Introduction to Hydromagnetic Dynamo Theory with Applications to the Sun and the Earth

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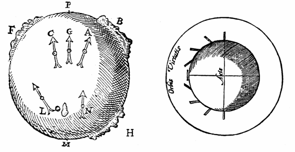
Outline

Geomagnetic field

1600 Gilbert, De Magnete: "Magnus magnes ipse est globus terrestris."
(The Earth's globe itself is a great magnet.)



1838 Gauss: Mathematical description of geomagnetic field

$$B = \sum_{l,m} \boldsymbol{B}_l^m = -\sum \boldsymbol{\nabla}\Phi_l^m = -R \sum \boldsymbol{\nabla} \left(\frac{R}{r}\right)^{l+1} P_l^m(\cos\vartheta)\left(g_l^m \cos m\phi + h_l^m \sin m\phi\right)$$

sources inside Earth

l number of nodal lines, m number of azimuthal nodal lines

$l = 1, 2, 3, \ldots$ dipole, quadrupole, octupole, ...

$m = 0$ axisymmetry, $m = 1, 2, \ldots$ non-axisymmetry

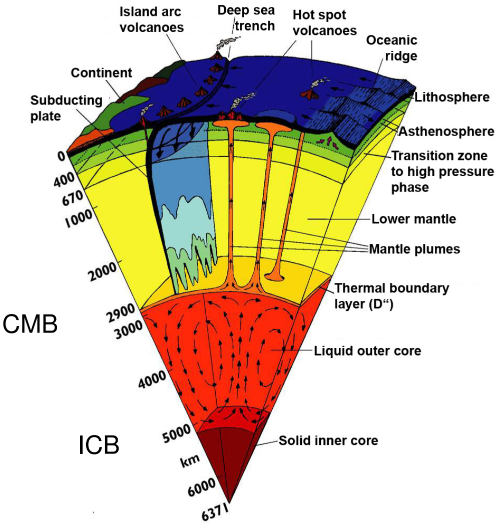
Earth: $g_1^0 \approx -0.3\,\mathrm{G}$, all other $|g_l^m|, |h_l^m| < 0.05\,\mathrm{G}$

mainly dipolar, dipole moment $\mu = R^3 \left[(g_1^0)^2 + (g_1^1)^2 + (h_1^1)^2\right]^{1/2} \approx 8{\cdot}10^{25}\,\mathrm{G\,cm^3}$

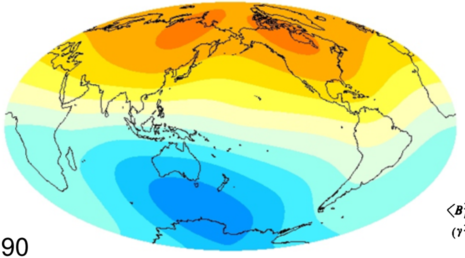
$\tan\psi = \left[(g_1^1)^2 + (h_1^1)^2\right]^{1/2} \Big/ g_1^0$, dipole tilt $\psi \approx 11^\circ$

dipole : quadrupole $\approx 1 : 0.14$ (at CMB)

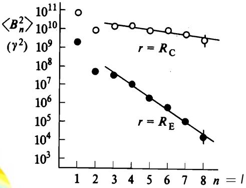
Internal structure of the Earth

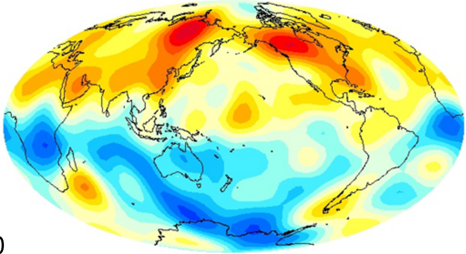


Spatial structure of geomagnetic field



B_r at surface 1990

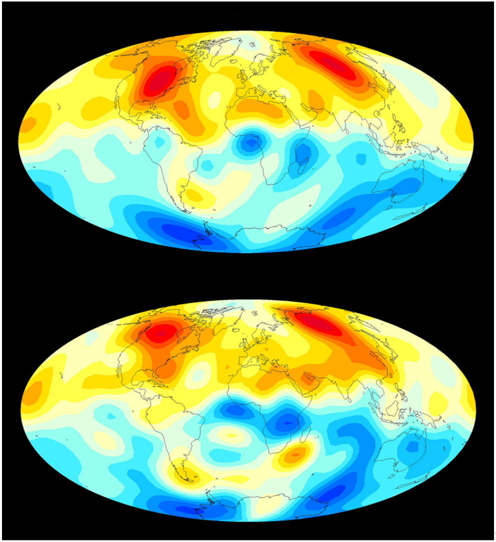




B_r at CMB 1990

Secular variation

B_r at CMB 1890

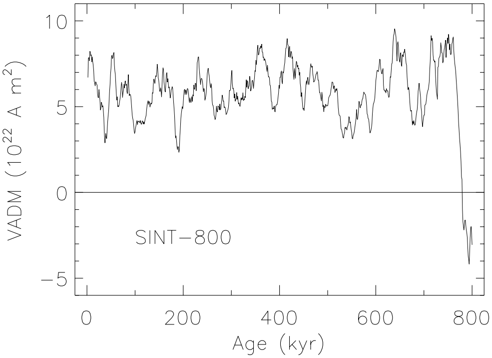


B_r at CMB 1990

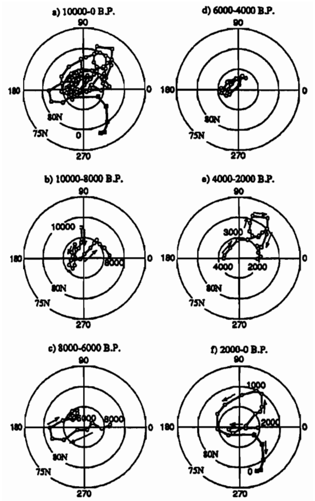
westward drift $0.18^\circ/\text{yr}$

$u \approx 0.5\,\text{mm}/\text{sec}$

Secular variation

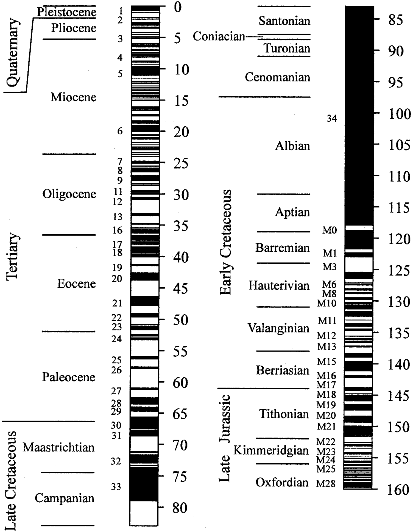


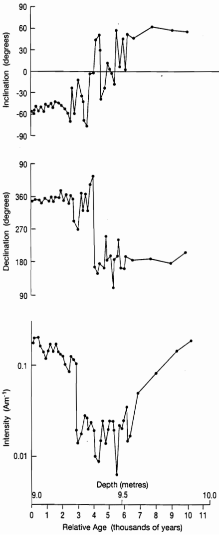
SINT-800 VADM (Guyodo and Valet 1999)



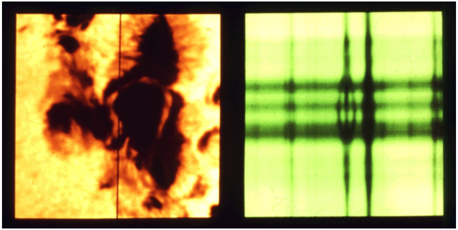
NGP (Ohno and Hamano 1992)

Polarity reversals

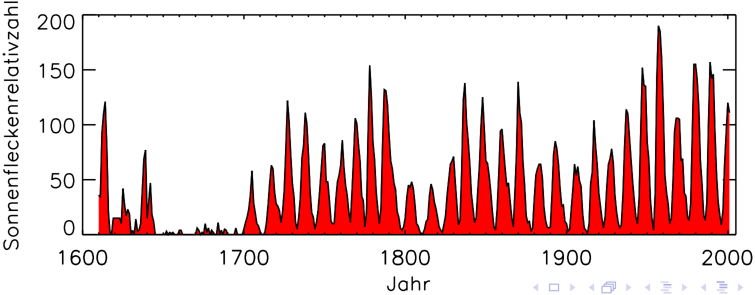




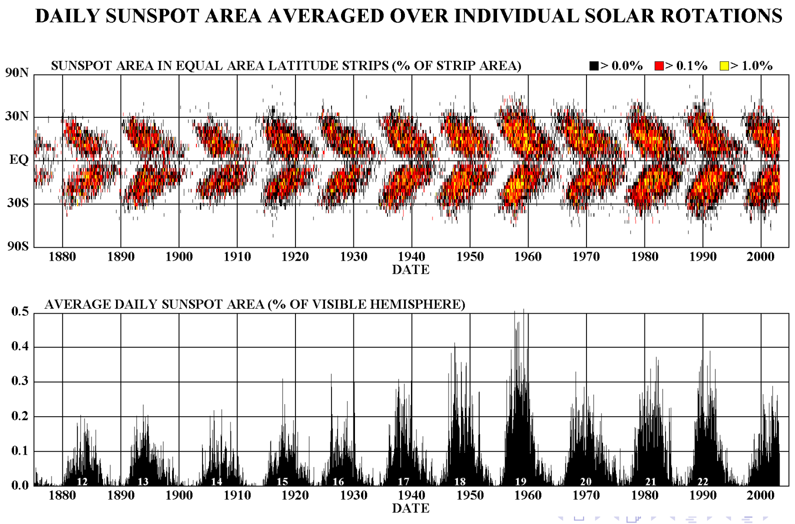
Solar activity cycle (Schwabe 1843, Wolf 1848)



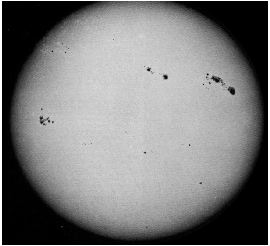
(Hale 1908)

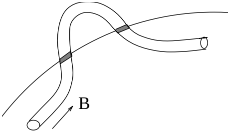


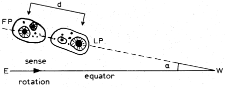
Butterfly diagram (Spörer ~1865, Maunder 1904)

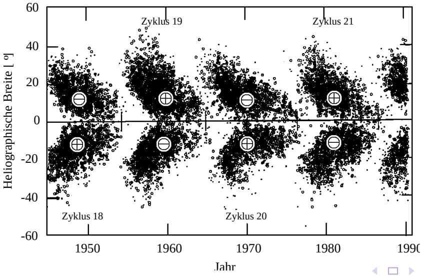


Polarity rules (Hale et al. 1919)

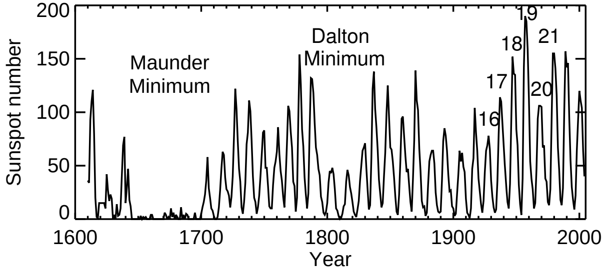


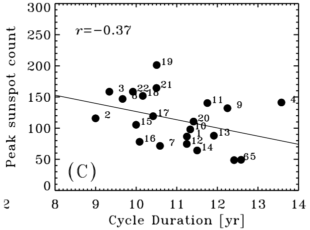






Variability of cycle length and strength

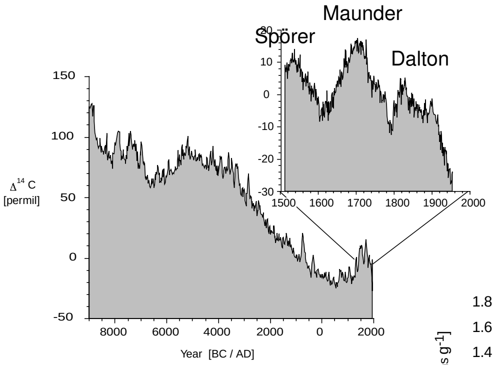


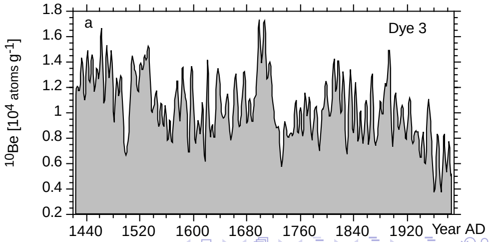


odd-even effect
(Gnevyshev and Ohl 1948)

Long-term variability / C14 and Be10

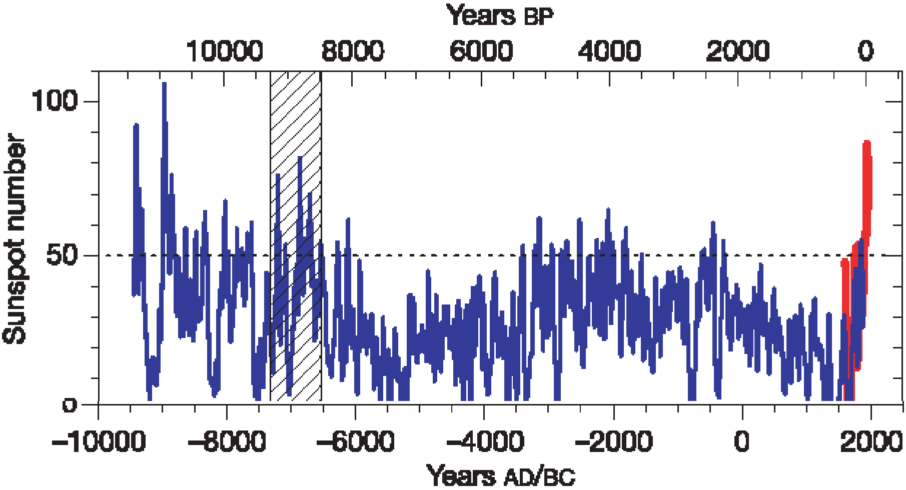
(Stuiver et al. 1998)





(Beer et al. 1994)

Solar activity in the last 11400 year



(Solanki et al. 2004)

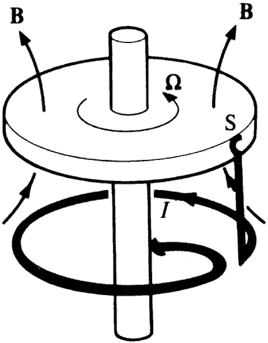
Dynamo hypothesis

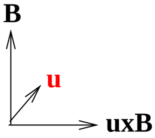
- Larmor (1919): Magnetic field of Earth and Sun maintained by self-excited dynamo
- Dynamo: $\boldsymbol{u}\times\boldsymbol{B} \curvearrowright \boldsymbol{j} \curvearrowright \boldsymbol{B} \curvearrowright \boldsymbol{u}$

  Faraday Ampere Lorentz

  motion of an electrical conductor in an 'inducing' magnetic field
  $\curvearrowright$ induction of electric current
- Self-excited dynamo: inducing magnetic field created by the electric current (Siemens 1867)
- Example: homopolar dynamo
- Homogeneous dynamo (no wires in Earth core or solar convection zone)
  $\curvearrowright$ complex motion necessary
- Kinematic ($\boldsymbol{u}$ prescribed, linear)
- Dynamic ($\boldsymbol{u}$ determined by forces, including Lorentz force, non-linear)

Homopolar dynamo





electromotive force $\boldsymbol{u}\times\boldsymbol{B}$ $\curvearrowright$ electric current through wire loop
$\curvearrowright$ induced magnetic field reinforces applied magnetic field

self-excitation if rotation $\Omega > 2\pi R/M$ is maintained
where R resistance, M inductance

Pre-Maxwell theory

Maxwell equations: cgs system, vacuum, $\boldsymbol{B} = \boldsymbol{H}$, $\boldsymbol{D} = \boldsymbol{E}$

$$c\nabla\times\boldsymbol{B} = 4\pi\boldsymbol{j} + \frac{\partial\boldsymbol{E}}{\partial t}\ , \quad c\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}\ , \quad \nabla\cdot\boldsymbol{B} = 0\ , \quad \nabla\cdot\boldsymbol{E} = 4\pi\lambda$$

Basic assumptions of MHD:

- $u \ll c$: system stationary on light travel time, no em waves
- high electrical conductivity: $\boldsymbol{E}$ determined by $\partial\boldsymbol{B}/\partial t$, not by charges λ

$$c\frac{E}{L} \approx \frac{B}{T} \curvearrowright \frac{E}{B} \approx \frac{1}{c}\frac{L}{T} \approx \frac{u}{c} \ll 1\ , \ E \text{ plays minor role} : \frac{e_{el}}{e_m} \approx \frac{E^2}{B^2} \ll 1$$

$$\frac{\partial\boldsymbol{E}/\partial t}{c\nabla\times\boldsymbol{B}} \approx \frac{E/T}{cB/L} \approx \frac{E}{B}\frac{u}{c} \approx \frac{u^2}{c^2} \ll 1\ , \ \text{displacement current negligible}$$

Pre-Maxwell equations:

$$c\nabla\times\boldsymbol{B} = 4\pi\boldsymbol{j}\ , \quad c\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}\ , \quad \nabla\cdot\boldsymbol{B} = 0$$

Pre-Maxwell theory

Pre-Maxwell equations Galilei-covariant:

$$\boldsymbol{E}' = \boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}\,, \quad \boldsymbol{B}' = \boldsymbol{B}\,, \quad \boldsymbol{j}' = \boldsymbol{j}$$

Relation between $\boldsymbol{j}$ and $\boldsymbol{E}$ by Galilei-covariant **Ohm's law:** $\boldsymbol{j}' = \sigma\boldsymbol{E}'$

in resting frame of reference, σ electrical conductivity

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\times\boldsymbol{B})$$

Magnetohydrokinematics:

$$c\nabla\times\boldsymbol{B} = 4\pi\boldsymbol{j}$$

$$c\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$

$$\nabla\cdot\boldsymbol{B} = 0$$

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\times\boldsymbol{B})$$

Magnetohydrodynamics:

additionally

Equation of motion
Equation of continuity
Equation of state
Energy equation

Induction equation

Evolution of magnetic field

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\nabla\times\boldsymbol{E} = -c\nabla\times\left(\frac{\boldsymbol{j}}{\sigma} - \frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}\right) = -c\nabla\times\left(\frac{c}{4\pi\sigma}\nabla\times\boldsymbol{B} - \frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}\right)$$

$$= \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \nabla\times\left(\frac{c^2}{4\pi\sigma}\nabla\times\boldsymbol{B}\right) = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \eta\nabla\times\nabla\times\boldsymbol{B}$$

with $\eta = \frac{c^2}{4\pi\sigma} = \text{const}$ magnetic diffusivity

induction, diffusion

$$\nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = -\boldsymbol{B}\,\nabla\cdot\boldsymbol{u} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$$

expansion/contraction, shear/stretching, advection

$\nabla\cdot\boldsymbol{B} = 0$ as initial condition, conserved

Alfven's theorem (Alfvén 1942)

Ideal conductor $\eta = 0$: $\dfrac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B})$

Magnetic flux through floating surface is constant : $\dfrac{d}{dt}\displaystyle\int_F \boldsymbol{B}\cdot d\boldsymbol{F} = 0$

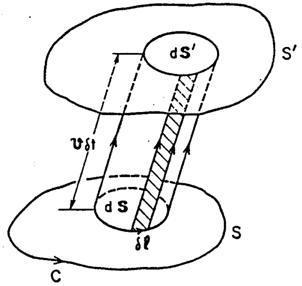
Proof:

$$0 = \int \nabla\cdot\boldsymbol{B}dV = \int \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{F} = \int_F \boldsymbol{B}(t)\cdot d\boldsymbol{F} - \int_{F'} \boldsymbol{B}(t)\cdot d\boldsymbol{F'} - \oint_C \boldsymbol{B}(t)\cdot d\boldsymbol{s}\times\boldsymbol{u}dt\,,$$

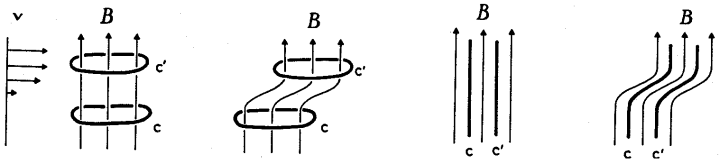
$$\int_{F'} \boldsymbol{B}(t+dt)\cdot d\boldsymbol{F'} - \int_F \boldsymbol{B}(t)\cdot d\boldsymbol{F} = \int_F \{\boldsymbol{B}(t+dt) - \boldsymbol{B}(t)\}\cdot \mathrm{d}\boldsymbol{F} - \oint_C \boldsymbol{B}\cdot d\boldsymbol{s}\times\boldsymbol{u}dt$$

$$= dt\left(\int \frac{\partial \boldsymbol{B}}{\partial t}\cdot \mathrm{d}\boldsymbol{F} - \oint_C \boldsymbol{B}\cdot d\boldsymbol{s}\times\boldsymbol{u}\right) = dt\left(\int \nabla\times(\boldsymbol{u}\times\boldsymbol{B})\cdot d\boldsymbol{F} - \oint_C \boldsymbol{B}\cdot d\boldsymbol{s}\times\boldsymbol{u}\right)$$

$$= dt\left(\oint_C \boldsymbol{u}\times\boldsymbol{B}\cdot d\boldsymbol{s} - \oint_C \boldsymbol{B}\cdot d\boldsymbol{s}\times\boldsymbol{u}\right) = 0$$



Alfven's theorem



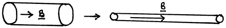
Frozen-in field lines

impression that magnetic field follows flow, but $\boldsymbol{E} = -\boldsymbol{u}\times\boldsymbol{B}$ and $\nabla\times\boldsymbol{E} = -c\partial\boldsymbol{B}/\partial t$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = -\boldsymbol{B}\,\nabla\cdot\boldsymbol{u} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$$

(i) star contraction: $\overline{B} \sim R^{-2}, \overline{\rho} \sim R^{-3} \curvearrowright \overline{B} \sim \overline{\rho}^{2/3}$

Sun $\curvearrowright$ white dwarf $\curvearrowright$ neutron star: ρ [g cm^{-3}]: $1 \curvearrowright 10^6 \curvearrowright 10^{15}$

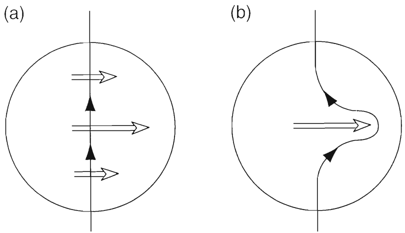
(ii) stretching of flux tube: 

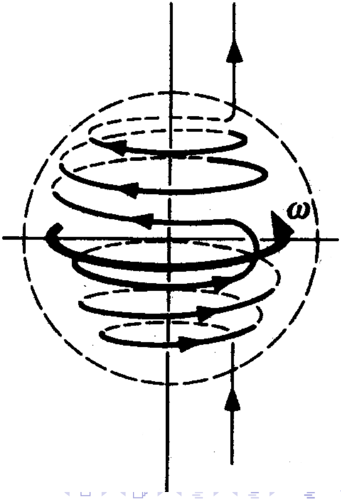
$Bd^2 = \text{const}$, $ld^2 = \text{const} \curvearrowright B \sim l$

(iii) shear, differential rotation

Differential rotation

$$\partial B_\phi / \partial t = r \sin\theta \, \boldsymbol{\nabla}\Omega \cdot \boldsymbol{B}_p$$





Magnetic Reynolds number

Dimensionless variables: length L, velocity u_0, time L/u_0

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) - R_m^{-1}\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{B} \quad \text{with} \quad R_m = \frac{u_0 L}{\eta}$$

as combined parameter

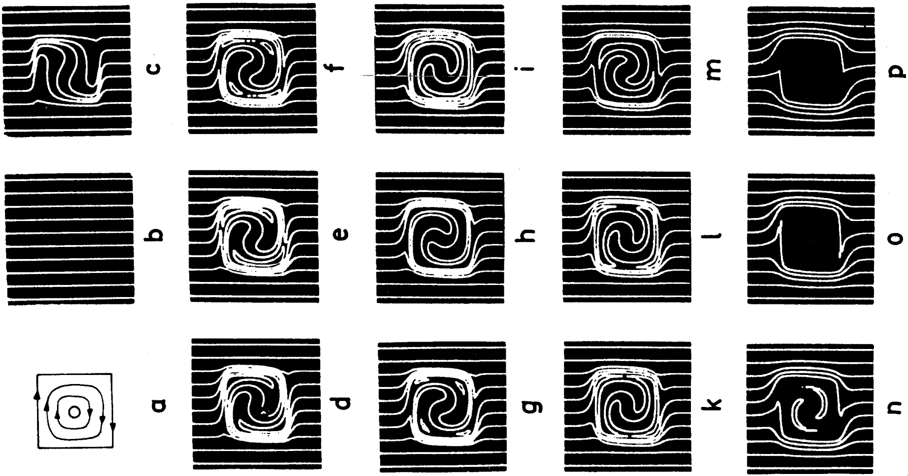
laboratorium: $R_m \ll 1$, cosmos: $R_m \gg 1$

induction for $R_m \gg 1$, diffusion for $R_m \ll 1$, e.g. for small L

example: flux expulsion from closed velocity fields

Flux expulsion

(Weiss 1966)



Poloidal and toroidal magnetic fields

Spherical coordinates (r, ϑ, φ)

Axisymmetric fields: $\partial/\partial\varphi = 0$

$$\boldsymbol{B}(r,\vartheta) = (B_r, B_\vartheta, B_\varphi)$$

$$\nabla\cdot\boldsymbol{B} = 0 \curvearrowright \frac{1}{r^2}\frac{\partial r^2 B_r}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial \sin\vartheta B_\vartheta}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\overbrace{\frac{\partial B_\varphi}{\partial\varphi}}^{=0} = 0$$

$\boldsymbol{B} = \boldsymbol{B}_p + \boldsymbol{B}_t$ poloidal and toroidal magnetic field

$\boldsymbol{B}_t = (0, 0, B_\varphi)$ satisfies $\nabla\cdot\boldsymbol{B}_t = 0$

$\boldsymbol{B}_p = (B_r, B_\vartheta, 0) = \nabla\times\boldsymbol{A}$ with $\boldsymbol{A} = (0, 0, A_\varphi)$ satisfies $\nabla\cdot\boldsymbol{B}_p = 0$

$$\boldsymbol{B}_p = \frac{1}{r\sin\vartheta}\left(\frac{\partial r\sin\vartheta A_\varphi}{r\partial\vartheta}, -\frac{\partial r\sin\vartheta A_\varphi}{\partial r}, 0\right)$$

axisymmetric magnetic field determined by the two scalars: $r\sin\vartheta A_\varphi$ and B_φ

Poloidal and toroidal magnetic fields

Axisymmetric fields:

$$\boldsymbol{j}_t = \frac{c}{4\pi}\nabla\times\boldsymbol{B}_p\,, \quad \boldsymbol{j}_p = \frac{c}{4\pi}\nabla\times\boldsymbol{B}_t$$

$r\sin\vartheta A_\varphi = \text{const}:$ field lines of poloidal field in meridional plane

field lines of $\boldsymbol{B}_t$ are circles around symmetry axis

Non-axisymmetric fields:

$$\boldsymbol{B} = \boldsymbol{B}_p + \boldsymbol{B}_t = \nabla\times\nabla\times(P\boldsymbol{r}) + \nabla\times(T\boldsymbol{r}) = -\nabla\times(\boldsymbol{r}\times\nabla P) - \boldsymbol{r}\times\nabla T$$

$\boldsymbol{r} = (r,0,0)\,,\quad P(r,\vartheta,\varphi)$ and $T(r,\vartheta,\varphi)$ defining scalars

$$\nabla\cdot\boldsymbol{B} = 0\,, \quad \boldsymbol{j}_t = \frac{c}{4\pi}\nabla\times\boldsymbol{B}_p\,, \quad \boldsymbol{j}_p = \frac{c}{4\pi}\nabla\times\boldsymbol{B}_t$$

$\boldsymbol{r}\cdot\boldsymbol{B}_t = 0$ field lines of the toroidal field lie on spheres, no r component

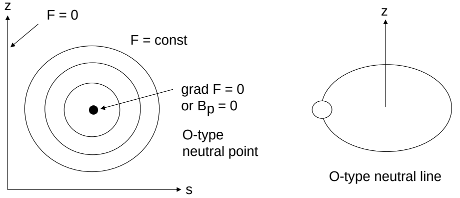
$\boldsymbol{B}_p$ has in general all three components

Cowling's theorem (Cowling 1934)

Axisymmetric magnetic fields can not be maintained by a dynamo.

Sketch of proof:

- electric currents as sources of the magnetic field only in finite space
- field line $F = 0$ along axis closes at infinity
- field lines on circular tori whose cross section are the lines $F = \text{const}$



- axisymmetry: closed neutral line
- around neutral line is $\nabla\times\boldsymbol{B} \neq 0 \curvearrowright j_\varphi \neq 0$, but there is no source of j_φ: $E_\varphi = 0$ because of axisymmetry and $(\boldsymbol{u}\times\boldsymbol{B})_\varphi = 0$ on neutral line for finite $\boldsymbol{u}$

Cowling's theorem – Formal proof

Consider vicinity of neutral line, assume axisymmetry

$$\oint B_p dl = \oint \boldsymbol{B}\cdot d\boldsymbol{l} = \int \nabla\times\boldsymbol{B}\, d\boldsymbol{f} = \frac{4\pi}{c}\int \boldsymbol{j}\cdot d\boldsymbol{f} = \frac{4\pi}{c}\int |j_\varphi| df$$

$$= \frac{4\pi\sigma}{c^2}\int |\boldsymbol{u}_p\times\boldsymbol{B}_p| df \leq \frac{4\pi\sigma}{c^2}\int u_p B_p df \leq \frac{4\pi\sigma}{c^2} u_{\mathrm{p,max}}\int B_p df$$

integration circle of radius ε

$$B_p 2\pi\varepsilon \leq \frac{4\pi\sigma}{c^2} u_{\mathrm{p,max}} B_p \pi\varepsilon^2 \quad \text{or} \quad 1 \leq \frac{2\pi\sigma}{c^2} u_{\mathrm{p,max}}\varepsilon$$

$\varepsilon \to 0 \curvearrowright u_{\mathrm{p,max}} \to \infty$

contradiction

Toroidal theorems

Toroidal velocity theorem (Elsasser 1947, Bullard & Gellman 1954)

A toroidal motion in a spherical conductor can not maintain a magnetic field by dynamo action.

Sketch of proof:

$$\frac{d}{dt}(\boldsymbol{r}\cdot\boldsymbol{B}) = \eta\boldsymbol{\nabla}^2(\boldsymbol{r}\cdot\boldsymbol{B}) \quad \text{for} \quad \boldsymbol{r}\cdot\boldsymbol{u} = 0$$

$$\curvearrowright \ \boldsymbol{r}\cdot\boldsymbol{B} \rightarrow 0 \quad \text{for} \quad t \rightarrow \infty \ \curvearrowright \ P \rightarrow 0 \ \curvearrowright \ T \rightarrow 0$$

Toroidal field theorem / Invisible dynamo theorem (Kaiser et al. 1994)

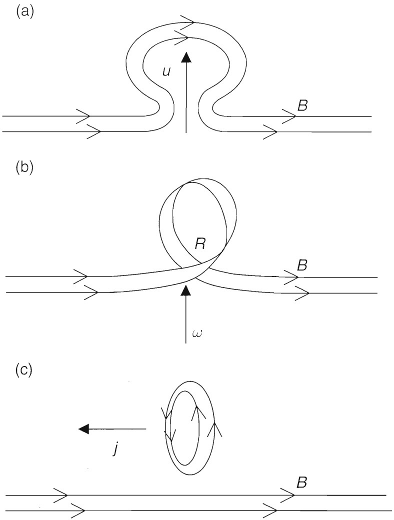
A purely toroidal magnetic field can not be maintained by a dynamo.

Parker's helical convection

velocity $\boldsymbol{u}$

vorticity $\omega = \nabla \times \boldsymbol{u}$

helicity $H = \boldsymbol{u} \cdot \omega$



(Parker 1955)

Mean-field theory

Statistical consideration of turbulent helical convection on mean magnetic field (Steenbeck, Krause and Rädler 1966)

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \eta\nabla\times\nabla\times\boldsymbol{B}$$

$\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u}'$, $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{B}'$ Reynolds rules for averages

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla\times(\overline{\boldsymbol{u}}\times\overline{\boldsymbol{B}} + \mathcal{E}) - \eta\nabla\times\nabla\times\overline{\boldsymbol{B}}$$

$\mathcal{E} = \overline{\boldsymbol{u}'\times\boldsymbol{B}'}$ mean electromotive force

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \nabla\times(\overline{\boldsymbol{u}}\times\boldsymbol{B}' + \boldsymbol{u}'\times\overline{\boldsymbol{B}} + \mathcal{G}) - \eta\nabla\times\nabla\times\boldsymbol{B}'$$

$\mathcal{G} = \boldsymbol{u}'\times\boldsymbol{B}' - \overline{\boldsymbol{u}'\times\boldsymbol{B}'}$ usually neglected, FOSA = SOCA

$\boldsymbol{B}'$ linear, homogeneous functional of $\overline{\boldsymbol{B}}$

approximation of scale separation: $\boldsymbol{B}'$ depends on $\overline{\boldsymbol{B}}$ only in small surrounding

Taylor expansion: $\left(\overline{\boldsymbol{u}'\times\boldsymbol{B}'}\right)_i = \alpha_{ij}\overline{B}_j + \beta_{ijk}\partial\overline{B}_k/\partial x_j + \ldots$

Mean-field theory

$$\left(\overline{\boldsymbol{u}' \times \boldsymbol{B}'}\right)_i = \alpha_{ij}\overline{B}_j + \beta_{ijk}\partial\overline{B}_k/\partial x_j + \ldots$$

α_{ij} and β_{ijk} depend on $\boldsymbol{u}'$

homogeneous, isotropic $\boldsymbol{u}'$: $\alpha_{ij} = \alpha\delta_{ij}$, $\beta_{ijk} = -\beta\varepsilon_{ijk}$ then

$$\overline{\boldsymbol{u}' \times \boldsymbol{B}'} = \alpha\overline{\boldsymbol{B}} - \beta\nabla\times\overline{\boldsymbol{B}}$$

Ohm's law: $\boldsymbol{j} = \sigma(\boldsymbol{E} + (\boldsymbol{u}\times\boldsymbol{B})/c)$

$$\overline{\boldsymbol{j}} = \sigma(\overline{\boldsymbol{E}} + (\overline{\boldsymbol{u}}\times\overline{\boldsymbol{B}})/c + (\alpha\overline{\boldsymbol{B}} - \beta\nabla\times\overline{\boldsymbol{B}})/c) \quad \text{and} \quad c\nabla\times\overline{\boldsymbol{B}} = 4\pi\overline{\boldsymbol{j}}$$

$$\overline{\boldsymbol{j}} = \sigma_{\text{eff}}(\overline{\boldsymbol{E}} + (\overline{\boldsymbol{u}}\times\overline{\boldsymbol{B}})/c + \alpha\overline{\boldsymbol{B}}/c)$$

$$\frac{\partial\overline{\boldsymbol{B}}}{\partial t} = \nabla\times(\overline{\boldsymbol{u}}\times\overline{\boldsymbol{B}} + \alpha\overline{\boldsymbol{B}}) - \eta_{\text{eff}}\nabla\times\nabla\times\overline{\boldsymbol{B}} \quad \text{with} \quad \eta_{\text{eff}} = \eta + \beta$$

Two effects:

(1) α – effect: $\overline{\boldsymbol{j}} = \sigma_{\text{eff}}\alpha\overline{\boldsymbol{B}}/c$

(2) turbulent diffusivity: $\beta \gg \eta$, $\eta_{\text{eff}} = \beta = \eta_T$

Sketch of dependence of α and β on u'

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \nabla\times(\overline{\boldsymbol{u}}\times\boldsymbol{B}' + \boldsymbol{u}'\times\overline{\boldsymbol{B}} + \mathcal{G}) - \eta\nabla\times\nabla\times\boldsymbol{B}'$$

simplifying assumptions: $\mathcal{G} = 0$, $\boldsymbol{u}'$ incompressible, isotropic , $\overline{\boldsymbol{u}} = 0$, $\eta = 0$

$$B'_k = \int_{t_0}^{t} \underbrace{\varepsilon_{klm}\varepsilon_{mrs}}_{\delta_{kr}\delta_{ls} - \delta_{ks}\delta_{lr}} \frac{\partial}{\partial x_l}(u'_r\overline{B}_s)d\tau + B'_k(t_0)$$

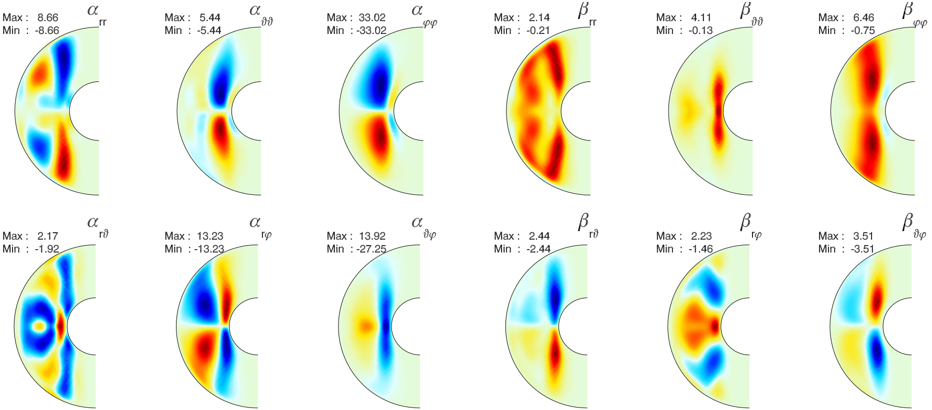
$$\mathcal{E}_i = \langle \boldsymbol{u}'\times\boldsymbol{B}'\rangle_i = \varepsilon_{ijk}\Big\langle u'_j(t)\Big[\int_{t_0}^{t}\Big(\frac{\partial u'_k}{\partial x_l}\overline{B}_l + u'_k\frac{\partial \overline{B}_l}{\partial x_l} - \frac{\partial u'_l}{\partial x_l}\overline{B}_k - u'_l\frac{\partial \overline{B}_k}{\partial x_l}\Big)d\tau + B'_k(t_0)\Big]\Big\rangle$$

$$= \varepsilon_{ijk}\int_{t_0}^{t}\Big[\underbrace{\Big\langle u'_j(t)\frac{\partial u'_k(\tau)}{\partial x_l}\Big\rangle}_{\curvearrowright\ \alpha}\overline{B}_l - \underbrace{\Big\langle u'_j(t)u'_l(\tau)\Big\rangle}_{\curvearrowright\ \beta}\frac{\partial \overline{B}_k}{\partial x_l}\Big]d\tau$$

isotropic turbulence: $\alpha = -\frac{1}{3}\overline{\boldsymbol{u}'\cdot\nabla\times\boldsymbol{u}'}\tau^* = -\frac{1}{3}\overline{H}\tau^*$ and $\beta = \frac{1}{3}u'^2\tau^*$

H helicity , τ^* correlation time

Mean-field coefficients derived from a MHD geodynamo simulation



(http://www.solar-system-school.de/alumni/schrinner.pdf, Schrinner et al. 2007)

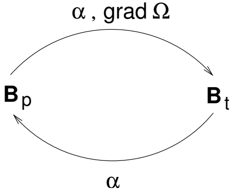
Mean-field dynamos

Dynamo equation: $$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla\times(\overline{\boldsymbol{u}}\times\overline{\boldsymbol{B}} + \alpha\overline{\boldsymbol{B}} - \eta_T\nabla\times\overline{\boldsymbol{B}})$$

- spherical coordinates, axisymmetry
- $\overline{\boldsymbol{u}} = (0, 0, \Omega(r,\vartheta) r\sin\vartheta)$
- $\overline{\boldsymbol{B}} = (0, 0, B(r,\vartheta,t)) + \nabla\times(0, 0, A(r,\vartheta,t))$

$$\frac{\partial B}{\partial t} = r\sin\vartheta(\nabla\times\boldsymbol{A})\cdot\nabla\Omega - \alpha\nabla_1^2 A + \eta_T\nabla_1^2 B$$

$$\frac{\partial A}{\partial t} = \alpha B + \eta_T\nabla_1^2 A \quad \text{with} \quad \nabla_1^2 = \nabla^2 - (r\sin\vartheta)^{-2}$$

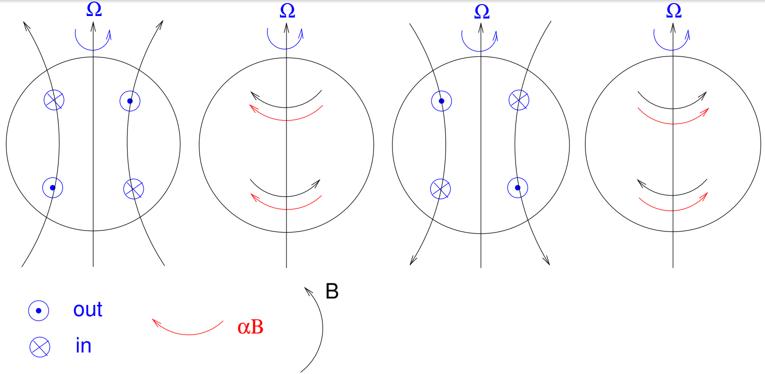


rigid rotation has no effect

no dynamo if $\alpha = 0$

$$\frac{\alpha\text{–term}}{\nabla\Omega\text{–term}} \approx \frac{\alpha_0}{|\nabla\Omega| L^2} \begin{cases} \gg 1 & \alpha^2\text{–dynamo with dynamo number } R_\alpha^2 \\ \sim 1 & \alpha^2\Omega\text{–dynamo} \\ \ll 1 & \alpha\Omega\text{–dynamo with dynamo number } R_\alpha R_\Omega \end{cases}$$

Sketch of an $\alpha\Omega$ dynamo

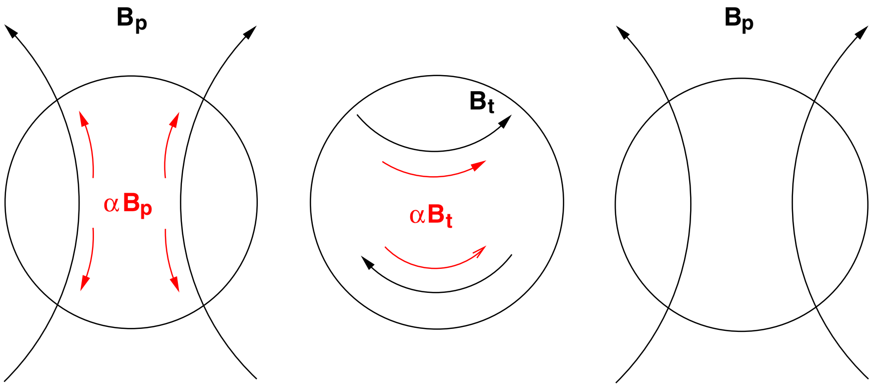


poloidal field	toroidal field by differential rotation; electric currents by α-effect	poloidal field by α-effect	toroidal field by differential rotation; electric currents by α-effect

$\partial\Omega/\partial r < 0$
$\alpha \sim \cos\theta$

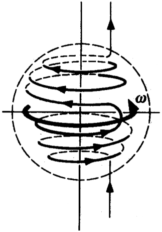
periodically alternating field, here antisymmetric with respect to equator

Sketch of an α^2 dynamo

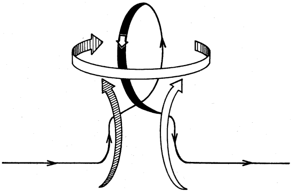


stationary field, here antisymmetric with respect to equator

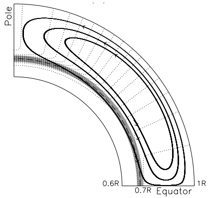
Basic solar dynamo incredients



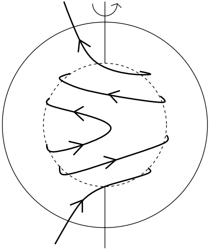
Differential rotation

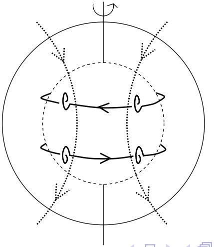


Helical motion



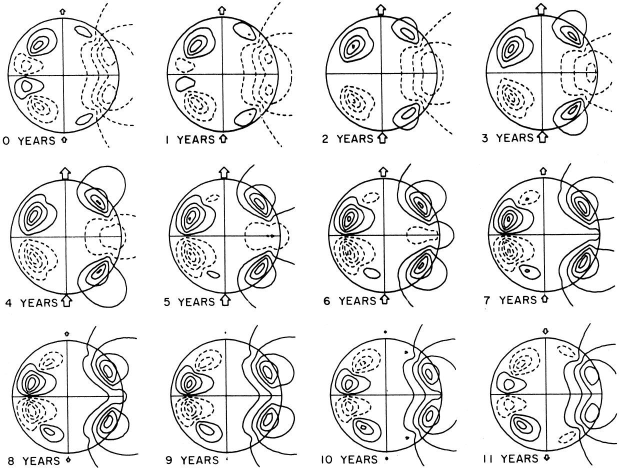
Meridional circulation





Convection zone dynamos

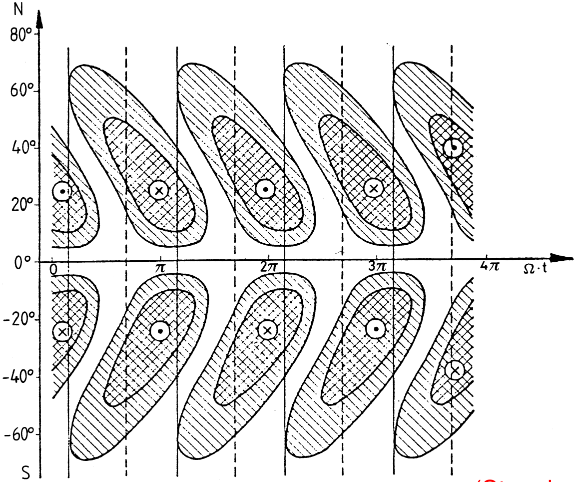
$\alpha\Omega$-dynamo in convection zone, $\Omega(r)$ with $\partial\Omega/\partial r < 0$, $\alpha \sim \cos\vartheta$, $\eta_T = 10^{10}\,\mathrm{cm}^2\mathrm{cm}^{-1}$



(Stix 1976)

Convection zone dynamos

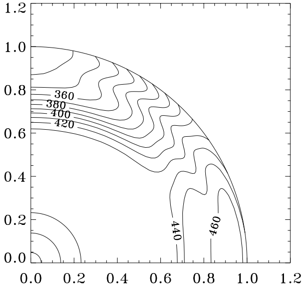
Theoretical butterfly diagram

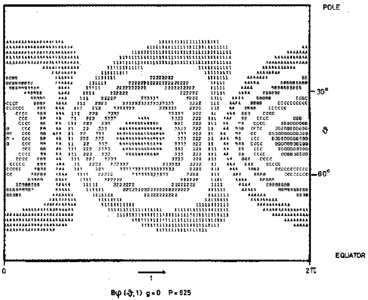


(Steenbeck and Krause 1969)

Difficulties of convection zone dynamos

- Intermittency: $B' \gg \langle B \rangle$
- Polarity rules: $B \sim 10^5$ G
- Magnetic buoyancy and storage problem: rise time $\ll$ cycle length
- Rotation law
- Butterfly diagram





Overshoot layer dynamos

- Favourable dynamo site:

  storage, reduced turbulent diffusivity, rotation, dynamic α-effect
- Dynamo action of magnetostrophic waves (Schmitt 1985):

  magnetic field layer unstable due to magnetic buoyancy

  $\rightarrow$ excitation of magnetostrophic waves in a fast rotating fluid

  $v_A^2 / v_{\rm rot} \approx v_{\rm mw} \ll v_A \ll v_{\rm rot} \ll v_S$

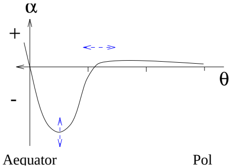
  mw are helical and induce an electromotive force

  $\rightarrow$ electric current parallel to toroidal magnetic field

  $\equiv$ dynamic α-effect: $\alpha \langle \boldsymbol{B} \rangle_{\rm tor} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle_{\rm tor}$

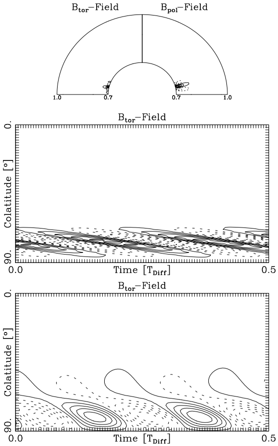
  not based on convection, applicable to strong fields

  superposition of most unstable waves:



Overshoot layer dynamos

- Dynamo model



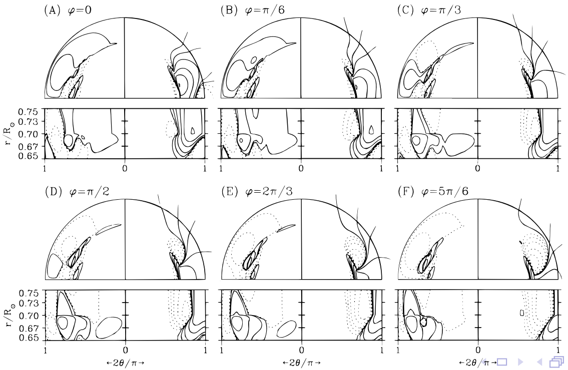
(Schmitt 1993)

- Disadvantages: overlapping wings, parity, α concentrated near equator
- Flux tube instability: $B > B_{\text{threshold}}$ (Ferriz-Mas et al. 1994)

Interface dynamos

Parker (1993), Charbonneau and MacGregor (1997), Zhang et al. (2004):

Dynamo on interface between
- convection zone: η large, α
- overshoot layer: η small, $\partial\Omega/\partial r$, most flux

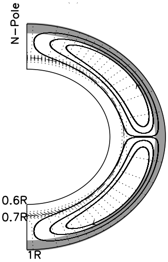


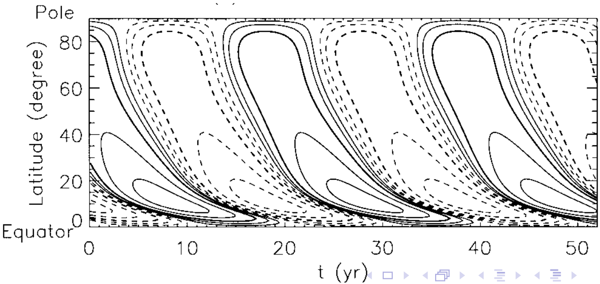
Flux transport dynamos

Durney (1995), Choudhari et al. (1995), Dikpati and Charbonneau (1999):

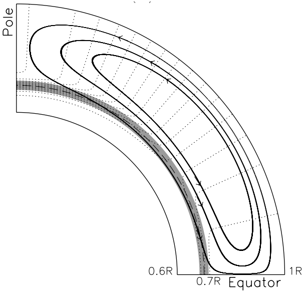
- regeneration of poloidal field through tilt of bipolar active regions close to surface (Babcock 1961, Leighton 1969)
- rotational shear in tachocline
- transport of magnetic flux by meridional circulation

$\curvearrowright$ determines migration direction and cycle period

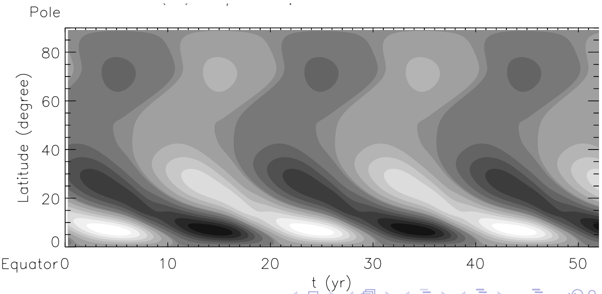




Overshoot layer dynamo with meridional circulation



(Dikpati and Gilman 2001)



MHD equations of rotating fluids in non-dimensional form

Navier-Stokes equation including Coriolis and Lorentz forces

$$E\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla^2\boldsymbol{u}\right) + 2\hat{\boldsymbol{z}}\times\boldsymbol{u} + \nabla\Pi = \frac{Ra\,E}{Pr}\frac{\boldsymbol{r}}{r_0}T + \frac{1}{Pm}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$$

Inertia Viscosity Coriolis Buoyancy Lorentz

Induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \frac{1}{Pm}\nabla\times\nabla\times\boldsymbol{B}$$

Induction Diffusion

Energy equation

$$\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T = \frac{1}{Pr}\nabla^2 T + Q$$

Incompressibility and divergence-free magnetic field

$$\nabla\cdot\boldsymbol{u} = 0 \;, \qquad \nabla\cdot\boldsymbol{B} = 0$$

Non-dimensional parameters

Control parameters (Input)

Parameter	Definition	Force balance	Model value	Earth value
Rayleigh number	$Ra = \alpha g_0 \Delta T d / \nu\kappa$	buoyancy/diffusivity	$1 - 50 Ra_{crit}$	$\gg Ra_{crit}$
Ekman number	$E = \nu / \Omega d^2$	viscosity/Coriolis	$10^{-6} - 10^{-4}$	10^{-14}
Prandtl number	$Pr = \nu / \kappa$	viscosity/thermal diff.	$2 \cdot 10^{-2} - 10^{3}$	$0.1 - 1$
Magnetic Prandtl	$Pm = \nu / \eta$	viscosity/magn. diff.	$10^{-1} - 10^{3}$	$10^{-6} - 10^{-5}$

Diagnostic parameters (Output)

Parameter	Definition	Force balance	Model value	Earth value
Elsasser number	$\Lambda = B^2 / \mu\rho\eta\Omega$	Lorentz/Coriolis	$0.1 - 100$	$0.1 - 10$
Reynolds number	$Re = ud / \nu$	inertia/viscosity	< 500	$10^{8} - 10^{9}$
Magnetic Reynolds	$Rm = ud / \eta$	induction/magn. diff.	$50 - 10^{3}$	$10^{2} - 10^{3}$
Rossby number	$Ro = u / \Omega d$	inertia/Coriolis	$3 \cdot 10^{-4} - 10^{-2}$	$10^{-7} - 10^{-6}$

Earth core values: $d \approx 2 \cdot 10^{5}$ m, $u \approx 2 \cdot 10^{-4}$ m s^{-1}, $\nu \approx 10^{-6}$ m^{2}s^{-1}

Proudman-Taylor theorem

Non-magnetic hydrodynamics in rapidly rotating system

$E \ll 1$, $Ro \ll 1$: viscosity and inertia small

balance between Coriolis force and pressure gradient

$$-\nabla p = 2\rho\boldsymbol{\Omega}\times\boldsymbol{u} \,, \quad \nabla\times: \quad (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{u} = 0$$

$\dfrac{\partial \boldsymbol{u}}{\partial z} = 0$ motion independent along axis of rotation, geostrophic motion

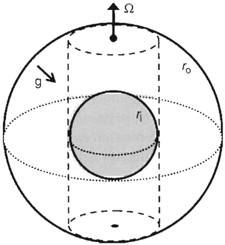
(Proudman 1916, Taylor 1921)

Ekman layer:

At fixed boundary $\boldsymbol{u} = 0$, violation of P.-T. theorem necessary for motion close to boundary allow viscous stresses $\nu\nabla^2\boldsymbol{u}$ for gradients of $\boldsymbol{u}$ in z-direction

Ekman layer of thickness $\delta_l \sim E^{1/2}L \sim 0.2$ m for Earth core

Convection in rotating spherical shell



inside tangent cylinder: $\boldsymbol{g} \parallel \boldsymbol{\Omega}$:

Coriolis force opposes convection

outside tangent cylinder:

P.-T. theorem leads to columnar convection cells

$\exp(im\varphi - \omega t)$ dependence at onset of convection,

$2m$ columns which drift in φ-direction

inclined outer boundary violates Proudman-Taylor theorem

$\curvearrowright$ columns close to tangent cylinder around inner core

inclined boundaries, Ekman pumping and inhomogeneous thermal buoyancy

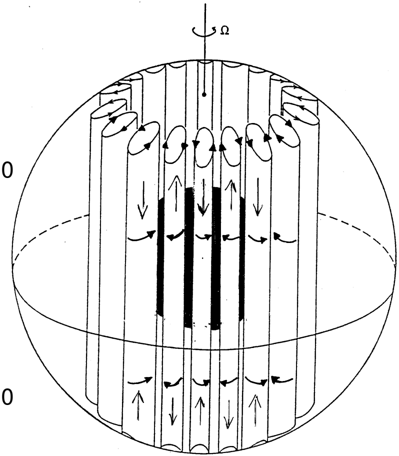
lead to secondary circulation along convection columns:

poleward in columns with $\omega_z < 0$, equatorward in columns with $\omega_z > 0$

$\curvearrowright$ negative helicity north of the equator and positive one south

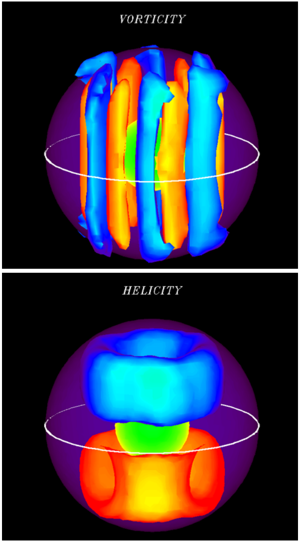
Convection in rotating spherical shell

$H < 0$



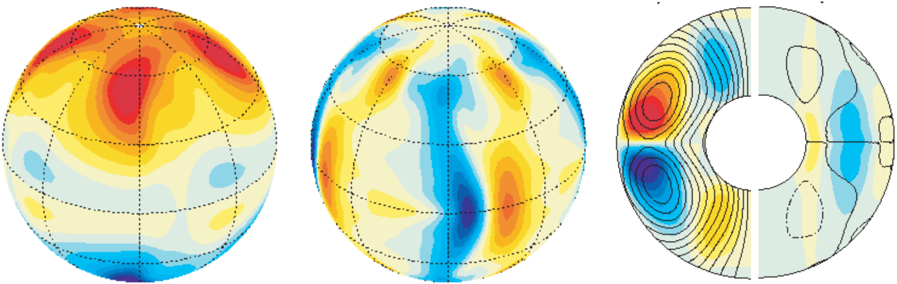
$H > 0$

$\omega_z > 0$ and < 0
cyclones / anticyclones



Benchmark dynamo

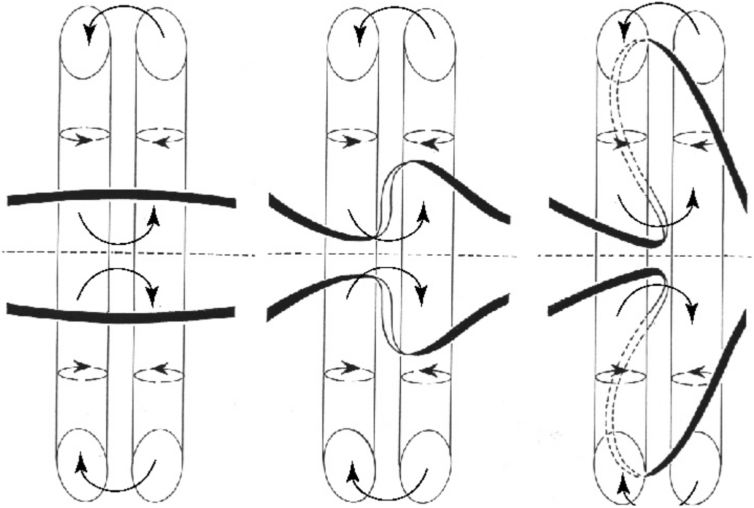
$$Ra = 10^5 = 1.8\, Ra_{\text{crit}} \,, \quad E = 10^{-3} \,, \quad Pr = 1 \,, \quad Pm = 5$$



radial magnetic field at outer radius | radial velocity field at $r = 0.83 r_0$ | axisymmetric magnetic field | axisymmetric flow

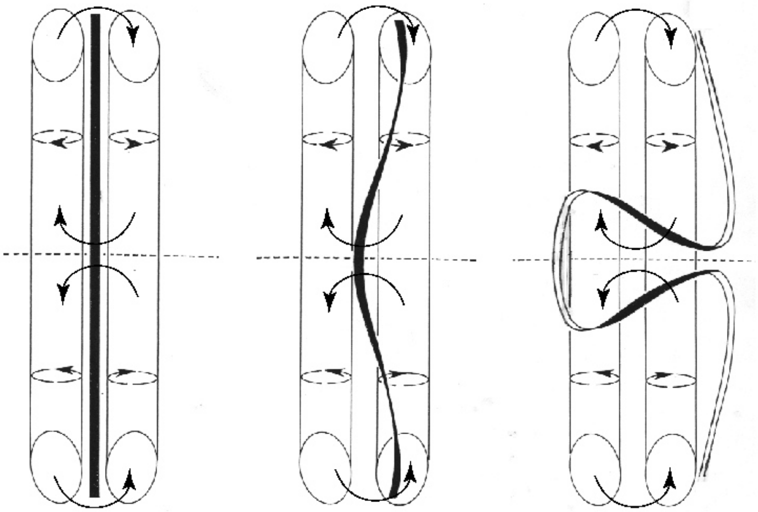
(Christensen et al. 2001)

Conversion of toroidal field into poloidal field



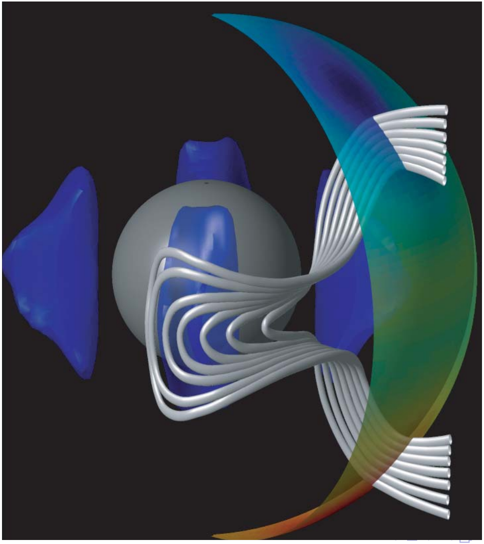
(Olson et al. 1999)

Generation of toroidal field from poloidal field



(Olson et al. 1999)

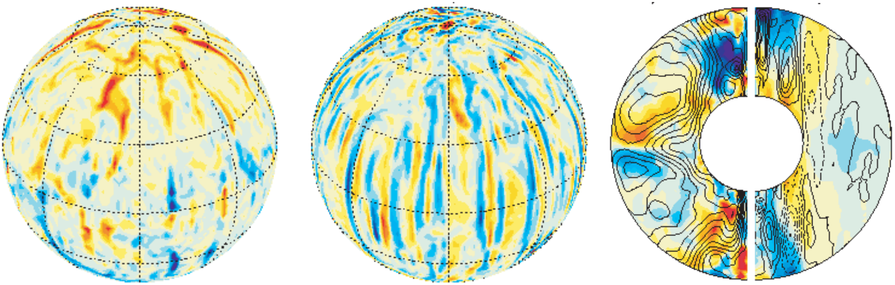
Field line bundle in the benchmark dynamo



(cf Aubert)

Strongly driven dynamo model

$$Ra = 1.2\times10^8 = 42\,Ra_{\rm crit}\ , \quad E = 3\times10^{-5}\ , \quad Pr = 1\ , \quad Pm = 2.5$$



radial magnetic field at outer radius

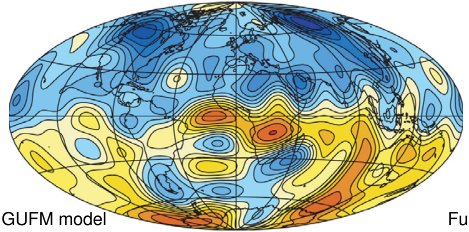
radial velocity field at $r = 0.93r_0$

axisymmetric magnetic field

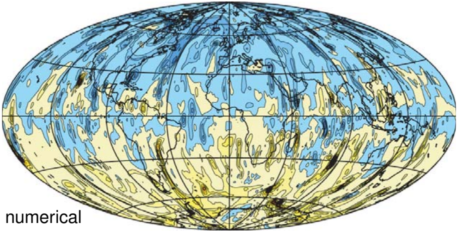
axisymmetric flow

(Christensen et al. 2001)

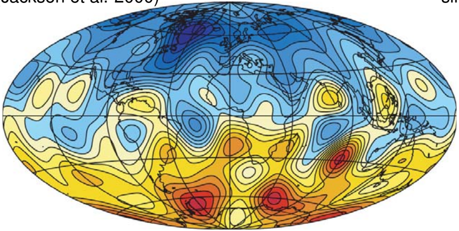
Comparison of the radial magnetic field at the CMB



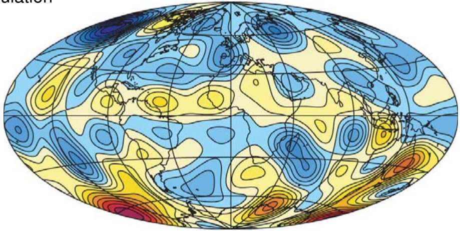
GUFM model
(Jackson et al. 2000)



Full numerical
simulation



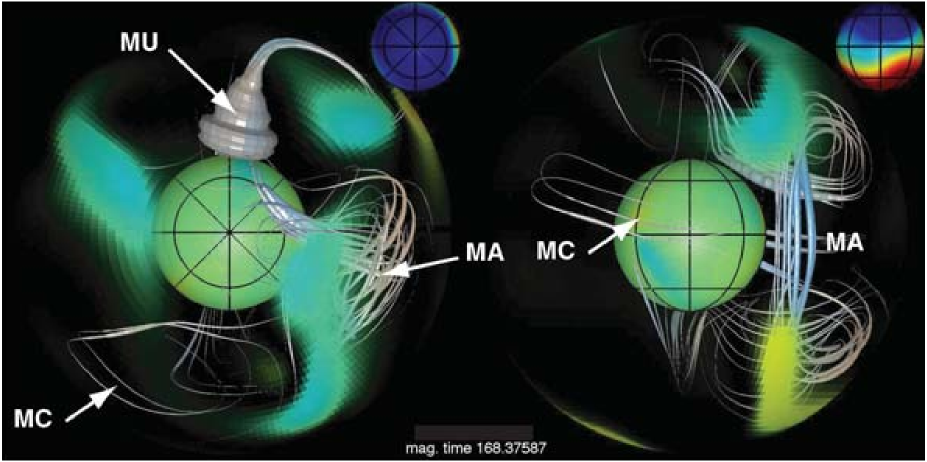
Spectrally filtered simulation at
$E = 3 \cdot 10^{-5}$, $Ra = 42\, Ra_{crit}$, $Pm = 1$, $Pr = 1$



Reversing dynamo at
$E = 3 \cdot 10^{-4}$, $Ra = 26\, Ra_{crit}$, $Pm = 3$, $Pr = 1$

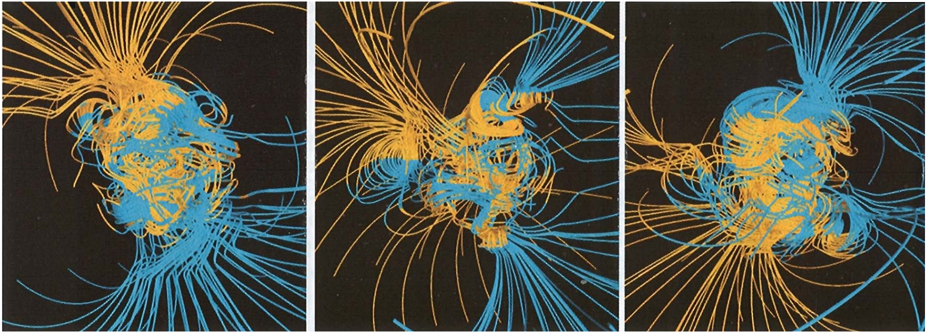
(Christensen & Wicht 2007)

Dynamical Magnetic Field Line Imaging / Movie 2



(Aubert et al. 2008)

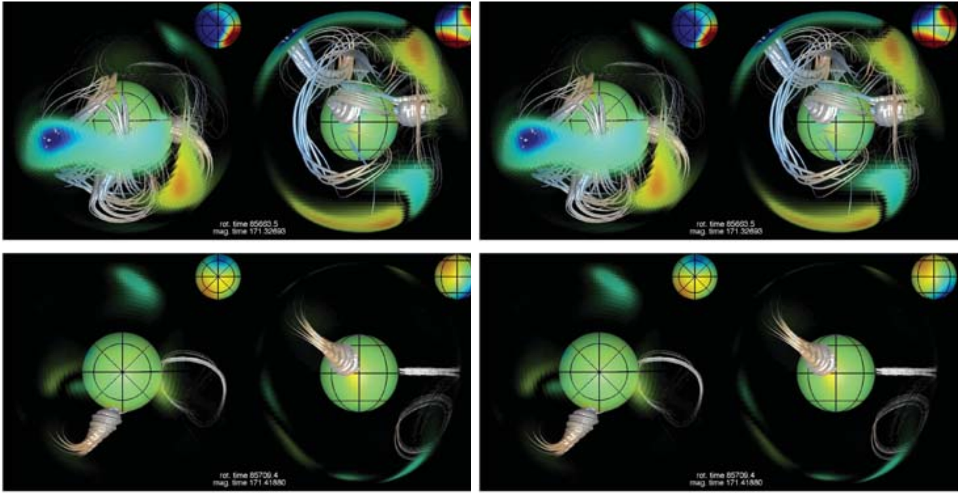
Reversals



500 years before midpoint midpoint 500 years after midpoint

(Glatzmaier and Roberts 1995)

Reversals



(Aubert et al. 2008)

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