MEAN-FIELD VIEW ON MAGNETOCONVECTION AND DYNAMO MODELS

M. Schrinner[1], *K.–H. Rädler*[2], *D. Schmitt*[1], *M. Rheinhardt*[3], *U. Christensen*[1]

[1] *Max Planck Institute for Solar System Research, 2 Max–Planck–Strasse, 37191 Katlenburg–Lindau, Germany*
[2] *Astrophysical Institute Potsdam, An der Sternwarte 16, 14482 Potsdam, Germany*
[3] *Institute for Geophysics, University Göttingen, Herzberger Landstrasse 180 37075 Göttingen, Germany*

1. Introduction. The mean–field concept has proved to be a useful tool for the investigation of magnetohydrodynamic processes with complex fluid motions. Within this concept mean fields are defined by averaging of the original fields, here indicated overbars. The magnetic field $\mathbf{B}$ and the fluid velocity $\mathbf{U}$ are according to $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{b}$ and $\mathbf{U} = \overline{\mathbf{U}} + \mathbf{u}$ split into mean fields, $\overline{\mathbf{B}}$ and $\overline{\mathbf{U}}$, and deviations from them, $\mathbf{b}$ and $\mathbf{u}$. The mean electromagnetic fields are governed by equations, which differ formally from Maxwell's and the completing constitutive equations for the original fields, or from the corresponding induction equation, only in one point. In the mean-field versions of the Ohm's law, and so also in the induction equation, an additional electromotive force $\mathcal{E} = \overline{\mathbf{u} \times \mathbf{b}}$ occurs. Under some simplifying assumptions it allows the representation

$$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\nabla_k\overline{B}_j \, . \tag{1}$$

Here the Cartesian coordinates and the summation convention are used. The coefficients a_{ij} and b_{ijk} are determined by $\mathbf{u}$ and $\overline{\mathbf{U}}$ and can depend on $\overline{\mathbf{B}}$ only via these quantities.

Although so far not generally justified, simple relation (1) for $\mathcal{E}$ has been used in almost all mean-field models of magnetohydrodynamic phenomena, in particular, in mean–field dynamo models. In a few cases now direct numerical simulations are available. So the possibility opens up to calculate the tensors a_{ij} and b_{ijk}, or equivalent quantities, with the fields $\mathbf{u}$ and $\mathbf{b}$ taken from these simulations. In this paper we deal with an example of magnetoconvection as investigated by Olsen et al. [1] and a geodynamo model by Christensen et al. [2]. In both cases we compare, with a view to the applicability of relation (1), the mean magnetic field resulting from the mean-field models using the so determined a_{ij} and b_{ijk} with that derived immediately from the numerical simulations.

2. The examples considered. In both cases, magnetoconvection and geodynamo, a rotating spherical shell of an electrically conducting fluid is considered. The behavior of the fluid velocity $\mathbf{U}$ and the magnetic field $\mathbf{B}$ are studied in the Boussinesq approximation. No–slip conditions are posed at the boundaries, which are in this respect considered as rigid bodies. All surroundings of the spherical shell are considered as electrically non-conducting so that the magnetic field continues as a potential field in both parts of the outer space. In the magnetoconvection case, an imposed toroidal magnetic field is assumed resulting from electic currents due to sources or sinks on the boundaries. The temperature of the fluid is assumed to be constant on each of the boundaries.

The non–dimensional basic equations contain four non–dimensional parameters, that is, the Ekman number $\mathrm{E} = \nu/\Omega D^2$, a modified Rayleigh number

$\mathrm{Ra} = \alpha_T g \Delta T D/\nu\Omega$, the Prandtl number $\mathrm{Pr} = \nu/\kappa$ and the magnetic Prandtl number $\mathrm{Pm} = \nu/\eta$. The typical magnitude B_0 of the imposed toroidal magnetic field is expressed by the Elsasser number $\Lambda = B_0^2/\varrho\mu\eta\Omega$. As usual, ν, η, κ, ϱ and α_T are kinematic viscosity, magnetic diffusivity, thermal conductivity, mass density and the thermal volume expansion coefficient of the fluid, and μ is its magnetic permeability, assumed to be equal to that of free space. Ω is the angular velocity responsible for the Coriolis force and g the gravitational acceleration. D means the thickness of the spherical shell and ΔT the difference of temperatures at the inner and the outer boundaries. In all simulations considered in the following, $D = 0.65 r_0$ is assumed, where r_0 is the radius of the outer boundary. In order to characterize the results of the simulations, we use, in particular, the magnetic Reynolds number $\mathrm{Rm} = uD/\eta$ with u interpreted as the r.m.s. value of $\mathbf{u}$.

For the numerical solution of the above equations a code is used, which was originally designed by Glatzmaier [3] and later modified by Christensen et al. [2].

3. The mean-field concept. When applying the mean-field concept to the these examples, we focus attention on the induction equation only. We refer now to a spherical coordinate system (r, ϑ, φ), the polar axis of which coincides with the rotation axis of the shell. A mean-vector field is defined by averaging the components of the original field with respect to the spherical coordinate system over all values of the azimuthal coordinate φ, e.g.,

$$\overline{\mathbf{B}} = \overline{B}_r(r,\vartheta)\,\mathbf{e}_r + \overline{B}_\vartheta(r,\vartheta)\,\mathbf{e}_\vartheta + \overline{B}_\varphi(r,\vartheta)\,\mathbf{e}_\varphi .$$

With this definition of mean fields the Reynolds averaging rules apply exactly. Of course, all mean fields are axisymmetric about the rotation axis of the shell.

Subjecting the induction equation to averaging, we obtain

$$\partial_t \overline{\mathbf{B}} - \nabla \times (\overline{\mathbf{U}} \times \overline{\mathbf{B}} + \mathcal{E}) - \eta \nabla^2 \overline{\mathbf{B}} = \mathbf{0}\,,\ \nabla \cdot \overline{\mathbf{B}} = 0\,, \tag{2}$$

with the crucial electromotive force $\mathcal{E} = \overline{\mathbf{u} \times \mathbf{b}}$ mentioned above. If $\mathbf{u}$ is given, the calculation of $\mathcal{E}$ further requires the knowledge of $\mathbf{b}$. From the original induction equation and (2) we may derive

$$\begin{aligned} \partial_t \mathbf{b} - \nabla \times (\overline{\mathbf{U}} \times \mathbf{b} + \mathbf{G}) - \eta \nabla^2 \mathbf{b} &= \nabla \times (\mathbf{u} \times \overline{\mathbf{B}}) \\ \mathbf{G} = \mathbf{u} \times \mathbf{b} - \overline{\mathbf{u} \times \mathbf{b}}\,, &\quad \nabla \cdot \mathbf{b} = 0\,. \end{aligned} \tag{3}$$

Cancelling $\mathbf{G}$ in the first line of (3) leads to the often used "first-order smoothing" approximation.

For the examples envisaged it can be easily justified that $\mathbf{b}$ vanishes if $\overline{\mathbf{B}}$ does so. If we further assume that $\overline{\mathbf{B}}$ varies only weakly in space and time, we may conclude that $\mathcal{E}$ takes the form (1). This assumption, however, remains to be checked. Stronger variations of $\overline{\mathbf{B}}$ in space would require to take higher-order derivatives into account. Relation (1) can also be written in the coordinate-independent form

$$\mathcal{E} = -\boldsymbol{\alpha} \cdot \overline{\mathbf{B}} - \boldsymbol{\gamma} \times \overline{\mathbf{B}} - \boldsymbol{\beta} \cdot (\nabla \times \overline{\mathbf{B}}) - \boldsymbol{\delta} \times (\nabla \times \overline{\mathbf{B}}) - \boldsymbol{\kappa} \cdot (\nabla \overline{\mathbf{B}})^{(\mathrm{s})} \tag{4}$$

which we refer to in the following; see, e.g., Rädler [4]. Here $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are symmetric second–rank tensors, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ vectors, $\boldsymbol{\kappa}$ is a third–rank tensor with some symmetries, and $(\nabla \overline{\mathbf{B}})^{(\mathrm{s})}$ the symmetric part of the gradient tensor of $\overline{\mathbf{B}}$. Like a_{ij} and b_{ijk} $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ are also determined by $\mathbf{u}$ and $\overline{\mathbf{U}}$. The $\boldsymbol{\alpha}$ term in (4) describes in general an anisotropic α–effect, the $\boldsymbol{\gamma}$ term describes an advection of the mean magnetic field like that by a mean motion of the fluid. The $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$

terms can be interpreted in the sense of an anisotropic electrical mean–field conductivity and the $\boldsymbol{\kappa}$ term covers various other influences on the mean fields. Due to the axisymmetry of $\overline{\mathbf{B}}$ some elements of $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ are meaningless, that is, $\mathcal{E}$ remains unchanged if these are changed.

In view of the calculation of $\mathcal{E}$ we note that in the examples considered the configurations of $\mathbf{U}$ and $\mathbf{B}$ rotate like a rigid body. This allows us to change to a rotating frame of reference, in which these fields and so also $\overline{\mathbf{B}}$ and $\mathcal{E}$ are steady. The result for $\mathcal{E}$ obtained in this rotating frame applies also in the original frame.

Two methods have been used for the calculation of the coefficients $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ on the basis of the numerical simulations addressed in Sec. 2. They are described in some more detail in the paper by Schrinner et al. [5]. Method (i) is based on full equation (3) for $\mathbf{b}$, specified to the steady case. This equation is solved numerically with $\mathbf{u}$ and $\overline{\mathbf{U}}$ taken from the simulations mentioned, but employing properly chosen "test fields" for $\overline{\mathbf{B}}$. With the results for $\mathbf{b}$ obtained in this way, $\mathcal{E}$ is calculated for each test field, and from these results the $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ are determined. Method (ii) ignores any mean fluid motion and uses the first–order smoothing approximation. The steady version of equation (3) for $\mathbf{b}$ with $\overline{\mathbf{U}} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}$ can be solved analytically for arbitrary $\mathbf{u}$ and $\overline{\mathbf{B}}$. On this basis $\mathcal{E}$, and so $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$, can be defined for arbitrary $\mathbf{u}$ in the usual way, and later be specified by choosing $\mathbf{u}$ according to the simulations mentioned.

We note that the $\mathbf{u}$ values needed for the determination of the coefficients $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ were in both methods taken from simulations with non–zero $\overline{\mathbf{B}}$. That is, the resulting coefficients are already subject to a quenching corresponding to this $\overline{\mathbf{B}}$. In a further study they should be compared with those for the limit of vanishing $\overline{\mathbf{B}}$.

4. Magnetoconvection. The simulation by Olsen et al. [1] has been considered with $\mathrm{E} = 10^{-3}$, $\mathrm{Ra} = 94$, $\mathrm{Pr} = \mathrm{Pm} = 1$ and with an imposed toroidal magnetic field corresponding to $\Lambda = 1$. In this case the intensity of the fluid motion is characterized by $\mathrm{Rm} \approx 12$.

The quantities $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ have been calculated by the two methods explained above. We consider the results of method (i) as most reliable. A mean-field model of magnetoconvection based on (2) and (4) with $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ obtained by method (i) reproduces very well the $\overline{\mathbf{B}}$-field obtained from the direct numerical simulations.

The results for $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ obtained by method (ii) fairly agree with those of method (i) as far as the profiles of these quantities are concerned, but overestimate their magnitudes typically by a few per cent. We recall that method (ii) uses the first-order smoothing approximation. In the steady case considered here it is surely justified for $\mathrm{Rm}' \ll 1$ (a sufficient condition), where $\mathrm{Rm}' = ul/\eta$, with u being again the r.m.s. value of $\mathbf{u}$ and l a characteristic length of the $\mathbf{u}$–field. It seems reasonable to assume that l is not much smaller than D, that is, Rm' is not much smaller than Rm. A more detailed consideration shows that in the case considered the results of method (ii) are well acceptable as long as Rm does not markedly exceed unity.

We have also investigated the quantity $\delta\mathcal{E} = (\mathcal{E}^{\mathrm{DNS}} - \mathcal{E}^{\mathrm{MF1}})/\sqrt{\langle(\mathcal{E}^{\mathrm{DNS}})^2\rangle}$. Here $\mathcal{E}^{\mathrm{DNS}}$ means the quantity $\mathcal{E}$ immediately extracted from the direct numerical simulation, and $\mathcal{E}^{\mathrm{MF1}}$ that is defined according to (4) (considering no higher than first-order derivatives of $\overline{\mathbf{B}}$) with $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ as obtained by method (i) and $\overline{\mathbf{B}}$ corresponding to the direct numerical simulations (or, what is here the same, to the mean-field model). Further $\langle\cdots\rangle$ means averaging over all r and ϑ of interest in a meridional plane. With the exception of a few small areas in this plane $|\delta\mathcal{E}|$

proves to be much smaller than unity. This indicates that representation (4) is indeed satisfactory for the example considered.

5. Geodynamo model. We consider now the case $\mathrm{E} = 10^{-3}$, $\mathrm{Ra} = 100$, $\mathrm{Pr} = 1$, $\mathrm{Pm} = 5$ and $\Lambda = 0$, in which the numerical simulations by Christensen et al. [6] indeed show a dynamo. The intensity of the fluid motions can be characterized by $\mathrm{Rm} \approx 40$.

In this case there is a clear difference in the results for $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ obtained by the two methods.

Several attempts have been made to reproduce the quasi–steady dynamo observed in the direct numerical simulations by a mean-field model using representation (4) of $\mathcal{E}$ with the calculated $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$. The results were not completely satisfying. The mean-field model with the most reliable choice of these quantities, that is, with those determined by method (i), proved to be slightly subcritical. The steady mean magnetic field extracted from the direct numerical simulations, however, is geometrically rather similar to the slowly decaying one of the mean-field model.

The quantity $\delta\mathcal{E}$ turns out to be larger than in the case of magnetoconvection by a factor in the order of 10. This seems to indicate that representation (4) no longer describes the real $\mathcal{E}$ reasonably. The neglect of higher than first-order derivatives of $\overline{\mathbf{B}}$ is no longer justified.

6. Summary. The two examples considered in this paper lead us to limits of the applicability of two simplifications frequently used in mean-field theory. Although in these examples the validity of the first–order smoothing approximation has proved not to be rigourously restricted to Rm much smaller than unity, it became clear that this approximation does not work well with Rm markedly exceeding the order of unity. In the second example, in addition, the traditional representation of the mean electromotive force considering no higher than first-order derivatives of the mean magnetic field seems to be no longer justified. Nonetheless, the results derived from our mean-field models match the azimuthal averages extracted from the direct numerical simulations surprisingly well.

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