ACOUSTIC SCATTERING BY FLUX TUBES: IS THE BORN APPROXIMATION VALID?

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ABSTRACT

With the aim of studying magnetic effects in time-distance helioseismology, we use the first-order Born approximation to compute the scattering of acoustic plane waves by a magnetic cylinder embedded in a uniform medium. We show, by comparison with the exact solution, that the travel-time shifts computed in the Born approximation are everywhere valid to first order in the ratio of the magnetic to the gas pressures. For arbitrary magnetic field strength, the Born approximation is not valid in the limit where the radius of the magnetic cylinder tends to zero. This contribution is a summary of a paper published in the Astrophysical Journal.

Key words: Flux tubes; Acoustics; Born approximation.

1. INTRODUCTION

Time-distance helioseismology [1] has been used to measure wave travel times in and around magnetic active regions and sunspots to estimate subsurface flows and wave-speed perturbations. A challenging problem is to estimate the subsurface magnetic field from travel times. In order to do so, one must understand the dependence of the travel times on the magnetic field.

The interaction of acoustic waves with sunspot magnetic fields is quite strong in the near surface layers. As a result, the effect of the magnetic field on the travel times is not expected to be small near the surface. Deeper inside the Sun, however, the ratio of the magnetic pressure to the gas pressure becomes small, and it is tempting to treat the effects of the magnetic field on the waves using perturbation theory. Of particular interest is the search for a magnetic field at the bottom of the convection zone. Such a linear inversion scheme has been proposed by Kosovichev & Duvall [2] for time-distance helioseismology using the ray approximation, but it needs to be extended to finite wavelengths. In relation to the linearity of magnetic field effects, Gizon, Hanasoge & Birch [3] have demonstrated the validity of the Born approximation using a simplified model of a flux tube; This paper attempts to summarize this work.

2. THE PROBLEM

We start with the ideal equations of magnetohydrodynamics. We solve the equations of continuity, momentum, magnetic induction, Gauss' law for the magnetic field and utilize the ideal gas law. We denote density by ρ, velocity by $\boldsymbol{v}$, pressure by p, temperature by T and the magnetic field by $\boldsymbol{B}$. We consider a magnetic cylinder with radius R and uniform magnetic field strength B_t embedded in an infinite, otherwise uniform, gravity free medium with constant density ρ_0, gas pressure p_0, and temperature T_0. We use a cylindrical coordinate system (r, θ, z) where r is the radial coordinate, θ is the azimuthal angle, and z is the vertical coordinate in the direction of the cylinder axis. We denote the corresponding unit vectors by $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{z}}$. All steady physical quantities are denoted with an overbar. In particular, we have

$$\overline{\boldsymbol{B}} = B_t \Theta(R - r)\hat{\boldsymbol{z}}, \quad (1)$$
$$\overline{\rho} = \rho_t \Theta(R - r) + \rho_0 \Theta(r - R), \quad (2)$$
$$\overline{p} = p_t \Theta(R - r) + p_0 \Theta(r - R), \quad (3)$$

where the Heaviside step function is defined by $\Theta(r) = 0$ if $r < 0$ and $\Theta(r) = 1$ if $r > 0$. The density and pressure inside the tube are ρ_t and p_t respectively. We assume that there is no mean flow in this problem, i.e. $\overline{\boldsymbol{v}} = 0$. We choose to study the case where the background temperature is the same inside and outside the magnetized region, resulting in constant sound speed everyewhere.

3. LINEAR WAVES

In this calculation, we only study linear waves on a steady background. The magnetic field $\overline{\boldsymbol{B}}$ and all other background quantities do not depend on z. Thus, a wave with a z dependence of the form $e^{ik_z z}$, where k_z is the wavenumber in the z direction, will have the same z dependence after interacting with the magnetic cylinder. Consequently, we study solutions where the pressure fluctuations are of the form

$$p'(\boldsymbol{r}, z, t) = \tilde{p}(\boldsymbol{r}) \exp(ik_z z - i\omega t), \quad (4)$$

where t is time, ω is the circular frequency of oscillation, $\boldsymbol{r} = (r, \theta)$ is a position vector perpendicular to the tube axis, p' is the pressure fluctuation about the background pressure, and $\tilde{p}(\boldsymbol{r})$ contains the horizontal dependence of the pressure fluctuation. All the other wave variables, ρ', $\boldsymbol{v}'$, and $\boldsymbol{B}'$ (fluctuating density, velocity and magnetic field, respectively) are written in the same form as equation (4). A plane wave can be expanded in cylindrical coordinates as a sum over azimuthal components (index m) according to:

$$\tilde{p}_{\rm inc}(\boldsymbol{r}) = P \sum_{m=-\infty}^{\infty} i^m J_m(kr) e^{im\phi}, \qquad (5)$$

where 'inc' denotes that the wave is incident, P is the amplitude of the wave, J_m denotes the Bessel function of order m, $\boldsymbol{k}$ is the horizontal wave vector, i.e. perpendicular to the axis of the flux tube, $k = |\boldsymbol{k}|$, and ϕ is the angle between $\boldsymbol{k}$ and $\boldsymbol{r}$. Applying the boundary condition that the total wave pressure, hydrodynamic plus magnetic, and the radial velocity must be continuous across the tube boundary, the *exact* linear (small amplitude waves) scattered wavefield due to the incident wave, given by equation (5), may be computed [4]:

$$\tilde{p}(\boldsymbol{r}) = \begin{cases} P \sum_m i^m B_m J_m(k_t r) e^{im\phi} \; (r < R) \\ \tilde{p}_{\rm inc} + P \sum_m i^m A_m H_m(kr) e^{im\phi} \; (r > R) \end{cases} \qquad (6)$$

where $H_m = H_m^{(1)}$ is the Hankel function of the first kind of order m, and A_m and B_m are listed in [4]. The quantity k_t is the horizontal wavenumber inside the tube.

4. THE BORN APPROXIMATION

Details regarding the application of the Born approximation to the problem at hand may be found in Gizon, Hanasoge & Birch [3] and references therein. Suffice it to say that after mathematical manipulation, we obtain an expression for the scattered wave-field akin to equation (6):

$$\begin{aligned} \tilde{p}_{\rm Born}(\boldsymbol{r}) &= \tilde{p}_{\rm inc} + P \sum_{m=-\infty}^{\infty} i^m e^{im\phi} \\ &\times \begin{cases} C_m J_m(kr) - \epsilon \frac{kr}{2} J_m'(kr) & r < R \\ A_m^{\rm Born} H_m(kr) & r > R, \end{cases} \end{aligned} \qquad (7)$$

where $\tilde{p}_{\rm Born}$ denotes the horizontal dependence of the total pressure fluctuation field (incident + scattered) obtained in the Born approximation, $J_m'(x)$ denotes the derivative of the function $J_m(x)$ with respect to x and the coefficients $A_m^{\rm Born}$ and C_m are described in Appendix C of Gizon, Hanasoge & Birch [3]. We show analytically that the Born approximation approaches the exact solution in the limit of small ϵ, where ϵ is a non-dimensional parameter defined as:

$$\epsilon = \frac{B_t^2}{4\pi\rho_0 c^2}, \qquad (8)$$

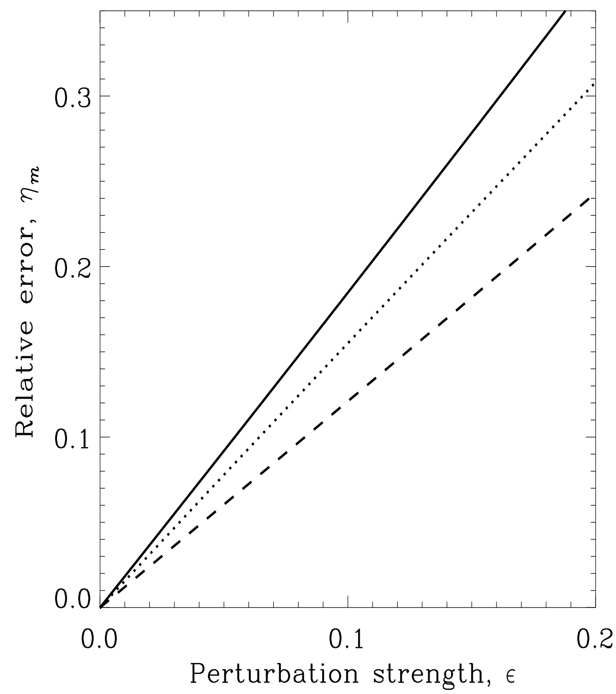

Figure 1. The fractional error $\eta_m = |A_m^{\rm Born} - A_m|/|A_m|$ as a function of ϵ for $m = 0$ (solid line), $m = 1$ (dotted line), and $m = 2$ (dashed line) for the case $R = 2$ Mm, $\omega/2\pi = 3$ mHz, and $k_z = 0$.

as may be evidenced in Figure 1. We also show the effect of the tube radius on $A_m^{\rm Born}/A_m$ for various m in Figure 2.

5. TRAVEL-TIMES

We define the travel-time shifts caused by the magnetic cylinder as the time $\delta t(\boldsymbol{r})$ which minimizes the function

$$X(t) = \int dt' \left[p'(\boldsymbol{r}, t') - p'_{\rm inc}(\boldsymbol{r}, t' - t)\right]^2, \qquad (9)$$

where p' is the full wavefield that includes both the incident wavepacket and the scattered wave packet caused by the magnetic field. The travel-time shifts can be computed in this way for either the exact solution or the Born-approximation and are shown in Figure 3(a). In addition, we also display comparisons with traveltimes obtained through the ray approximation in Figure 3(b).

6. CONCLUSIONS

Near the photosphere, ϵ is not small. It has been suggested by many authors that in this case the Born approximation will fail. An exception is the claim by Rosenthal

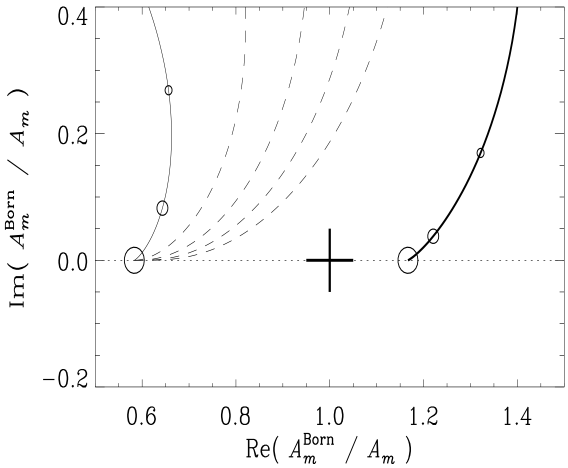

Figure 2. Ratio $A_m^{\rm Born}/A_m$ in the complex plane at fixed $\epsilon = 1$ and $k_z = 0$. The ratio is plotted for varying values of the tube radius in the cases $m = 0$ (thick line), $m = 1$ (thin line), and $2 \leq m \leq 5$ (dashed lines). The big circles show the limit $kR \to 0$. If the Born approximation were correct for small tube radii, the big circles would coincide with the cross. The small and medium-size circles are for $kR = 1$ and $kR = 1/2$ respectively.

[5] that the Born approximation will remain valid for kG magnetic fibrils in the limit where the radius of the magnetic element is much smaller than the wavelength. Contrary to this claim, we have shown in Figure 2 that even in this simplified model, the error is quite significant.

The sensitivity of travel times to local perturbations in internal solar properties can be described through linear sensitivity functions, also called travel-time kernels. The present work suggests that travel-time kernels for the subsurface magnetic field will be useful for probing depths greater than a few hundred km beneath the photosphere, at least in the case when the travel times are measured between surface points that are not in magnetic regions. One should be careful, however, not to draw definitive conclusions from the simple model we have studied, given the complexity of the real solar problem.

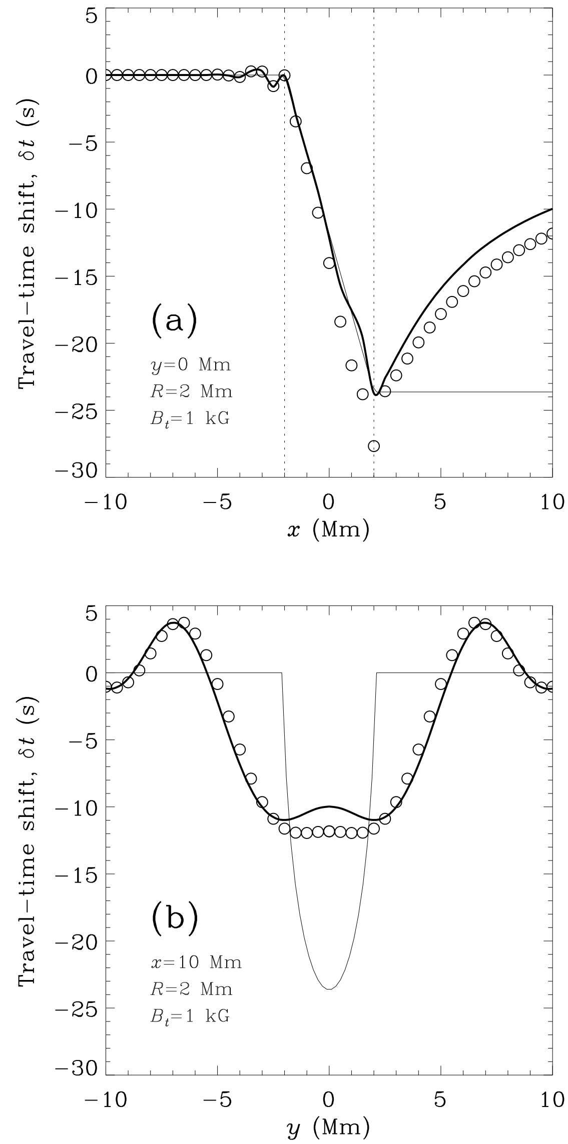

Figure 3. Local travel-time shifts $\delta t(\boldsymbol{r})$ caused by the magnetic cylinder ($\epsilon = 0.13$). The travel times are measured at positions $\boldsymbol{r}$ in a plane perpendicular to both the cylinder axis. The incoming wavepacket moves in the $+\hat{\mathbf{x}}$ direction. The radius of the tube is $R = 2$ Mm and the tube axis is $(x, y) = (0, 0)$. In both panels the heavy solid line is the exact travel-time shifts, the circles are the Born travel-time shifts, and the light line gives the ray approximation. Panel (a) shows the travel-time shifts as a function of x at fixed $y = 0$. Panel (b) shows the travel-time shifts as a function of y at fixed $x = 10$ Mm. The Born approximation is reasonable for this value of ϵ. The ray-approximation does not capture finite-wavelength effects and fails to describe wavefront healing.

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