Mathematical description of geomagnetic field

Maxwell: $\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

Insulator (mantle, air), slow changes $\rightsquigarrow \vec{\nabla} \times \vec{B} = 0$

Potential field: $\vec{B} = -\vec{\nabla}\phi \quad \rightsquigarrow \quad \nabla^2 \phi = 0$

General solution of Laplace's equation in spherical coordinates for magnetic potential:

$$\phi = a \sum_{l=1}^{\infty} \left(\frac{a}{r}\right)^{l+1} \sum_{m=0}^{l} \left(g_l^m \cos m\lambda + h_l^m \sin m\lambda\right) P_l^m(\cos\vartheta)$$

a: Earth's radius $\quad r$: radius $\quad \lambda$: longitude $\quad \vartheta$: colatitude

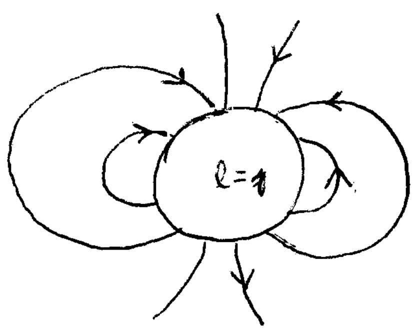
P_l^m: associated Legendre functions $\quad g_l^m, h_l^m$: Gauss coefficients

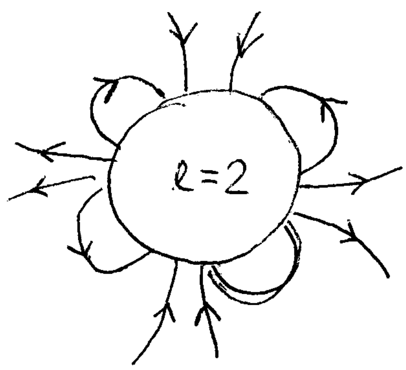
$$B_r = -\frac{\partial \phi}{\partial r} = \sum_{l=1}^{\infty} \left(\frac{a}{r}\right)^{l+2} \sum_{m=0}^{l} \left(g_l^m \cos m\lambda + h_l^m \sin m\lambda\right) P_l^m(\cos\vartheta)$$

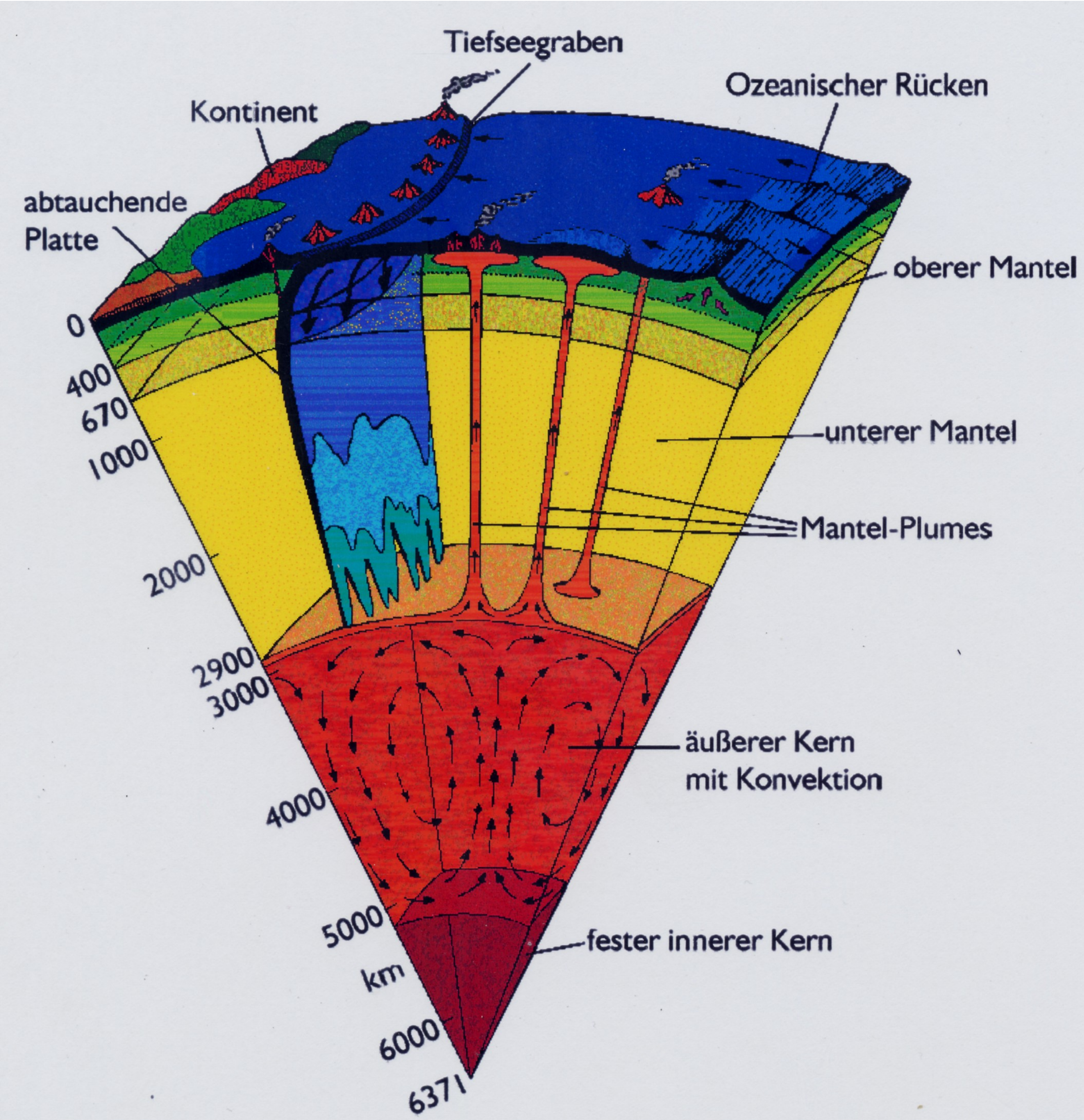
$l=1$: dipole, $l=2$: quadrupole, $l=3$: octupole etc

1980: $\quad g_1^0 = -29988\ nT \quad g_1^1 = -1957\ nT \quad h_1^1 = 5606\ nT$

$\quad |g_2^m, h_2^m| \lesssim 3000\ nT$

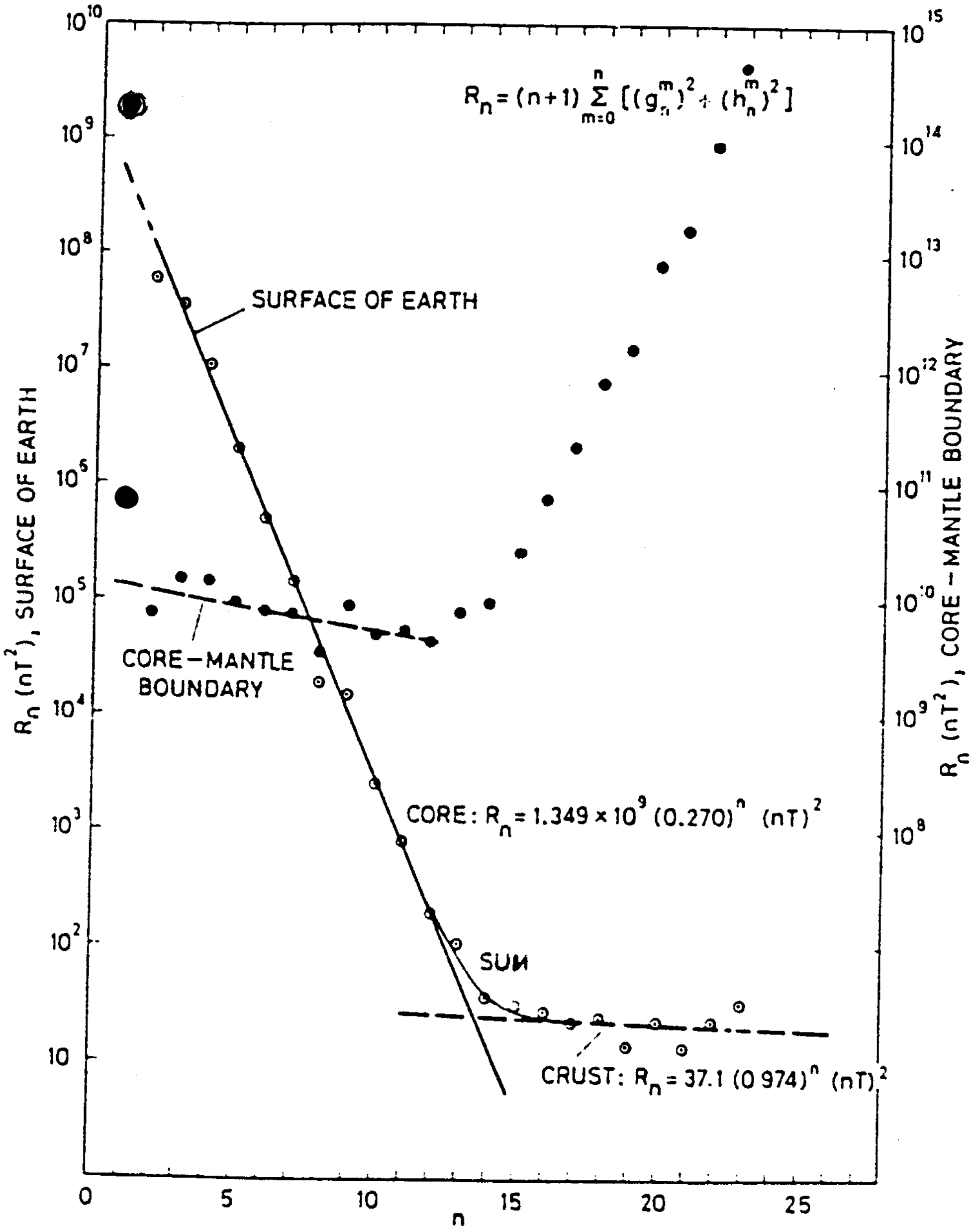




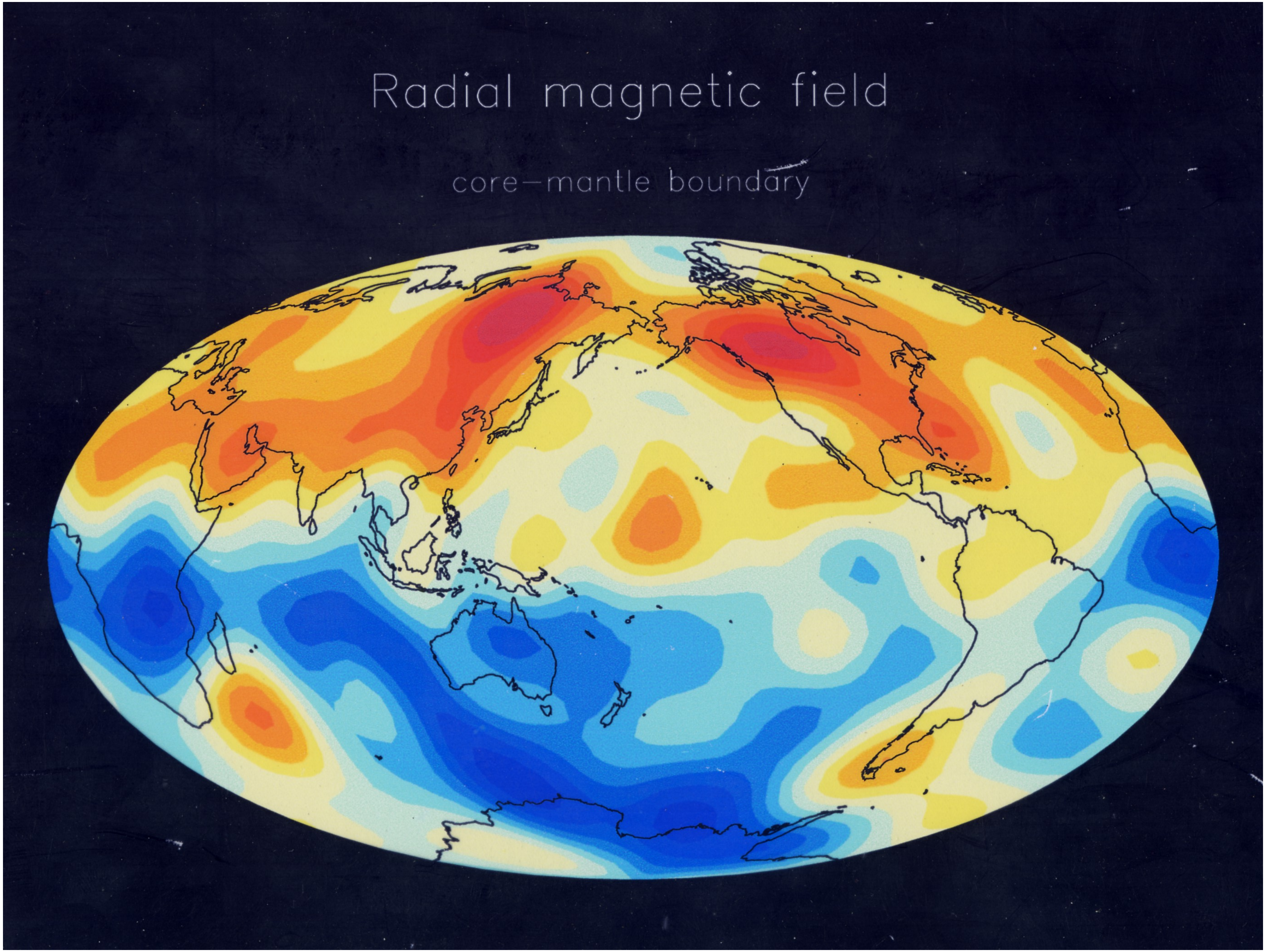


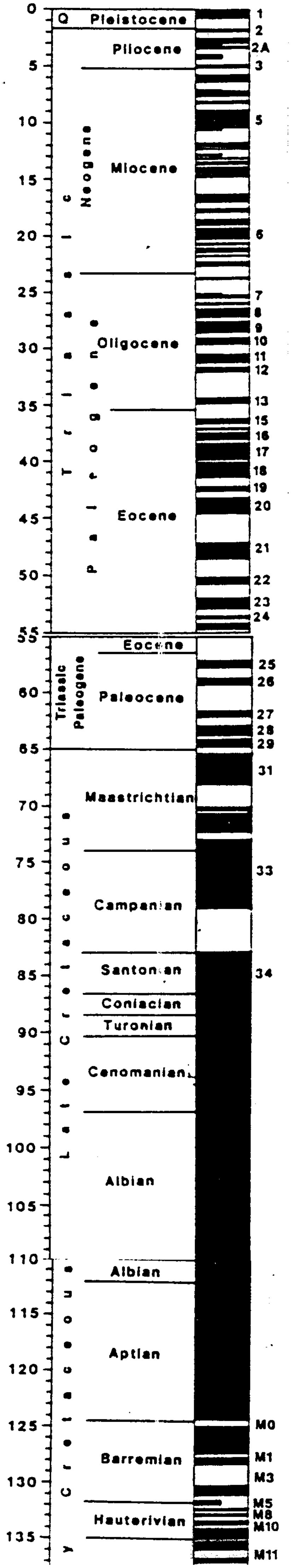
Tiefseegraben
Ozeanischer Rücken
Kontinent
abtauchende Platte
oberer Mantel
0
400
670
1000
unterer Mantel
Mantel-Plumes
2000
2900
3000
äußerer Kern mit Konvektion
4000
5000
km
fester innerer Kern
6000
6371

Magsat 1980



Kugelfunktionsgrad


Radial magnetic field
core–mantle boundary


Q
Pleistocene
Pliocene
Miocene
Neogene
Tertiary
Oligocene
Paleogene
Eocene
Eocene
Triassic
Paleogene
Paleocene
Maastrichtian
Campanian
Santonian
Coniacian
Turonian
Cenomanian
Late Cretaceous
Albian
Albian
Aptian
Barremian
Hauterivian
Early Cretaceous
0
5
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75
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90
95
100
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110
115
120
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130
135
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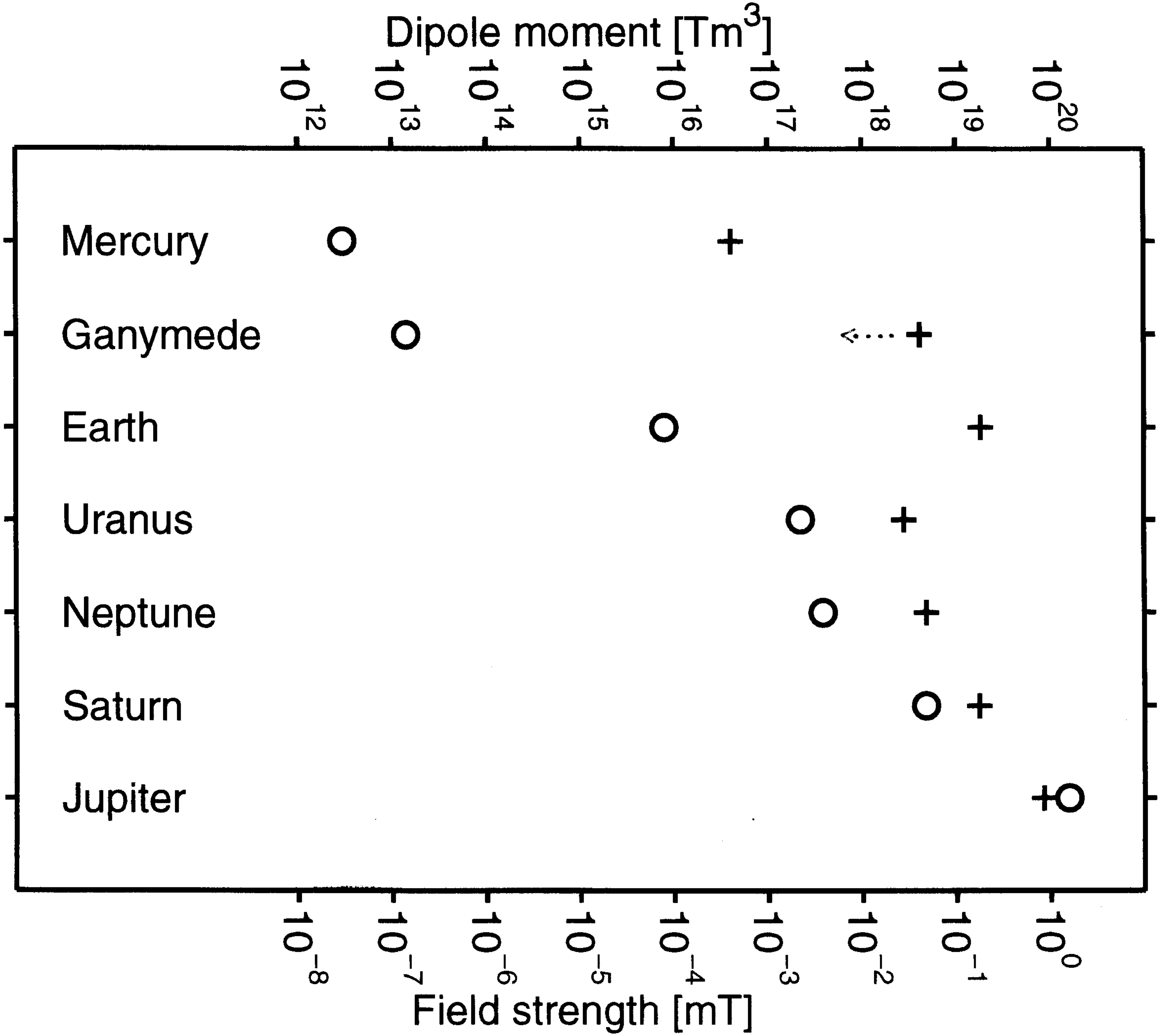
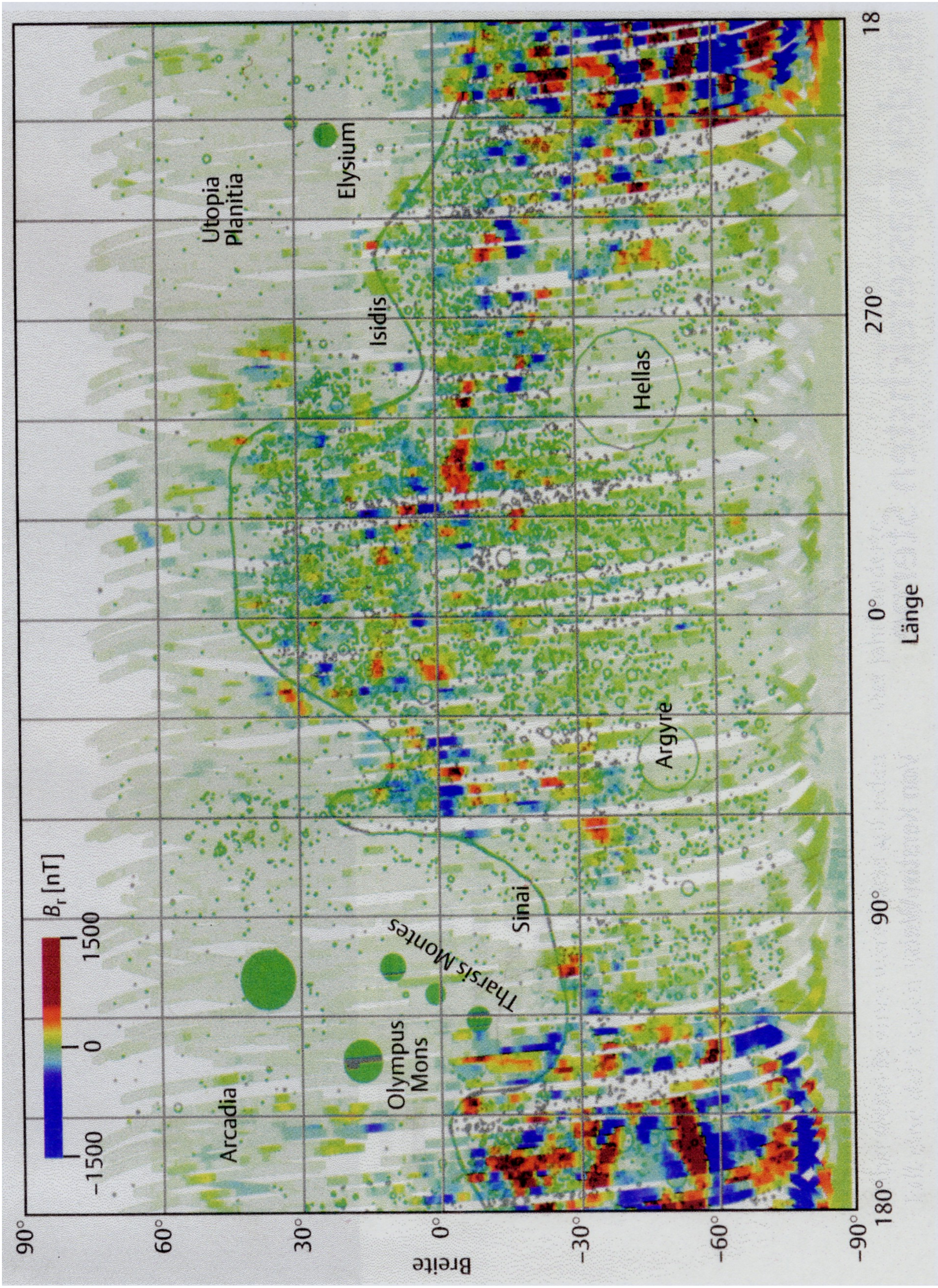
M0
M1
M3
M5
M8
M10
M11

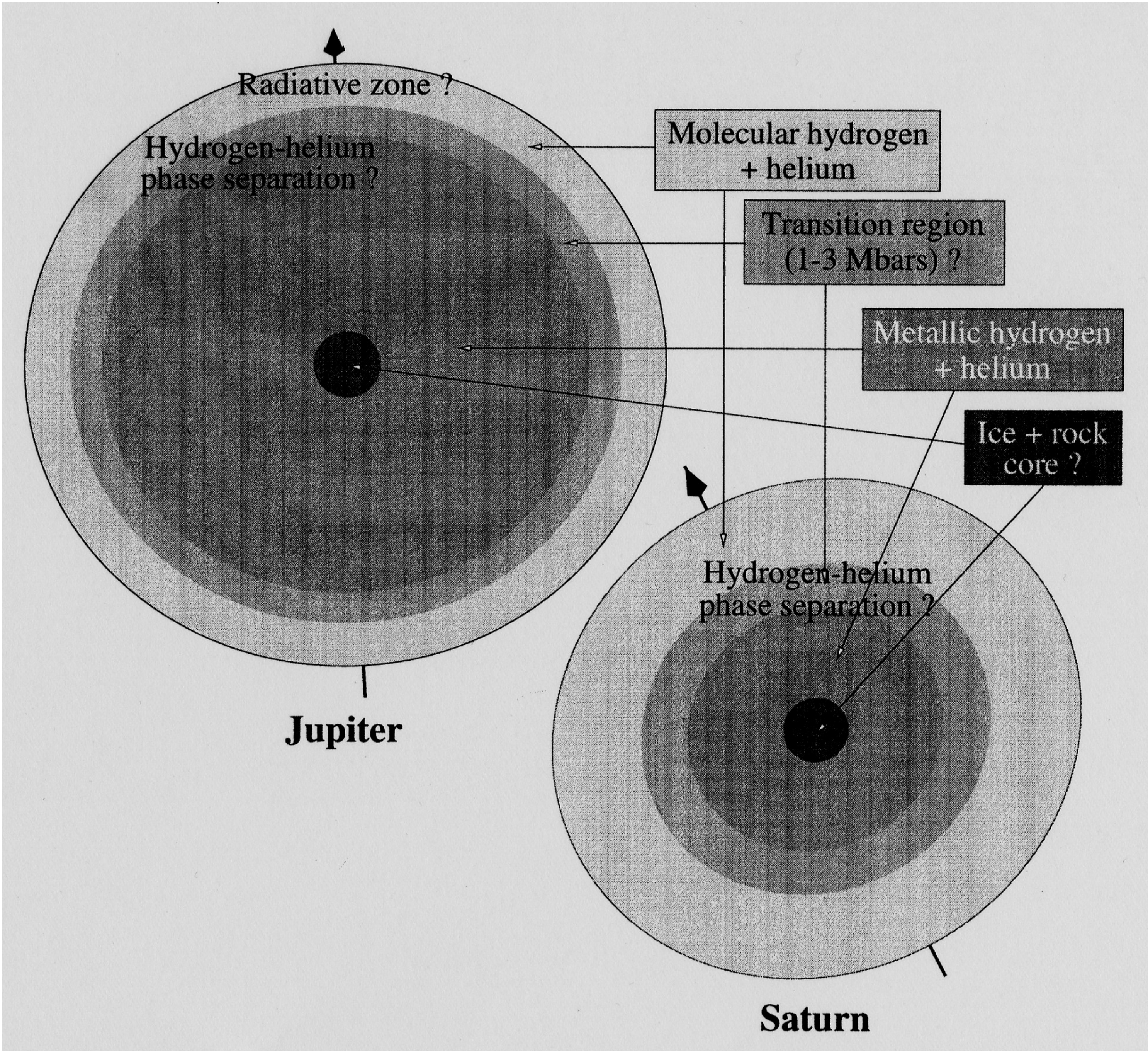
Summary of the Magnetic fields of the Planets.

Planet	Dipole moment. (Tm^3)	Tilt	Offset	Dipole: Quadrupole ratio.	Polarity Sign of g_1^0 term)	Normal solar wind pressure. dyne cm^{-2}	Sub-solar stagnation point.	Dynamo Present	Dynamo Past
Mercury	$\sim 3x10^{12}$	14°?	?	?	-ve	$6.3x10^{-8}$	$\sim 1R_{MY}$	Yes ?	
Venus	$<8x10^{10}$ (induced)	n/a	n/a	n/a	n/a	$2.6x10^{-8}$	$< 1R_V$	No	?
Earth	$7.8x10^{15}$	10.8°	$\sim 0.01R_E$	1:0.16	-ve	$1.33x10^{-8}$	$\sim 10.4R_E$	Yes	Yes
Moon	$<1.3x10^8$ (remanent)	n/a	n/a	n/a	n/a	$1.33x10^{-8}$	n/a	No	Yes ?
Mars	$\sim 10^{11}$ (induced)	n/a	n/a	n/a	n/a	$5.75x10^{-9}$	$\sim 1.5R_{MA}$	No	Yes
Jupiter	$1.56x10^{20}$	9.40°	$0.12R_J$	1:0.22	+ve	$5x10^{-10}$	$\sim 50R_J$	Yes	
Io	$\sim 8x10^{12}$ (induced?)	0°	?	?	-ve	n/a	n/a	?	
Europa	$\sim 7x10^{11}$ (induced?)	~45°	?	?	-ve	n/a	n/a	No ?	
Ganymede	$1.4x10^{13}$	~10°	?	?	-ve	n/a	n/a	Yes ?	
Saturn	$4.72x10^{18}$	< 1°	$0.04R_S$	1:0.076	+ve	$1.45x10^{-10}$	$\sim 20R_S$	Yes	
Uranus	$3.83x10^{17}$	58.6°	$0.35R_U$	1:0.70	+ve	$3.60x10^{-11}$	$18.04R_U$	Yes	
Neptune	$2.16x10^{17}$	46.9°	$0.48R_N$	1:1	+ve	$1.46x10^{-11}$	$\sim 26R_N$	Yes	

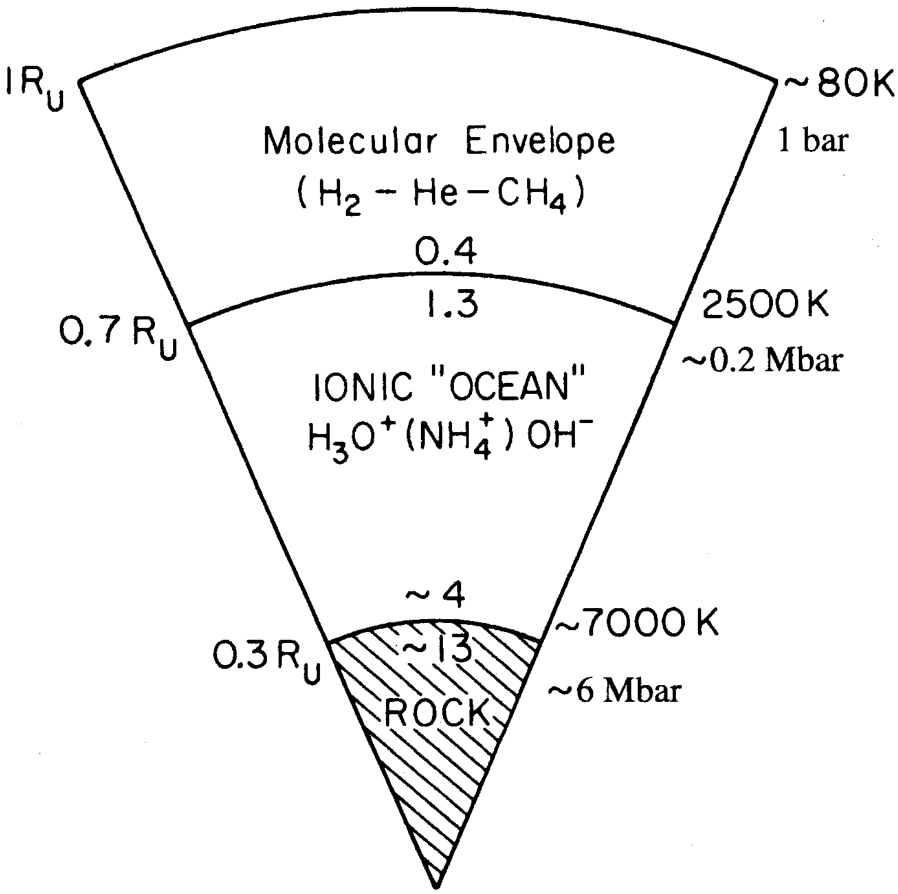

Dipole moment [Tm^3]
10^{12}
10^{13}
10^{14}
10^{15}
10^{16}
10^{17}
10^{18}
10^{19}
10^{20}
Mercury
Ganymede
Earth
Uranus
Neptune
Saturn
Jupiter
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
Field strength [mT]

Mars Magnetic Anomalies

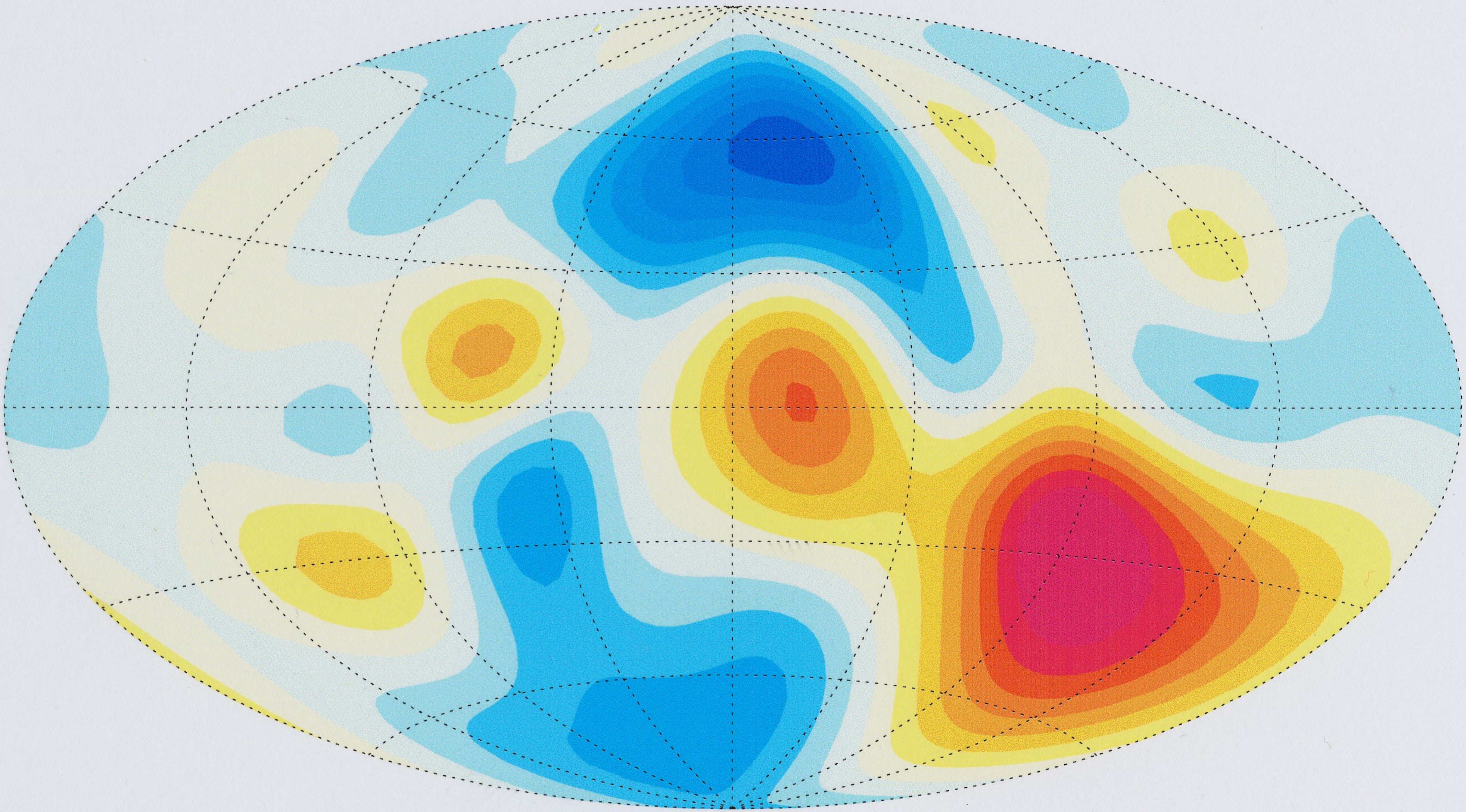



Radiative zone ?
Hydrogen-helium phase separation ?
Molecular hydrogen + helium
Transition region (1-3 Mbars) ?
Metallic hydrogen + helium
Ice + rock core ?
Hydrogen-helium phase separation ?
Jupiter
Saturn

(a) URANUS/NEPTUNE



Neptune



Equations governing fluid flow of a conducting fluid in a rotating system

Navier-Stokes equation for an inertial system:

$$\underbrace{\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right)}_{\mathcal{D}\vec{v}/\mathcal{D}t} = -\vec{\nabla}p + \rho\nu\nabla^2\vec{v} + \rho\vec{g} + \vec{F}$$

Inertial acceleration — Viscosity — Gravity — Other forces

Continuity: $\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\cdot\vec{v}) = 0$; incompressible $\vec{\nabla}\cdot\vec{v} = 0$

Transformation to rotating system; angular velocity $\vec{\Omega} = \Omega\vec{e}_z$

$$\left[\mathcal{D}\vec{v}/\mathcal{D}t\right]_{inertial} = \left[\mathcal{D}\vec{v}/\mathcal{D}t\right]_{rot} + \underbrace{2\vec{\Omega}\times\vec{v}_R}_{\text{Coriolis}} + \underbrace{\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}_{\text{Centrifugal}}$$

Lorentz force: $\vec{F} = \vec{j}\times\vec{B} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}$

Combine centrifugal + gravity forces to "effective gravity" $\vec{g}\,'$

Scale all properties: velocity scale U, length scale L, time scale L/U etc $\rightsquigarrow$ dimensionless equation

$$Ro\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right) = -\vec{\nabla}p + 2\vec{e}_z\times\vec{v} + E\,\nabla^2 v + \vec{F} + \rho\vec{g}\,'$$

$Ro = \frac{U}{\Omega L}$: Rossby number $\sim \frac{\text{Inertial forces}}{\text{Coriolis force}}$

$E = \frac{\nu}{\Omega L^2}$: Ekman number $\sim \frac{\text{Viscous forces}}{\text{Coriolis force}}$

Estimates:

	Ro	E
Sun	0.3	3×10^{-14}
Earth's core	10^{-6}	10^{-15}

Proudman-Taylor theorem:

Primary force balance between Coriolis and pressure:

$$2\,\vec{e}_z \times \vec{v} = \vec{\nabla} p$$

Take curl:

$$\frac{\partial}{\partial z}\,\vec{v} = 0$$

Flow cannot change in direction of rotation axis!

Convection in rotating systems:

Solve Navier-Stokes equation coupled with equation for advection and diffusion of temperature (or composition)

Boussinesq-approximation: assume constant density in all terms, except in term for gravity forces to allow for buoyancy driving a flow.

Because flow velocity is not known a-priori, use different scaling (time scale L^2/κ, κ: thermal diffusivity):
(or L^2/ν ν: viscosity)

$$E\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right) = -\vec{\nabla} p + 2\vec{e}_z \times \vec{v} + E\,\nabla^2 \vec{v} + Ra \cdot E \cdot \frac{\vec{r}}{r_c} + \dots$$

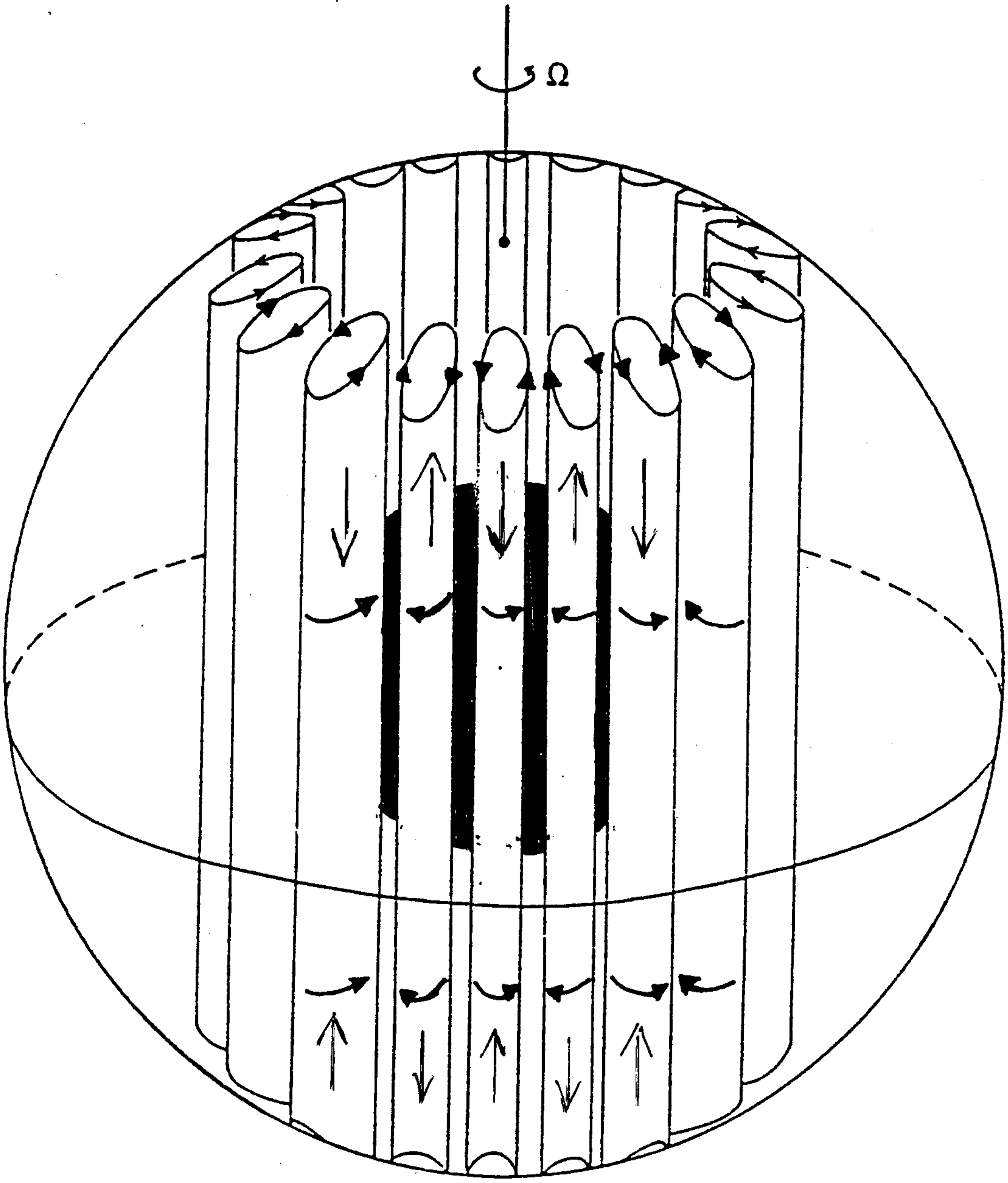
$$\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla} T = \frac{1}{Pr}\cdot \nabla^2 T$$

$$\vec{\nabla}\cdot\vec{v} = 0$$

Prandtl number $Pr = \nu/\kappa$ — α: thermal expans. coeff.

Rayleigh number $Ra = \dfrac{\alpha g \Delta T L^3}{\kappa \nu}$ — ΔT: temp. difference

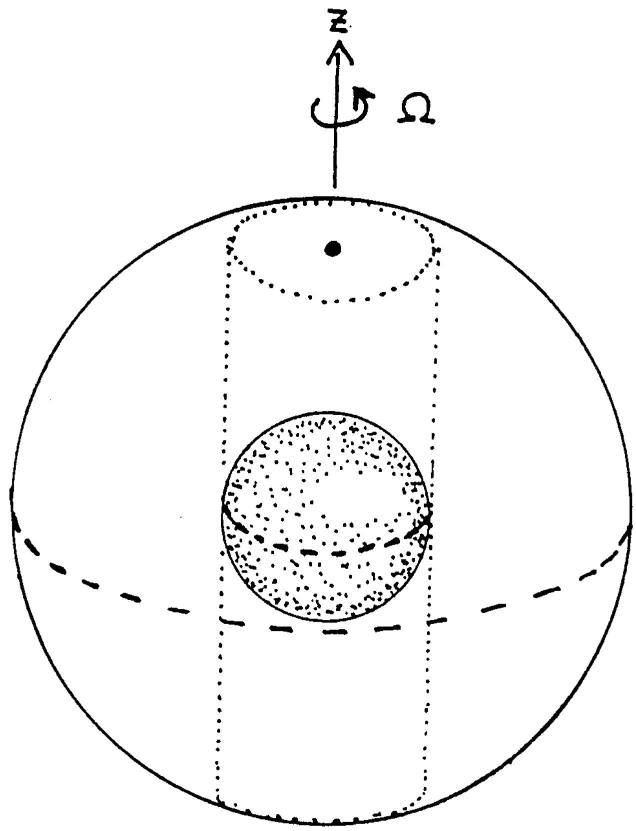
critical Rayleigh number $Ra_c \sim E^{-4/3}$


Ω

Outline of geodynamo models

Thermal (chemical) convection and magnetic induction in a rotating and conducting spherical shell

- three-dimensional and time-dependent
- based on first principles only : fundamental MHD equations
- But: some parameters not „Earth-like“
- differences between models:
  parameter values
  mechanical/thermal/magnetic boundary conditions
  parameterisations of turbulence (hyperviscosity)



$$E\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} - \nabla^2 \vec{u}\right) + \vec{\nabla} p + 2\vec{e}_z \times \vec{u} = Ra \overset{E}{\vee} T \frac{\vec{r}}{r_o} + \frac{1}{Pm} (\vec{\nabla} \times \vec{B}) \times \vec{B}$$

Inertia Viscosity Coriolis Buoyancy Lorentz

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \frac{1}{Pm} \nabla^2 \vec{B}$$

Advection & Field generation Diffusion

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \frac{1}{Pr} \nabla^2 T$$

$$\vec{\nabla} \cdot \vec{u} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

Non-dimensional parameters

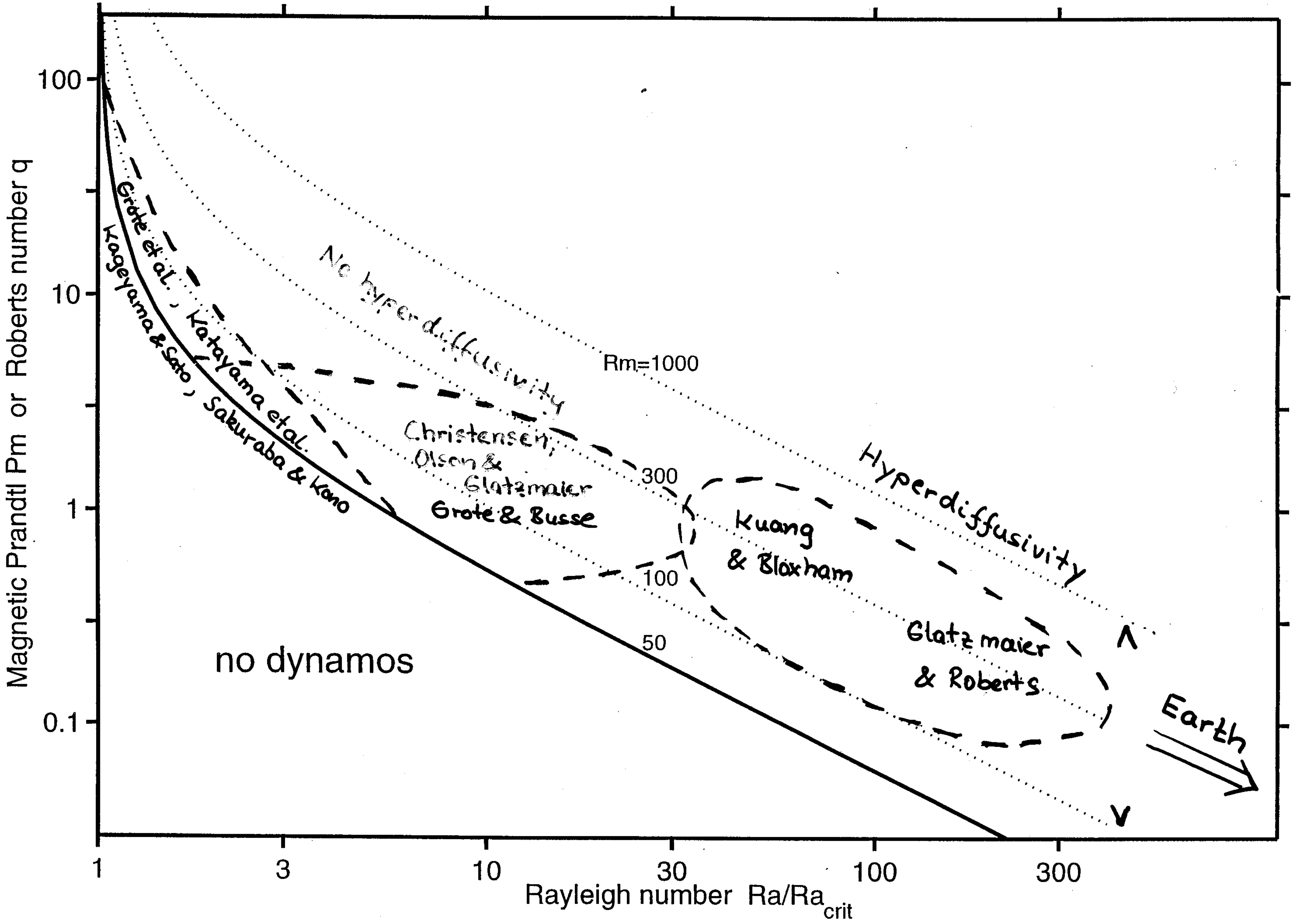
Control parameters (input)

Parameter	Definition	Force balance	Model value	Earth value
Rayleigh number	$Ra = \alpha g \Delta T d^3 / \kappa \nu$	buoyancy/diffusion	- 40× critical	?
Ekman number	$E = \nu / \Omega\, d^2$	viscosity/Coriolis	$\geq 3\times10^{-5}$	10^{-14}
Prandtl number	$Pr = \nu/\kappa$	viscosity/thermal diff.	0.3 – 10	0.1 - 1
Magnetic Prandtl	$Pm = \nu/\eta$	viscosity/magnetic diff.	0.5 – 5	10^{-6}

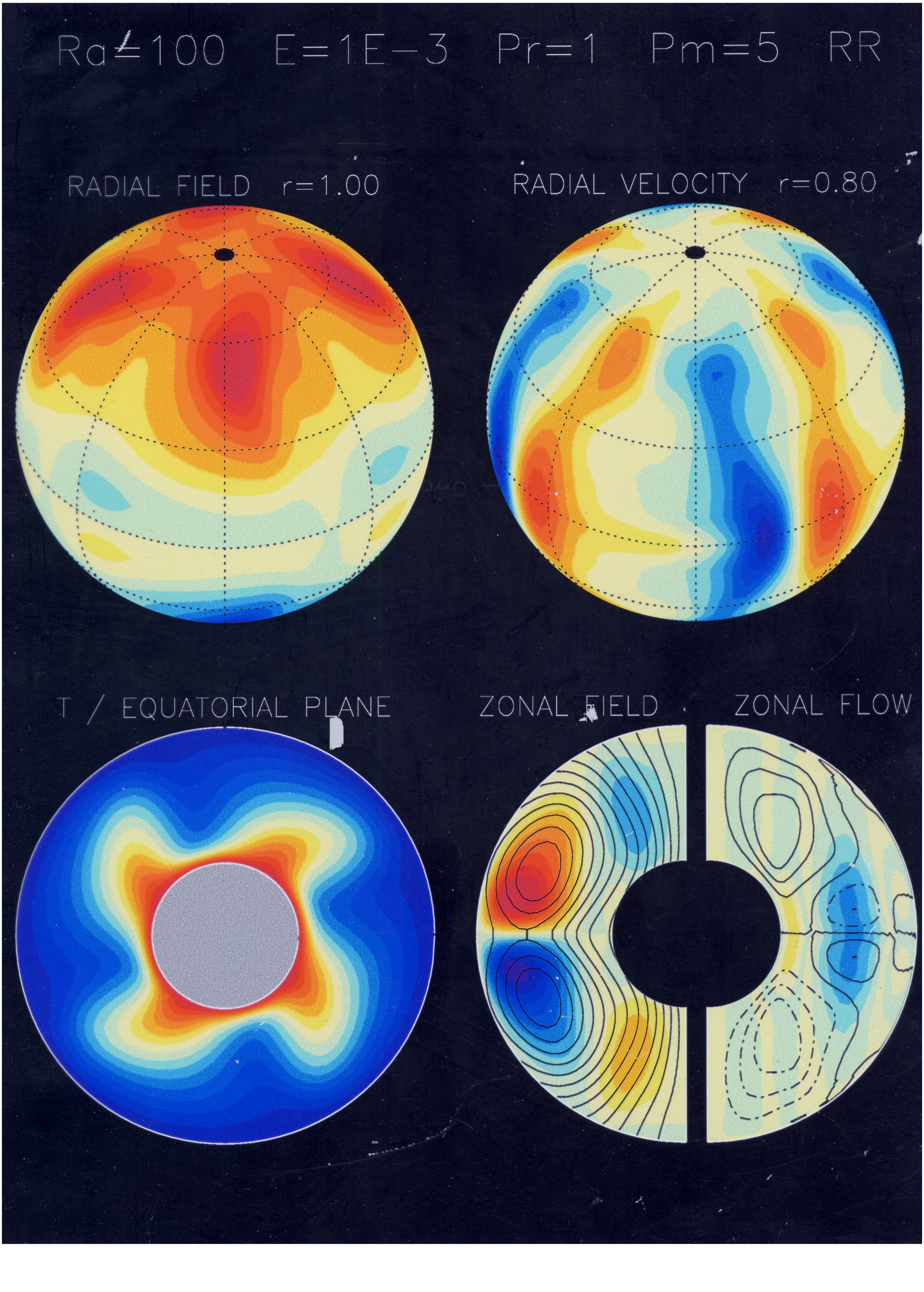
Diagnostic parameters (output)

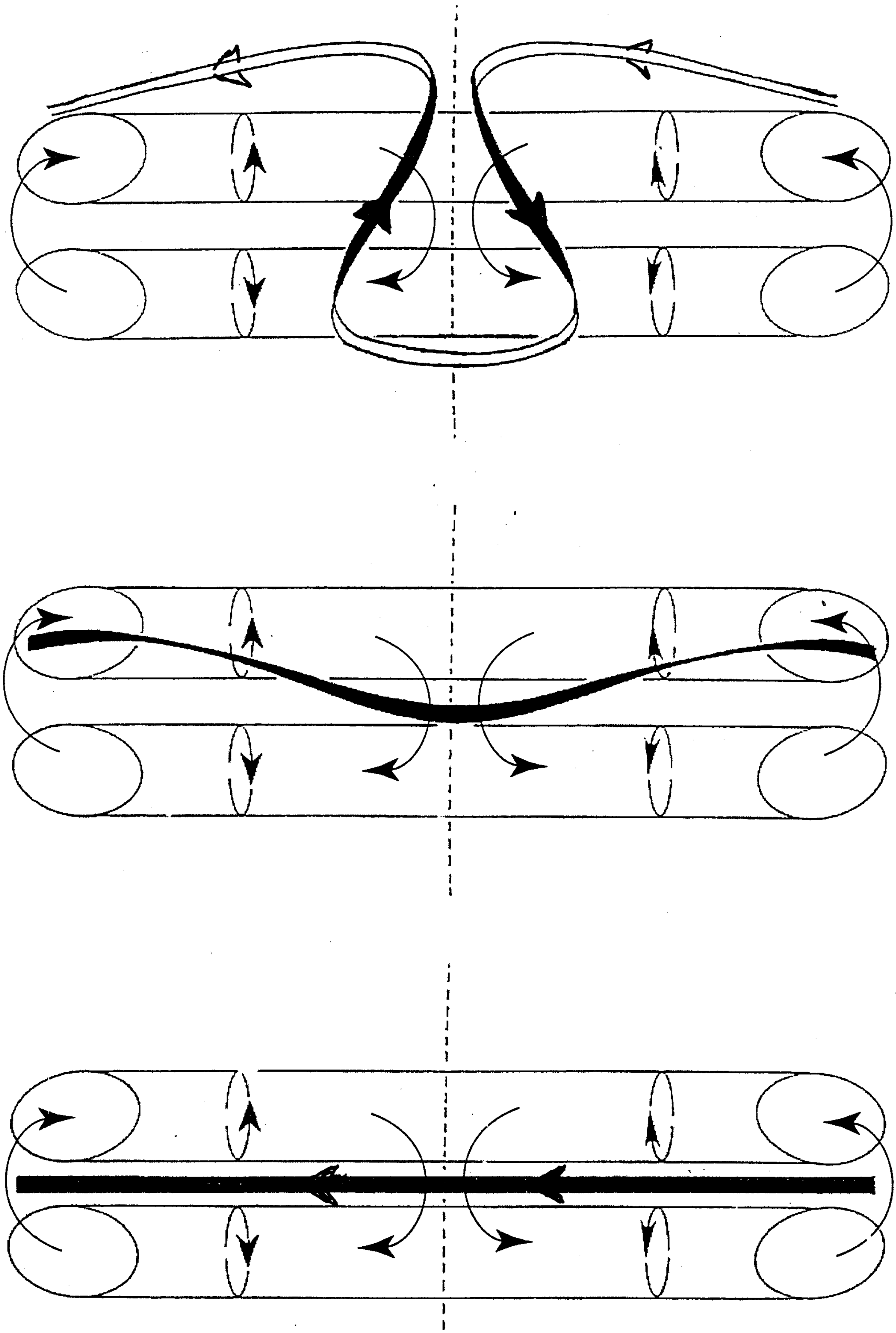
Parameter	Definition	Force balance	Model value	Earth value
Elsasser number	$\Lambda = B^2/\mu\rho\eta\Omega$	Lorentz / Coriolis	0.15 – 15	0.1 – 10 ?
Reynolds number	$Re = Ud/\nu$	inertia/viscosity	8 – 200	10^8
Magnetic Reynolds	$Rm = Ud/\eta$	advection/magnetic diff.	40 – 400	100 - 1000
Rossby number	$Ro = U/\Omega d$	inertia/Coriolis	$10^{-3} - 10^{-2}$	10^{-6}

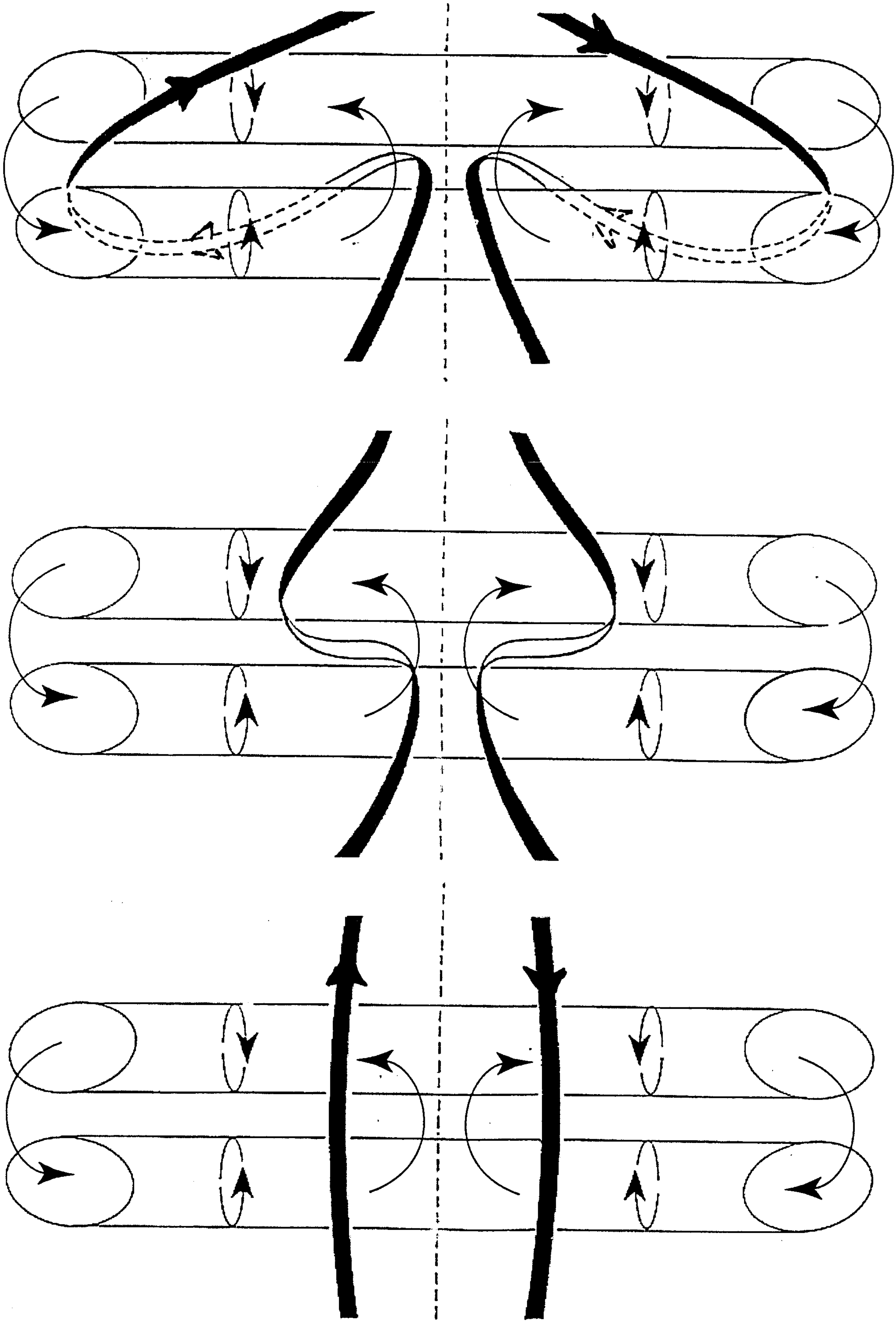
α: thermal expansion coeff.
ΔT: temperature contrast
Ω: rotational frequency
g: gravity
ν: kinematic viscosity
μ: magn. permeability
d: shell thickness
κ: thermal diffusivity
U: velocity
ρ: density
B: magnetic field
η: magnetic diffusivity



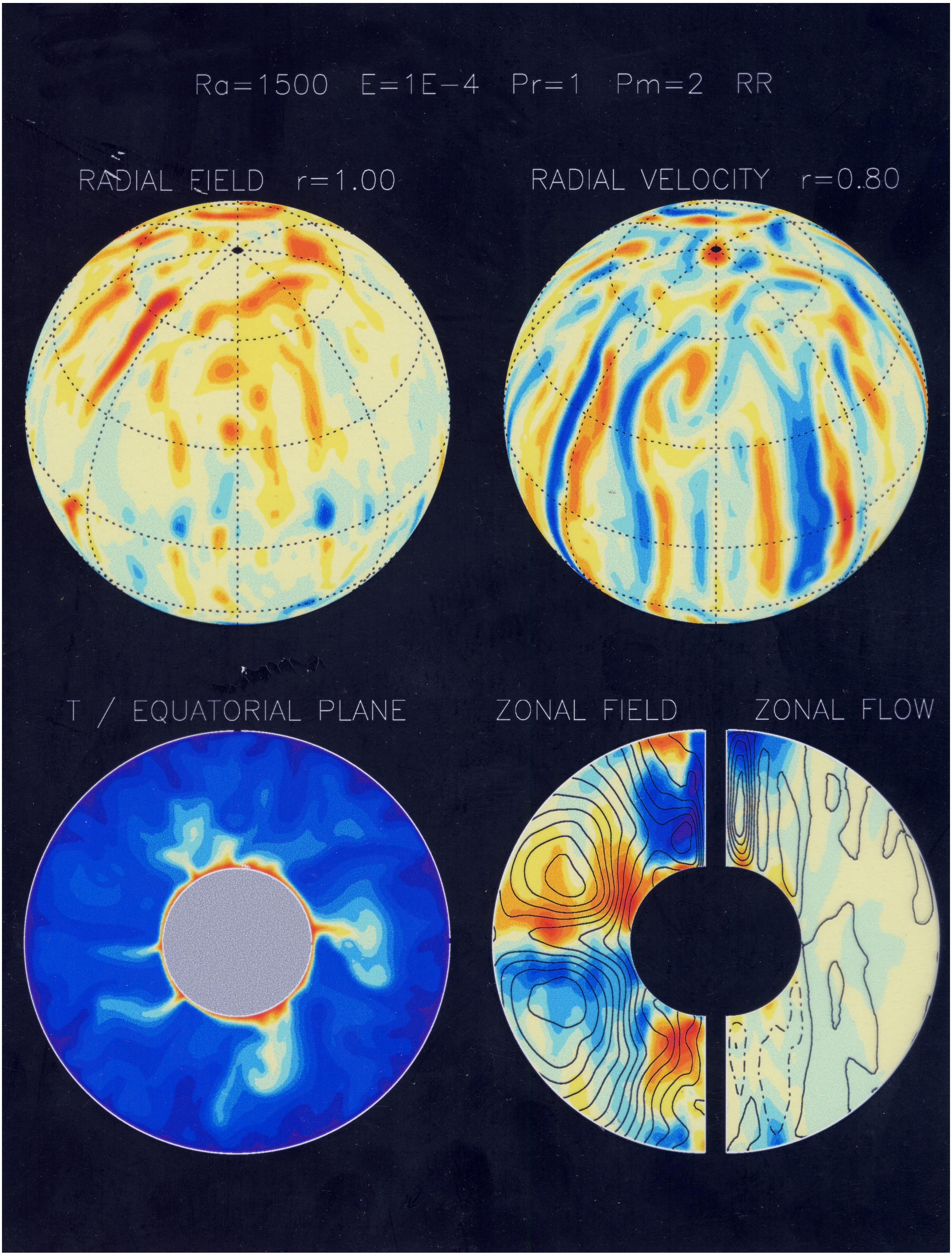
Magnetic Prandtl Pm or Roberts number q
100
10
1
0.1
1
3
10
30
100
300
Rayleigh number Ra/Ra crit
Rm=1000
300
100
50
no dynamos
Grote et al. , Katayama et al.
Kageyama & Sato , Sakuraba & Kano
No hyperdiffusivity
Christensen, Olson & Glatzmaier
Grote & Busse
Kuang & Bloxham
Hyperdiffusivity
Glatzmaier & Roberts
Earth

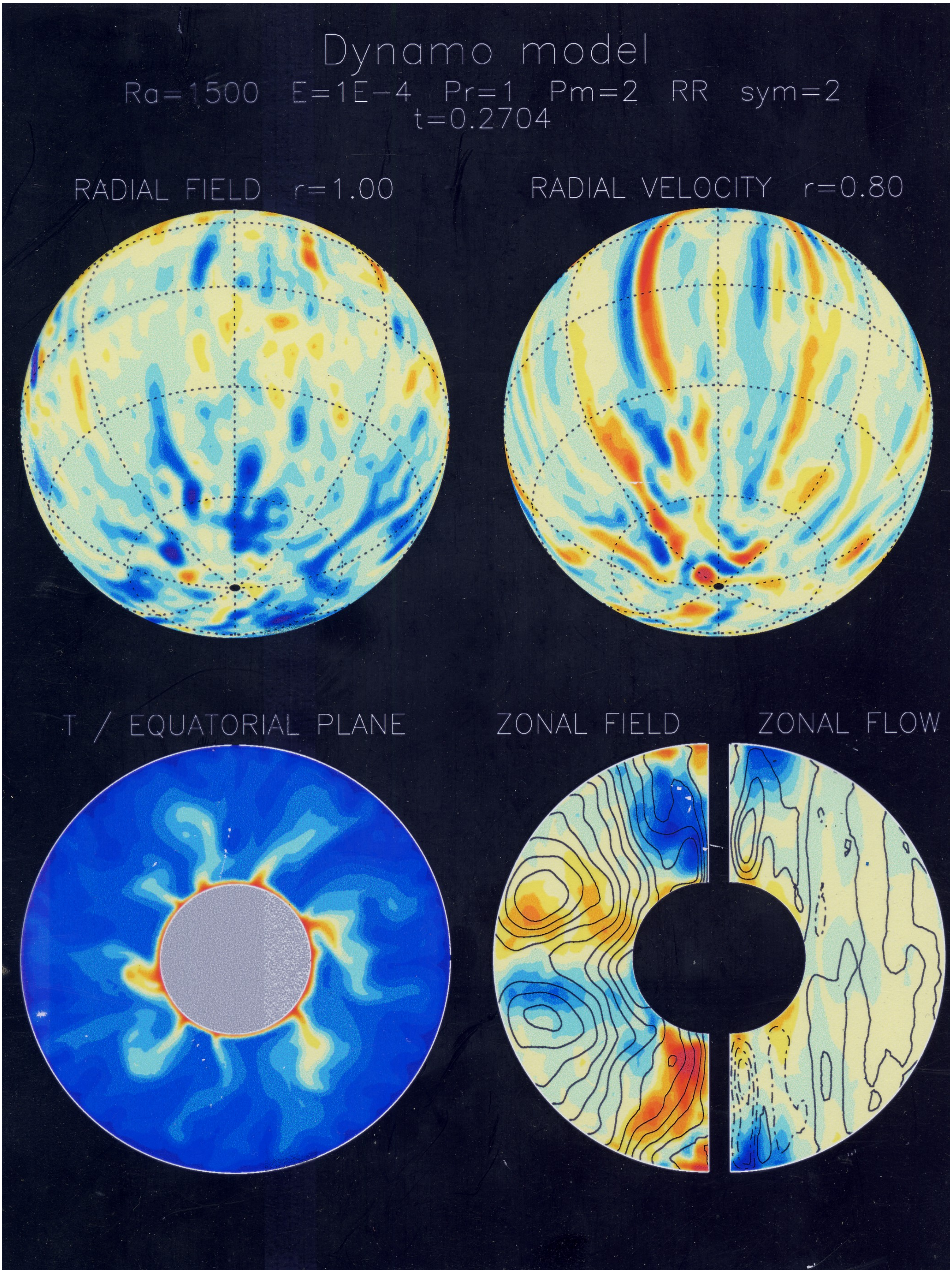

Ra=100 E=1E-3 Pr=1 Pm=5 RR
RADIAL FIELD r=1.00
RADIAL VELOCITY r=0.80
T / EQUATORIAL PLANE
ZONAL FIELD
ZONAL FLOW

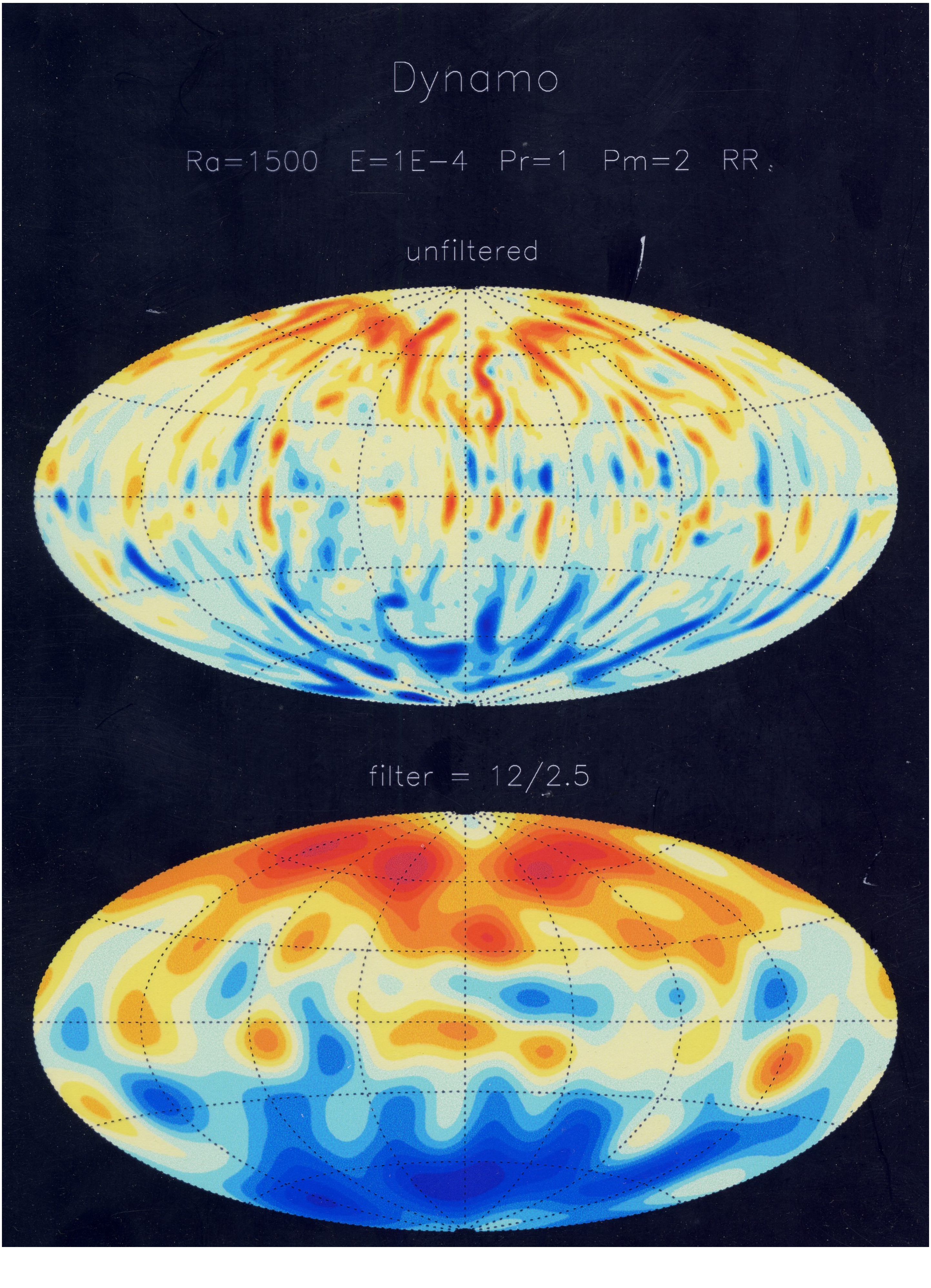




Yellow: Isosurface of anticyclonic z-vorticity

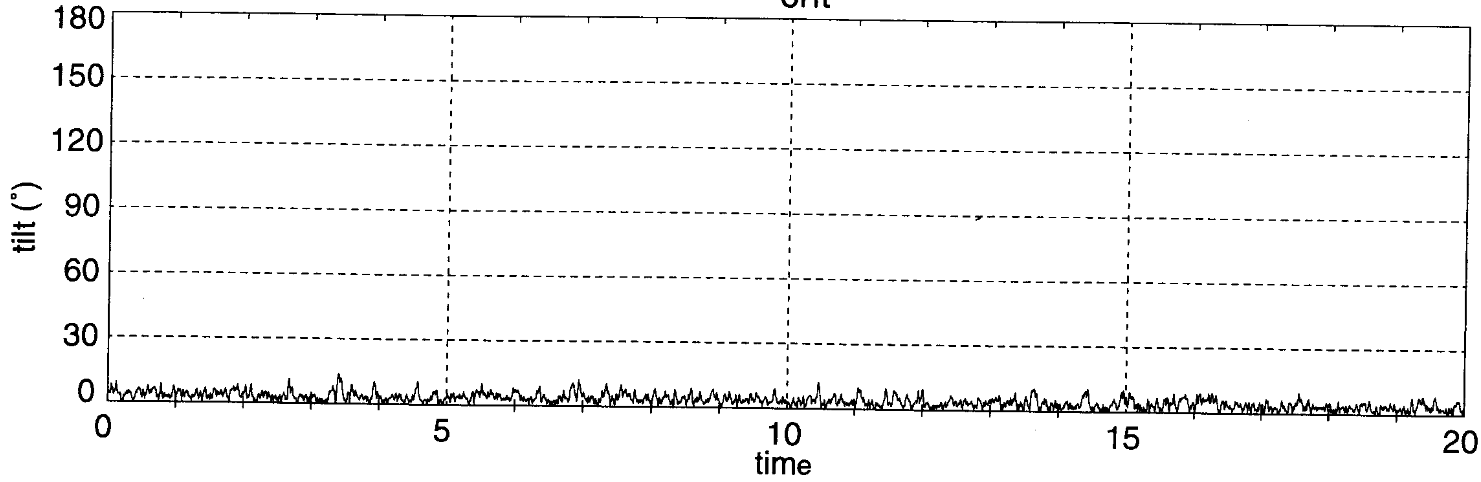

Ra=1500 E=1E-4 Pr=1 Pm=2 RR
RADIAL FIELD r=1.00
RADIAL VELOCITY r=0.80
T / EQUATORIAL PLANE
ZONAL FIELD
ZONAL FLOW


Dynamo model
Ra=1500 E=1E-4 Pr=1 Pm=2 RR sym=2
t=0.2704
RADIAL FIELD r=1.00
RADIAL VELOCITY r=0.80
T / EQUATORIAL PLANE
ZONAL FIELD
ZONAL FLOW

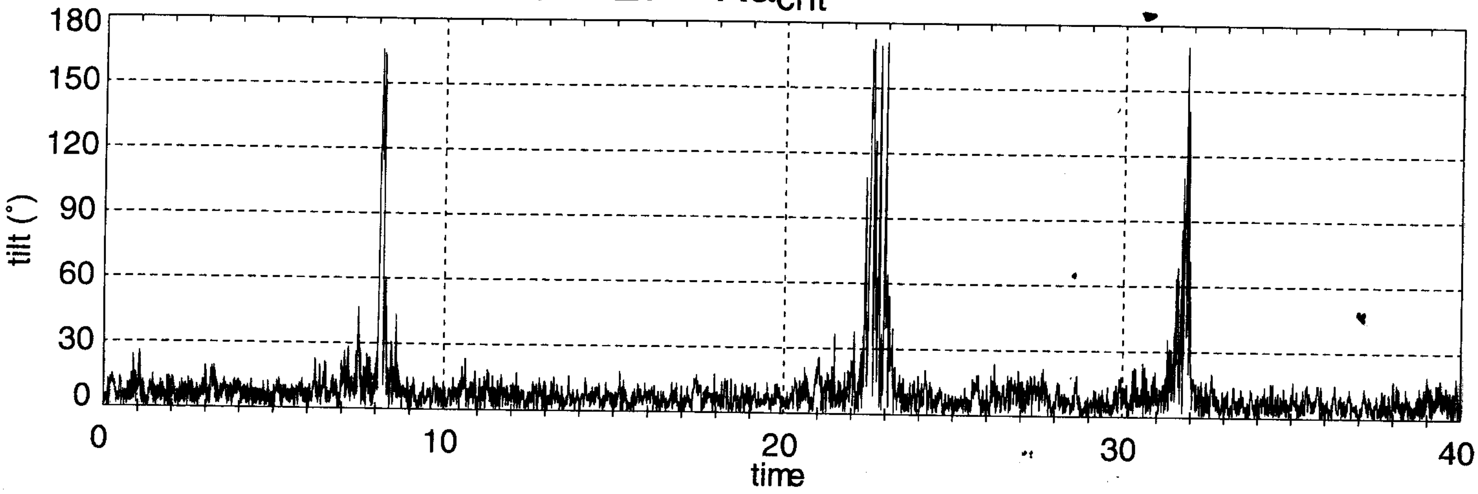

Dynamo
Ra=1500 E=1E-4 Pr=1 Pm=2 RR
unfiltered
filter = 12/2.5

tilt of dipole axis versus time
chemical convection, $E=3\cdot10^{-4}$, Pm=3

Ra = 17 * Ra_{crit}



Ra = 27 * Ra_{crit}



Ra = 35 * Ra_{crit}