theta \end{cases} \]
Power spectral density functions  
(also called autospectral density functions)

describe the general frequency composition of the data in terms of the spectral density of its mean square value

mean square value in frequency range \((f, f + \Delta f)\) :

\[
\Psi_x^2(f, \Delta f) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t, f, \Delta f)^2 dt
\]

portion of \(x(t)\) in \((f,f+\Delta f)\)

definition of power spectral density function:

\[
\Psi_x^2(f, \Delta f) \approx G_x(f) \Delta f
\]

important property: spectral density function is related to the autocorrelation function by a Fourier transform:

\[
G_x(f) = \lim_{\Delta f \to 0} \frac{\Psi_x^2(f, \Delta f)}{\Delta f} = \lim_{\Delta f \to 0} \frac{1}{\Delta f} \left[ \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t, f, \Delta f)^2 dt \right]
\]

\[
G_x(f) = 2 \int_{-\infty}^{\infty} R_x(\tau) e^{-i2\pi f \tau} d\tau = 4 \int_0^{\infty} R_x(\tau) \cos 2\pi f \tau \ d\tau
\]
Illustration: PSD

Dirac delta function at $f=f_0$

sine wave

+ random noise

narrowband noise

"white" noise:
spectrum is uniform over all frequencies

broadband noise

power spectral density functions

(a)

(b)

(c)

(d)
Joint properties of random data

until now: described properties of an individual random process

**Joint probability density functions**
- joint properties in the amplitude domain

**Cross-correlation functions**
- joint properties in the time domain

**Cross-spectral density functions**
- joint properties in the frequency domain

\[ T_{xy} = \sum_{i=1}^{k} \Delta t_i \]

joint probability measurement
Cross-correlation function

describes the general dependence of one data set to another

\[ R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} x(t) y(t + \tau) \, dt \]

similar to autocorrelation function

\[ R_{xy}(\tau) = 0 \quad \text{functions are uncorrelated} \]

typical cross-correlation plot (cross-correlogram): sharp peaks indicate the existence of a correlation between \( x(t) \) and \( y(t) \) for specific time displacements
Applications

Measurement of time delays

2 signals:
- different offset
- different S/N
- time delay 4s

often used:
'discrete' cross correlation coefficient
lag = l, for l >= 0:

\[ R_{xy}(l) = \frac{\sum_{k=1}^{N-l} (x_k - \bar{x})(y_{k+l} - \bar{y})}{\sqrt{\sum_{k=1}^{N} (x_k - \bar{x})^2 \sum_{k=1}^{N} (y_k - \bar{y})^2}} \]
**Applications**

Detection and recovery from signals in noise

- 3 signals:
  - noise free replica of the signal (e.g. model)
  - 2 noisy signals

Cross correlation can be used to determine if theoretical signal is present in data
Pre-processing Operations

- sampling considerations
- trend removal
- filtering methods

**Sampling**

cutoff frequency (=Nyquist frequency or folding frequency)

\[ f_c = \frac{1}{2h} \]
**Trend removal**

often desirable before performing a spectral analysis

**Least-square method:**

time series: \( u(t) \)

desired fit (e.g. polynomial):

\[
\hat{u} = \sum_{k=0}^{K} b_k (nh)^k \quad n=1,2,\ldots, N
\]

Lsq-Fit: minimize

\[
Q(b) = \sum_{n=1}^{N} (u_n - \hat{u}_n)^2
\]

→ set partial derivatives to 0:

\[
\frac{\partial Q}{\partial b_l} = \sum_{n=1}^{N} 2(u_n - \hat{u}_n)(nh)^l
\]

→ K+1 equations:

\[
\sum_{k=0}^{K} b_k \sum_{n=1}^{N} (nh)^{k+l} = \sum_{n=1}^{N} u_n (nh)^l
\]
Digital filtering

- Original data
- Highpass data
- Lowpass data
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Exercises
Fourier Series and Fast Fourier Transforms

Standard Fourier series procedure:

If a transformed sample record \( x(t) \) is periodic with a period \( T_p \) (fundamental frequency \( f_1 = 1/T_p \)), then \( x(t) \) can be represented by the Fourier series:

\[
x(t) = \frac{a_0}{2} + \sum_{q=1}^\infty \left( a_q \cos 2\pi q f_1 t + b_q \sin 2\pi q f_1 t \right)
\]

where

\[
a_q = \frac{2}{T} \int_0^T x(t) \cos 2\pi q f_1 t \, dt \quad q = 0, 1, 2, \ldots
\]

\[
b_q = \frac{2}{T} \int_0^T x(t) \sin 2\pi q f_1 t \, dt \quad q = 1, 2, 3, \ldots
\]
Fourier series procedure - method

sample record of finite length, equally spaced sampled:

\[ x_n = x(nh) \quad n = 1, 2, \ldots, \]

Fourier series passing through these N data values:

\[
x(t) = A_0 + \sum_{q=1}^{N/2} A_q \cos\left(\frac{2\pi q t}{T_p}\right) + \sum_{q=1}^{N/2-1} B_q \sin\left(\frac{2\pi q t}{T_p}\right)
\]

Fill in particular points: \( t = nh, \quad n = 1, 2, \ldots, N, \quad T_p = Nh, \quad x_n = x(nh) = \ldots \)

- coefficients \( A_q \) and \( B_q \):

\[
A_0 = \frac{1}{N} \sum_{n=1}^{N} x_n = \bar{x} \quad A_{N/2} = \frac{1}{N} \sum_{n=1}^{N} x_n \cos n\pi
\]

\[
A_q = \frac{2}{N} \sum_{n=1}^{N} x_n \cos\left(\frac{2\pi q n}{N}\right) \quad q = 1, 2, \ldots, \frac{N}{2} - 1
\]

\[
B_q = \frac{2}{N} \sum_{n=1}^{N} x_n \sin\left(\frac{2\pi q n}{N}\right) \quad q = 1, 2, \ldots, \frac{N}{2} - 1
\]

inefficient & slow \( \Rightarrow \) Fast Fourier Transforms developed
Fourier Transforms - Properties

**Linearity**

\[
\{ x_n \} \quad \overset{DFT}{\leftrightarrow} \quad \{ X_k \}
\]
\[
\{ y_n \} \quad \overset{DFT}{\leftrightarrow} \quad \{ Y_k \}
\]
\[
a \{ x_n \} + b \{ y_n \} \quad \overset{DFT}{\leftrightarrow} \quad a \{ X_k \} + b \{ Y_k \}
\]

**Symmetry**

\[
\{ X_k \} = \{ X^*_k \}
\]
\[
\Re \{ X_k \} \text{ is even} \quad \Rightarrow \quad \Im \{ X_k \} \text{ is odd}
\]

**Circular time shift**

\[
\{ x_{n-n_0} \} \quad \overset{DFT}{\leftrightarrow} \quad \{ e^{-ikn_0} X_k \}
\]
\[
\{ e^{ik_0n} y_n \} \quad \overset{DFT}{\leftrightarrow} \quad \{ Y_{k-k_0} \}
\]
Using FFT for Convolution

\[ r * s \equiv \int_{-\infty}^{\infty} r(\tau) s(t-\tau) d\tau \]

Convolution Theorem:

\[ r * s \overset{FT}{\iff} R(f) S(f) \]

Fourier transform of the convolution is product of the individual Fourier transforms

*discrete case:*

\[ (r * s)_j \equiv \sum_{k=-N/2+1}^{N/2} s_{j-k} r_k \]

Convolution Theorem:

\[ \sum_{k=-N/2+1}^{N/2} s_{j-k} r_k \overset{FT}{\iff} R_n S_n \]

constraints:
- duration of \( r \) and \( s \) are not the same
- signal is not periodic

(original data)

(response function)

(convolved data)

(note how the response function for negative times is wrapped around and stored at the extreme right end of the array)
Treatment of end effects by zero padding

constraint 1: simply expand response function to length N by padding it with zeros

constraint 2: extend data at one end with a number of zeros equal to the max. positive / negative duration of r (whichever is larger)
FFT for Convolution

1. zero-pad data
2. zero-pad response function
   (-> data and response function have N elements)
3. calculate FFT of data and response function
4. multiply FFT of data with FFT of response function
5. calculate inverse FFT for this product

Deconvolution

-> undo smearing caused by a response function

use steps (1-3), and then:
4. divide FFT of convolved data with FFT of response function
5. calculate inverse FFT for this product
Correlation / Autocorrelation with FFT

definition of correlation / autocorrelation see first lecture

\[ Corr(g, h) = g^*h = \int_{-\infty}^{\infty} g(t+\tau)h(\tau) d\tau \]

**Correlation Theorem:**

\[ Corr(g, h) \Leftrightarrow FT G(f) H^*(f) \]

Auto-Correlation:

\[ Corr(g, g) \Leftrightarrow |G(f)|^2 \]

discrete correlation theorem:

\[ Corr(g, h)_j \equiv \sum_{k=0}^{N-1} g_{j+k} h_k \]

\[ \Leftrightarrow FT G_k H_k^* \]
Fourier Transform - problems

spectral leakage

\[ T = n T_p \]

\[ T \neq n T_p \]
Reducing leakage by windowing (1)

Applying windowing (apodizing) function to data record:

\[ \tilde{x}(t) = x(t)w(t) \quad \text{(original data record } \times \text{ windowing function)} \]

\[ \tilde{x}_n = x_n w_n \]

**No Window:**

**Bartlett Window:**

\[ w(t) = 1 - \frac{|t-T/2|}{T/2} \]

**Hamming Window:**

\[ w(t) = 0.54 + 0.46 \cos(\pi t) \]

**Blackman Window:**

\[ w(t) = 0.42 + 0.5 \cos(\pi t) + 0.08 \cos(2\pi t) \]

**Hann Window:**

\[ w(t) = \cos^2\left(\frac{\pi t}{2}\right) \]

**Gaussian Window:**

\[ w(t) = \exp\left(-0.5(a t)^2\right) \]

\[ a = \text{constant} \]
reducing leakage by windowing (2)

without windowing

with Gaussian windowing
No constant sampling frequency

Fourier transformation requires constant sampling (data points at equal distances)
-> not the case for most physical data

Solution: Interpolation

- linear:
  linear interpolation between $y_k$ and $y_{k+1}$
  IDL> idata=interpol(data,t,t_reg)

- quadratic:
  quadratic interpolation using $y_{k-1}$, $y_k$, and $y_{k+1}$
  IDL> idata=interpol(data,t,t_reg,/quadratic)

- least-square quadratic
  least-square quadratic fit using $y_{k-1}$, $y_k$, $y_{k+1}$ and $y_{k+2}$
  IDL> idata=interpol(data,t,t_reg,/lsq)

- spline
  IDL> idata=interpol(data,t,t_reg,/spline)
  IDL> idata=spline(t,data,t_reg[,tension])

important:
interpolation changes sampling rate!
-> careful choice of new (regular) time grid necessary!
Fourier Transform on irregular gridded data - Interpolation

- original data: sine wave + noise
- FT of original data
- irregular sampling of data (measurement)
- interpolation: linear, lsq, spline, quadratic
- 're-sampling'
- FT of interpolated data
Noise removal

**Frequency threshold (lowpass)**
- make FT of data
- set high frequencies to 0
- transform back to time domain
Noise removal

signal threshold for weak frequencies (dB-threshold)

- make FT of data
- set frequencies with amplitudes below a given threshold to 0
- transform back to time domain
Optimal Filtering with FFT

normal situation with measured data:
- underlying, uncorrupted signal $u(t)$
- + response function of measurement $r(t)$
- = smeared signal $s(t)$
- + noise $n(t)$
- = smeared, noisy signal $c(t)$

**estimate true signal $u(t)$ with:**

$$
\hat{U}(f) = \frac{C(f)\Phi(f)}{R(f)}
$$

$\Phi(f), \varphi(t) =$ optimal filter
(Wiener filter)
Calculation of optimal filter

reconstructed signal and uncorrupted signal should be close in least-square sense:

\[ \text{minimize} \quad \int_{-\infty}^{\infty} |\tilde{u}(t) - u(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{U}(f) - U(f)|^2 df \]

\[ \Rightarrow \quad \frac{\partial}{\partial \Phi(f)} \left[ \frac{[S(f) + N(f)] \Phi(f)}{R(f)} - \frac{S(f)}{R(f)} \right]^2 = 0 \]

\[ \Rightarrow \quad \Phi(f) = \frac{|S(f)|^2}{|S(f)|^2 + |N(f)|^2} \]

additional information:

power spectral density can often be used to disentangle noise function \( N(f) \) from smeared signal \( S(f) \)
Using FFT for Power Spectrum Estimation

discrete Fourier transform of \( c(t) \)

\[ C_k = \sum_{j=0}^{N-1} c_j e^{2\pi i j k / N} \quad k = 0, \ldots, N-1 \]

-> periodogram estimate of power spectrum:

\[
\begin{align*}
P(0) &= P(f_0) = \frac{1}{N^2} |C_0|^2 \\
P(f_k) &= \frac{1}{N^2} \left[ |C_k|^2 + |C_{N-k}|^2 \right] \\
P(f_c) &= P(f_{N/2}) = \frac{1}{N^2} |C_{N/2}|^2
\end{align*}
\]
Spectral Analysis and Time Series

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Exercises
Introduction to Wavelets

- why wavelet transforms?
- fundamentals: FT, short term FT and resolution problems
- multiresolution analysis: continuous wavelet transform
- multiresolution analysis: discrete wavelet transform
Fourier: lost time information

6 Hz, 4 Hz, 2 Hz, 1 Hz

6 Hz + 4 Hz + 2 Hz + 1 Hz
Solution: Short Time Fourier Transform (STFT)

perform FT on 'windowed' function:

- example: rectangular window
- move window in small steps over data
- perform FT for every time step

\[ STFT(f, t') = \int_x [x(t) \omega(t - t')] e^{-i2\pi ft} dt \]
Short Time Fourier Transform

STFT-spectrogram shows both time and frequency information!
Short Time Fourier Transform: Problem

narrow window function -> good time resolution
wide window function -> good frequency resolution

Gauss-functions as windows
Solution: Wavelet Transformation

time vs. frequency resolution is intrinsic problem (Heisenberg Uncertainty Principle)
approach: analyze the signal at different frequencies with different resolutions

-> multiresolution analysis (MRA)

Continuous Wavelet Transform

similar to STFT:
- signal is multiplied with a function (the wavelet)
- transform is calculated separately for different segments of the time domain

but:
- the FT of the windowed signals are not taken (no negative frequencies)
- The width of the window is changed as the transform is computed for every single spectral component
Continuous Wavelet Transform

\[ CWT_x^\psi = \psi_x^\psi(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \psi^* \left( \frac{t - \tau}{s} \right) dt \]

\( \tau \) ... translation parameter, \( s \) ... scale parameter

\( \psi(t) \) ... mother wavelet (= small wave)

mother wavelet:
- finite length (compactly supported) -> 'let'
- oscillatory -> 'wave'
- functions for different regions are derived from this function -> 'mother'

scale parameter \( s \) replaces frequency in STFT
The Scale

similar to scales used in maps:
- high scale = non detailed global view (of the signal)
- low scale = detailed view

in practical applications:
- low scales (= high frequencies) appear usually as short bursts or spikes
- high scales (= low frequencies) last for entire signal

scaling dilates (stretches out) or compresses a signal:
- $s > 1$ -> dilation
- $s < 1$ -> compression
Computation of the CWT

signal to be analyzed: $x(t)$, mother wavelet: Morlet or Mexican Hat

- start with scale $s=1$ (lowest scale, highest frequency)
  - $\rightarrow$ most compressed wavelet
- shift wavelet in time from $t_0$ to $t_1$
- increase $s$ by small value
- shift dilated wavelet from $t_0$ to $t_1$
- repeat steps for all scales
CWT - Example

signal $x(t)$

axes of CWT: translation and scale (not time and frequency)
translation -> time
scale -> $1$/frequency
every box corresponds to a value of the wavelet transform in the time frequency plane

- all boxes have constant area
  $\Delta f \Delta t = \text{const.}$
- low frequencies: high resolution in $f$, low time resolution
- high frequencies: good time resolution

STFT: time and frequency resolution is constant (all boxes are the same)
Wavelets: Mathematical Approach

WL-transform:

\[ CWT_x^\psi = \Psi_x^\psi(\tau, s) = \frac{1}{\sqrt{|s|}} \int_t x(t) \psi^* \left( \frac{t-\tau}{s} \right) dt \]

Mexican Hat wavelet:

\[ \psi(t) = \frac{1}{\sqrt{s \pi \sigma^3}} e^{\frac{-t^2}{2\sigma^2}} \left( \frac{t^2}{\sigma^2} - 1 \right) \]

Morlet wavelet:

\[ \psi(t) = e^{iat} e^{-\frac{t^2}{2\sigma}} \]

inverse WL-transform:

\[ x(t) = \frac{1}{C_{\psi}^2} \int_\tau \int_s \Psi_x^\psi(\tau, s) \frac{1}{S^2} \psi \left( \frac{t-\tau}{s} \right) d\tau ds \]

admissibility condition:

\[ C_{\psi} = \left\{ 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi \right\}^{1/2} < \infty \] with \( \hat{\psi}(\xi) \Leftrightarrow \psi(t) \)
Discretization of CWT: Wavelet Series

-> sampling the time – frequency (or scale) plane

advantage:

- sampling high for high frequencies (low scales)
  scale \( s_1 \) and rate \( N_1 \)

- sampling rate can be decreased for low
  frequencies (high scales)
  scale \( s_2 \) and rate \( N_2 \)

\[
\begin{align*}
N_2 &= \frac{s_1}{s_2} N_1 \\
N_2 &= \frac{f_2}{f_1} N_1
\end{align*}
\]

continuous wavelet \[ \psi_{\tau,s} = \frac{1}{\sqrt{s}} \psi \left( \frac{t-\tau}{s} \right) \]
discrete wavelet \[ \psi_{j,k}(t) = s_0^{-j/2} \psi(s_0^{-j} t - k \tau_0), \quad \psi_{j,k} \text{ orthonormal} \]

\[ \Psi_x^{\psi,j,k} = \int x(t) \psi^*(j,t) dt \]

\[ x(t) = c_\varphi \sum_j \sum_k \Psi_x^{\psi,j,k} \psi_{j,k}(t) \]

WL-transformation
reconstruction of signal
Discrete Wavelet Transform

Discretized continuous wavelet transform is only a sampled version of the CWT.

The discrete wavelet transform (DWT) has significant advantages for implementation in computers.

excellent tutorial:
http://users.rowan.edu/~polikar/WAVELETS/WTtutorial.html

IDL-Wavelet Tools:
IDL> wv_applet

Wavelet expert at MPS:
Rajat Thomas
end of Wavelets
Exercises

Part I: Fourier Analysis
(Andreas Lagg)

Instructions:
http://www.linmpi.mpg.de/~lagg

Part II: Wavelets
(Rajat Thomas)

Seminar room
Time: 15:00
Exercise: Galileo magnetic field

data set from Galileo magnetometer
  (synthesized)
file: gll_data.sav, contains:
- total magnetic field
- radial distance
- time in seconds

Tips:
- restore,'gll_data.sav'
- use IDL-FFT
- remember basic plasma physics formula for the ion cyclotron wave:

\[ \omega_{\text{gyro}} = \frac{qB}{m}, \quad f_{\text{gyro}} = \frac{\omega_{\text{gyro}}}{2\pi} \]

your tasks:
- Which ions are present?
- Is the time resolution of the magnetometer sufficient to detect electrons or protons?

Background:
If the density of ions is high enough they will excite ion cyclotron waves during gyration around the magnetic field lines. This gyration frequency only depends on mass per charge and on the magnitude of the magnetic field.

In a low-beta plasma the magnetic field dominates over plasma effects. The magnetic field shows only very little influence from the plasma and can be considered as a magnetic dipole.

http://www.sciencemag.org/cgi/content/full/274/5286/396
Random Data: Analysis and Measurement Procedures
Bendat and Piersol, Wiley Interscience, 1971

The Analysis of Time Series: An Introduction
Chris Chatfield, Chapman and Hall / CRC, 2004

Time Series Analysis and Its Applications
Shumway and Stoffer, Springer, 2000

Numerical Recipies in C
Cambridge University Press, 1988-1992
http://www.nr.com/

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Robi Polikar, 2001
http://users.rowan.edu/~polikar/WAVELETS/WTtutorial.html