INTRODUCTION

TO

GENERAL AND MEAN-FIELD

DYNAMO THEORY

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1.

COSMIC MAGNETIC FIELDS AND THE QUESTION OF THEIR ORIGIN

Table 1. Magnetic fields of various cosmic objects and their spatial extents. All values of the magnetic flux densities and the linear dimensions of the objects have to be understood as orders of magnitude only.

Object	Magnetic flux density [T]	Linear dimension of the object [m]	Symmetry and time behaviour of the magnetic field
Earth	10^{-4}	10^{7} (10^{4} km)	slight deviations from symmetry about rotation axis and equatorial plane, non-oscillatory, reversals
Planets	$10^{-8} \cdots 10^{-3}$	$10^{6} \cdots 10^{8}$ ($10^{3} \cdots 10^{5}$ km)	various degrees of symmetry
Sun	some 10^{-1} (in spots)	10^{9} (10^{6} km)	slight deviations from symmetry about rotation axis and equatorial plane oscillatory, magnetic cycle, grand minima
Cool stars (F, G)		10^{9} (10^{6} km)	sun-like magnetic cycles
Hot stars (A, B)	1	10^{9} (10^{6} km)	oblique rotators
White dwarfs	10^{4}	10^{5} (100 km)	
Neutron stars	10^{8}	10^{4} (10 km)	oblique rotators
Galaxies	10^{-9}	10^{21} (30 kpc)	"axisymmetric" and "bisymmetric" structures

Suggestions
concerning the origin
of cosmic magnetic fields

$$\nabla \times H = j$$
$$\nabla \cdot B = 0$$

- Permanently magnetized matter
  ferromagnetism
  Curie-temperature $< 1000\,^{\circ}C$

$$j = 0$$
$$B = \mu_0 (H + \underline{M})$$

- Electric currents

  — relic currents / fields
  decay time $\tau = \mu_0 \sigma L^2$
  Earth : 10^4 yrs
  Sun-like bodies : $\gtrsim$ age of the universe

$$j = \sigma \underline{E}$$
$$B = \mu_0 H$$

  — currents driven by electromotive forces
  independent of electromagnetic fields,
  e.g.,
  - chemical inhomogeneities
  - chemical and temperature
    inhomogeneities (Seebeck-effect)
  - Nichols-Tolman-effect

  ⋮

$$j = \sigma (E + \underline{E^{(e)}})$$
$$B = \mu_0 H$$

  — currents driven by fluid motions
  in a magnetic field

  __DYNAMO__

$$j = \sigma (E + \underline{u \times B})$$
$$B = \mu_0 H$$

2.

THE IDEA OF THE SELF-EXCITING DYNAMO

Sir Joseph Larmor 1919

How Could a Rotating Body such as the Sun Become a Magnet ?

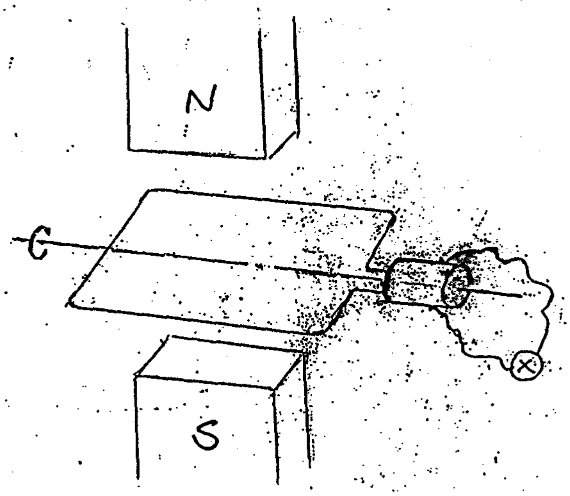
Rep. 87th Meeting Brit. Assoc. Adv. Sci.
Bournemouth 1919, Sept. 9 - 13,
John Murray London, pp 159 - 160

... internal motion induces an electrical field
acting on the moving matter ...

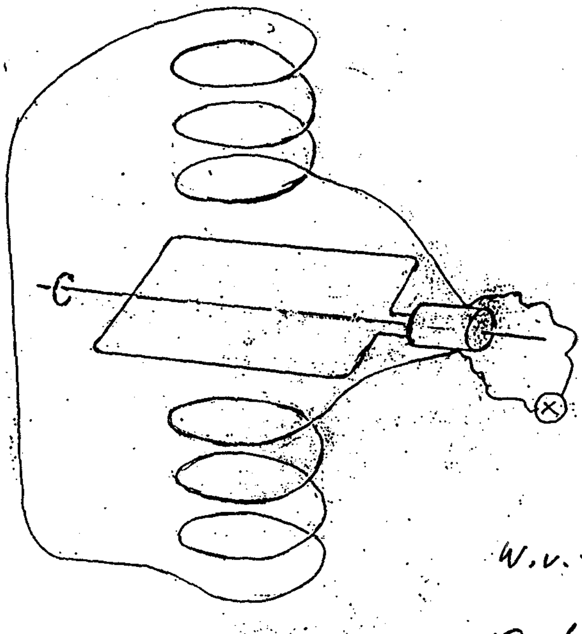
... if a conducting path ... happens to be open,
an electric current will flow round it,
which may in turn increase the inducing magnetic field.

In this way it is possible for the internal cyclic motion
to act after the manner
of the cycle of a self - exciting dynamo,
and maintain a permanent magnetic field
from insignificant beginnings,
at the expense of some of the energy
of the internal circulation.

... earth's magnetic field ...
... it would require fluidity and residual circulation
in deepseated regions.



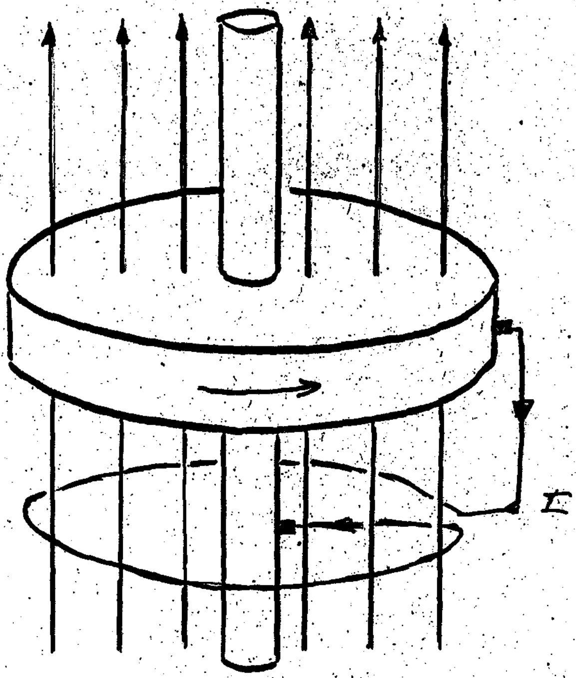
EXTERNALLY
EXCITED
DYNAMO



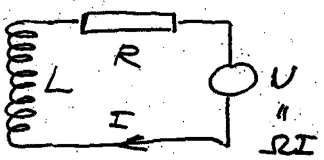
SELF-
EXCITING
DYNAMO

W. v. Siemens 1867
(1857?)
Ch. Wheatstone 1867

DISK DYNAMO



angular velocity Ω



$$L\dot{I} + RI = \Omega I$$

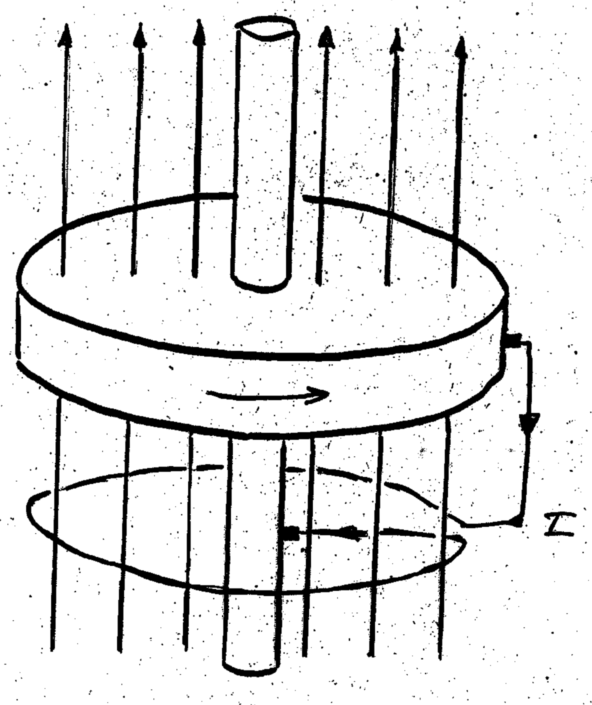
$$L\dot{I} + (R-\Omega) I = 0$$

$$I(t) = I(0) \exp\left(-\frac{R-\Omega}{L} t\right)$$

$R > \Omega$ DECAY

$R \leq \Omega$ DYNAMO

DISK DYNAMO

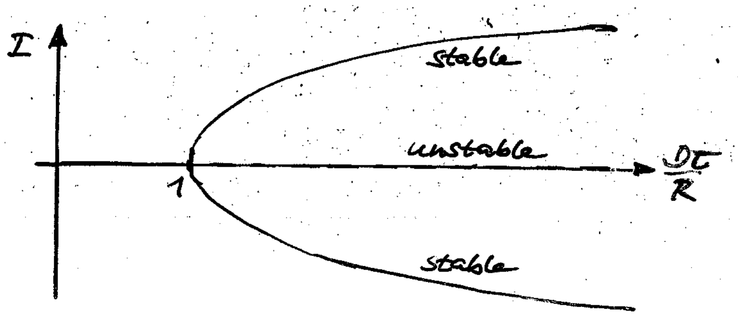


torque D

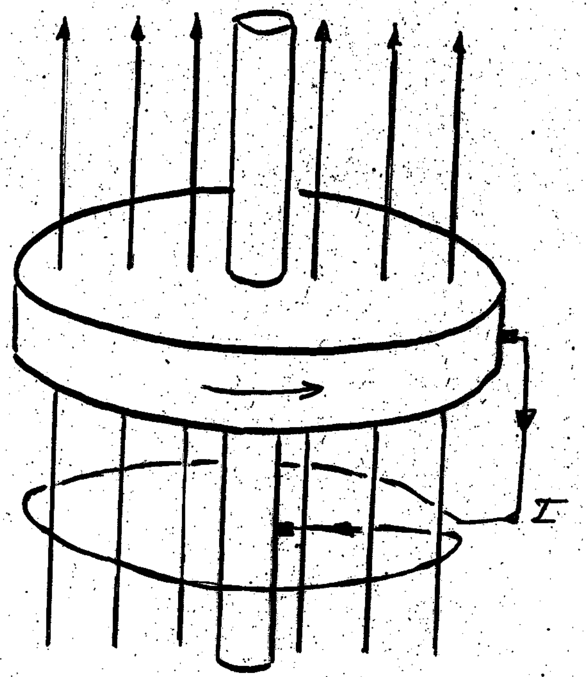
$$L\dot{I} + (R - \Omega) I = 0$$

$$\dot{\Omega} + \frac{1}{\tau}\Omega = \frac{D}{1 + CI^2}$$

Steady states $\dot{I} = \dot{\Omega} = 0$



DISK DYNAMO

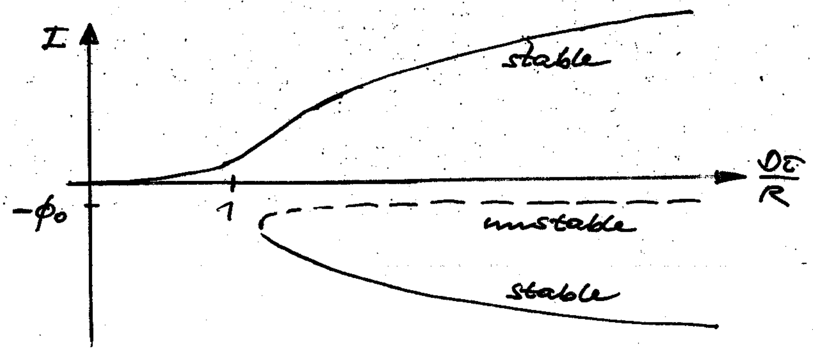


torque D

$$L\dot{I} + (R - \Omega)I = \Omega\phi_0$$ ← imposed magnetic field

$$\dot{\Omega} + \frac{1}{\tau}\Omega = \frac{D}{1 + cI^2}$$

Steady cases $\dot{I} = \dot{\Omega} = 0$



● 3.

ELEMENTS OF MAGNETOFLUIDDYNAMICS

● 3.1.

Maxwell equations and constitutive equations in the magnetofluiddynamic approximation

- flat space
- high electric conductivity
  $\varepsilon_0/\sigma \ll T$ $\quad T$ characteristic time of the considered process
- non-relativistic velocities
  $(u/c)^2$ negligible in comparison to 1

$\nabla \times E = -\partial_t B$ Faraday's law

$\nabla \cdot B = 0$

$\nabla \times H = j$ Ampere's law

$B = \mu H$

$j = \sigma(E + u \times B)$ Ohm's law

Lorentz e.m.f. $\quad + E^{(e)}$ external e.m.f.

international system of units $(m, kg, s, V, A, \dots)$

- approximations!

$$\left[\begin{array}{l} \nabla \cdot D = \rho_{el} \\ D = D(E, B) \end{array}\right] \text{— of secondary interest}$$

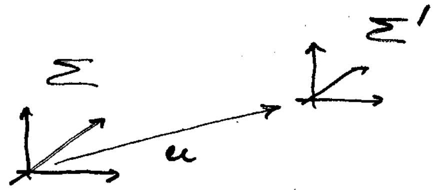
Transformation laws

$B' = B$

$H' = H$

$j' = j$

$E' = E + u \times B$



• 3.2.

Induction equation

$$\nabla\times(\eta\nabla\times B - u\times B) + \frac{\partial B}{\partial t} = 0 + \nabla\times E^{(e)}$$

$\nabla\cdot B = 0$ $\eta = 1/\mu\sigma$ magnetic difusivity

$\eta = \text{const} \rightsquigarrow$

$$\eta\nabla^2 B + \nabla\times(u\times B) - \frac{\partial B}{\partial t} = 0 - \nabla\times E^{(e)}$$

Advection / diffusion ratio:

$$\frac{|u\times B|}{|\eta\nabla\times B|} = \frac{UL}{\eta_c}$$

U characteristic value of $|u|$

L characteristic length scale

η_c characteristic value of η

$R_m = \frac{UL}{\eta_c}$ magnetic Reynolds number

$R_m \ll 1$ diffusion-dominated processes

$R_m \gg 1$ advection-dominated processes

Time scales

$T_\eta = L^2/\eta_c$ diffusive time scale

$T_U = L/U$ kinetic time scale

Dimensionless variables

$x \to Lx$ $t \to tT$ $u \to uU$ $\eta = \tilde{\eta}\eta_c$

$T = T_\eta \rightsquigarrow \nabla\times(\tilde{\eta}\nabla B - R_m u\times B) + \frac{\partial B}{\partial t} = 0$

$T = T_U \rightsquigarrow R_m^{-1}\nabla\times(\tilde{\eta}\nabla\times B) - \nabla\times(u\times B) - \frac{\partial B}{\partial t} = 0$

Table 2. Values of the magnetic Reynolds number R_m and the diffusion and kinematic times T_η and T_u for laboratory devices as well as the Earth and the Sun. As a comparison value for the electric conductivities σ we note that for copper: $6 \cdot 10^7$ S/m. For the laboratory devices U and L are arbitrarily chosen. For the Earth's core U gives a plausible magnitude of the internal motion, and L corresponds to about one third of the radius. As far as the the convection zone of the Sun is concerned, for granules U and L give their typical scales at the surface, and for sunspots L reflects their typical horizontal extension at the surface. For the consideration concerning the interior of the Sun, L is taken as roughly one third of the solar radius. More comments concerning the values for the Earth's core and the Sun's interior are given in Section 3.4, and concerning the values for the Sun's convection zone in Section 5.7.

	σ [S/m] η [m^2/s]	U [m/s]	L [m]	R_m	T_η [s]	T_u [s]
Mercury 18°C	$1.04 \cdot 10^6$ $7.65 \cdot 10^{-1}$	1	1	1.3	1.3	1
Sodium 100°C	$1.03 \cdot 10^7$ $7.73 \cdot 10^{-2}$	1	1	12.9	12.9	1
Earth's core	$3 \cdot 10^5$ 2.65	10^{-3}	10^6	$3.8 \cdot 10^2$	$3.8 \cdot 10^{11}$ ($1.2 \cdot 10^4$ yrs)	10^9 (32 yrs)
Sun's convection zone	$3 \cdot 10^3$ $2.65 \cdot 10^2$					
granules		$2 \cdot 10^2$	$2 \cdot 10^6$	$1.5 \cdot 10^6$	$1.5 \cdot 10^{10}$ ($4.8 \cdot 10^2$ yrs)	10^4 (2.8 h)
sunspots			10^7		$3.8 \cdot 10^{11}$ ($1.2 \cdot 10^4$ yrs)	
Sun's interior	10^8 $8.0 \cdot 10^{-3}$		$2 \cdot 10^8$		$5.0 \cdot 10^{18}$ ($1.6 \cdot 10^{11}$ yrs)	

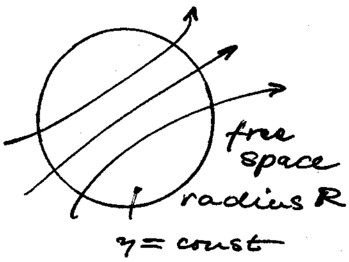
• 3.3.

The case $R_m = 0$

$$\nabla \times (\eta \nabla \times B) + \partial_t B = 0$$
$$\nabla \cdot B = 0$$

Magnetic fields are bound to decay.

Free decay of a magnetic field penetrating a spherical conductor



modes

$$B(x,t) = \hat{B}(x) e^{-\lambda t}$$

slowest-decaying mode is a dipole field with

$$\lambda = \pi^2 \eta / R^2$$
$$T_{decay} = R^2 / \pi^2 \eta$$

Earth's core

$R = 3 \cdot 10^6 m$

$\sigma = 3 \cdot 10^5 S/m$

$\eta = 2.65 m^2/s \quad \rightsquigarrow T_{decay} = 1.1 \cdot 10^4 yrs$

Sun-like body

$R = 7 \cdot 10^8 m$

$\sigma = 10^8 S/m$

$\eta = 8 \cdot 10^{-3} m^2/s \quad \rightsquigarrow T_{decay} = 2 \cdot 10^{11} yrs$

● 3.4.

Magnetic flux

$$\phi_m = \int_{\mathcal{S}} B \cdot ds$$

ϕ_m is the same for all surfaces coinciding in $\partial\mathcal{S}$

Time variation

of the magnetic flux through a »material surface« (i.e., a surface moving with the fluid)

$$\frac{d\phi_m}{dt} = \int_{\mathcal{S}} \Big(\underbrace{\frac{\partial B}{\partial t} - \nabla\times(u\times B)}_{-\nabla\times(\underbrace{E + u\times B}_{j/\sigma})}\Big)\cdot ds = -\int_{\partial\mathcal{S}} \frac{j}{\sigma}\cdot dl$$

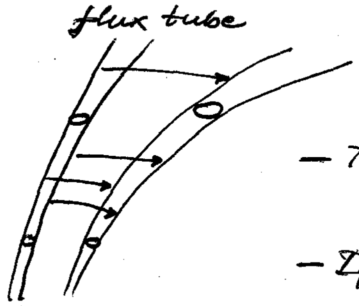
●

The high-conductivity limit $R_m \to \infty$

$$\left.\begin{array}{l} \dfrac{\partial B}{\partial t} - \nabla\times(u\times B) = 0 \\[2ex] \dfrac{d\phi_m}{dt} = 0 \end{array}\right]$$ equivalent to each other

○

Frozen-in magnetic field lines



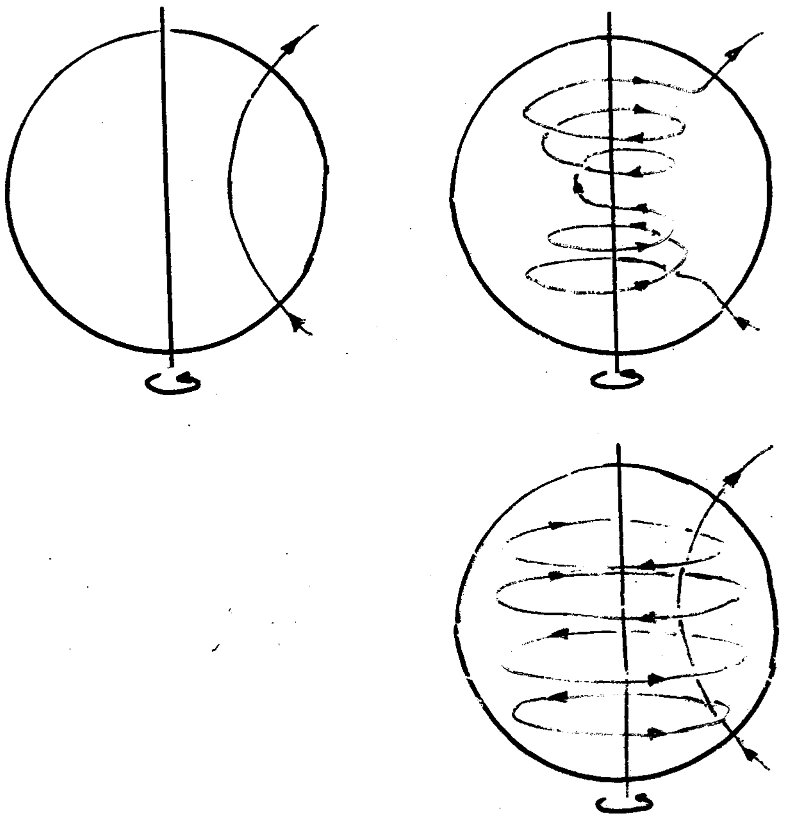
– ϕ_m coincides for all cross-sections of a given flux tube

– The fluid flow transform a flux tube again in a flux tube.

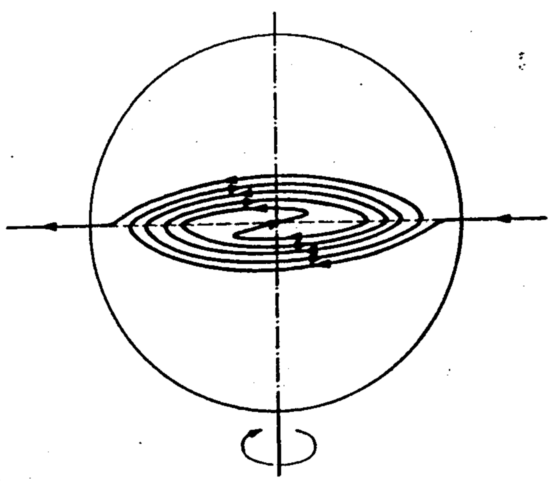
– If, at a given time, two fluid elements are connected by a field line they are always connected by a field line.

● 3.5. Advection of magnetic fields

○ 3.5.1. Differential rotation and axisymmetric magnetic field



○ 3.5.2. Differential rotation and non-axisymmetric magnetic field



• 3.6.

Magnetic energy

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu}\right) = -j\cdot E - \operatorname{div} S \qquad S = E\times H$$

Poynting vector

$$= -\underbrace{\frac{j^2}{\sigma}}_{\text{Joule heat production}} - \underbrace{u\cdot(j\times B)}_{\text{work done by Lorentz force (if >0) or against it (if <0)}} - \operatorname{div} S$$

work done by Lorentz force (if >0)
or against it (if <0)

Joule heat production

conducting body V

outer space
$E = O(r^{-2})$, $H = O(r^{-3})$

$$\underbrace{\frac{d}{dt}\int_\infty \frac{B^2}{2\mu}\,dv}_{\text{total magnetic energy}} = -\int_V \frac{j^2}{\sigma}\,dv - \int_V u\cdot(j\times B)\,dv$$

total
magnetic energy

$$-\int_V u\cdot(j\times B)\,dv$$
$$= \frac{1}{\mu}\int_V \left(e_{ij} - \frac{1}{2}(\nabla\cdot u)\delta_{ij}\right)B_i B_j\,dv - \frac{1}{\mu}\int_{\partial V}(u\cdot B)(B\cdot ds)$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ rate of strain tensor

Growth of total magnetic energy
requires stretching of magnetic field lines

● 3.7.

Fluiddynamic equations

Rotating frame of reference,
angular velocity Ω

$$\varrho(\partial_t u + (u\cdot\nabla)u) = -\nabla p - 2\varrho\Omega\times u + F^{(fr)} + F^{(ext)} + F^{(elm)}$$

← momentum balance

mass density

pressure, centrifugal term included

frictional force

external force

electromagnetic force

$$\partial_t\varrho + \nabla\cdot(\varrho u) = 0$$

← mass balance

$$\varrho = \varrho(p,T)$$

← equation of state

$$\varrho c_v(\partial_t T + u\cdot\nabla T) = \nabla(\tilde{\kappa}\nabla T) - q$$

← heat transport equation

specific heat capacity

heat conduction coefficient

heat production

$$F_i^{(fr)} = \frac{\partial S_{ij}}{\partial x_j}$$

$$S_{ij} = \varrho\nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \varrho\nu'(\nabla\cdot u)\delta_{ij}$$

kinematic viscosity

$q = \dots$ (external heating, viscos and Joule heating)

• 3.2.

Lorentz force

$$F^{(elm)} \overset{\text{magnetofluiddynamic approximation}}{=} \underset{\text{Lorentz force}}{F_L} \quad (\rho_e E \text{ ignored})$$

$$= j \times B$$

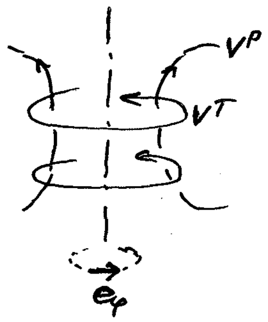
$$= \frac{1}{\mu}(\nabla \times B) \times B$$

$$= \underset{\text{tension}}{\frac{1}{\mu}(B \cdot \nabla) B} - \underset{\text{pressure force}}{\frac{1}{2\mu}\nabla B^2}$$

$$F_{Li} = \frac{\partial M_{ij}}{\partial x_j}, \quad \underset{\text{Maxwell stresses}}{M_{ij} = \frac{1}{\mu}\left(B_i B_j - \frac{1}{2} B^2 \delta_{ij}\right)}$$

3.9. A useful mathematical tool: poloidal and toroidal vector fields

3.9.1. Axisymmetric case



$V = V^P + V^T$

$V^P = V - (V\cdot e_\varphi)\, e_\varphi$ POLOIDAL

$V^T = (V\cdot e_\varphi)\, e_\varphi$ TOROIDAL

E.g., spherical coordinate system (r, ϑ, φ)

$V^P = (V_r, V_\vartheta, 0)$

$V^T = (0, 0, V_\varphi)$

- $V = 0 \curvearrowright V^P = V^T = 0$
- fV^P poloidal, fV^T toroidal, f any scalar
- $r \times V^P$ toroidal, $r \times V^T$ poloidal, r any poloidal vector, e.g., radius vector
- $\nabla\cdot V^T = 0$
- $\nabla\times V^P$ toroidal, $\nabla\times V^T$ poloidal

$\Big] \curvearrowright$ $\nabla^2 V^P$ poloidal, $\nabla^2 V^T$ toroidal

- $\nabla\times V^T = 0 \curvearrowright V^T = 0$
- $V^P\cdot V^T = 0$

Special case $\nabla\cdot V = 0 \curvearrowright \nabla\cdot V^P = 0$

$V^P = \nabla\times \hat{V}^T$

o 3.9.2.

General case

$V = V^P + V^T$ not $|V|$

$V^P = rV + \nabla W$ POLOIDAL

$V^T = r \times \nabla U$ TOROIDAL

r radius vector

Spherical coordinate system (r, ϑ, φ)

$V^P = \left(rV + \frac{\partial W}{\partial r}, \frac{1}{r}\frac{\partial W}{\partial \vartheta}, \frac{1}{r\sin\vartheta}\frac{\partial W}{\partial \varphi} \right)$

$V^T = \left(0, -\frac{1}{\sin\vartheta}\frac{\partial U}{\partial \varphi}, \frac{\partial U}{\partial \vartheta} \right)$

- $V = 0$ on $r = const$ $\curvearrowright$ $V^P = V^T = 0$ on $r = const$
- fV^P poloidal, fV^T toroidal
  if f independent of ϑ and φ
- $r \times V^P$ toroidal, $r \times V^T$ poloidal
  if r radius vector
- $\nabla \cdot V^T = 0$
- $\nabla \times V^P$ toroidal, $\nabla \times V^T$ poloidal

$\curvearrowright$ $\nabla^2 V^P$ poloidal, $\nabla^2 V^T$ toroidal

- $r \cdot (\nabla \times V^T) = 0$ on $r = const$ $\curvearrowright$ $V^T = 0$ on $r = const$
- $\langle V^P \cdot V^T \rangle = 0$, where $\langle \cdots \rangle$ average over $r = const$

Special case $\nabla \cdot V = 0$ $\curvearrowright$ $\nabla \cdot V^P = 0$

$V^P = \nabla \times (r \times \nabla \hat{U})$

$V^T = r \times \nabla U$

3.9.3

Applications to $\mathbf{B}$

$$\mathbf{B} = \mathbf{B}^P + \mathbf{B}^T$$

- Axisymmetric case

$$\mathbf{B}^P = \nabla\times(A\mathbf{e}_\varphi) = -\frac{1}{r\sin\vartheta}\nabla(r\sin\vartheta A)$$

$$\mathbf{B}^T = B\mathbf{e}_\varphi \quad \text{not } |\mathbf{B}|$$

$r\sin\vartheta A(r_0,\vartheta_0)$ magnetic flux through circle $r=r_0$, $\vartheta=\vartheta_0$

$r\sin\vartheta A(r,\vartheta) = \text{const}$ defines poloidal field lines

- General case

$$\mathbf{B}^P = -\nabla\times(\mathbf{r}\times\nabla S) = -\mathbf{r}\Delta S + \nabla\frac{\partial}{\partial r}(rS)$$

$$\mathbf{B}^T = -\mathbf{r}\times\nabla T$$

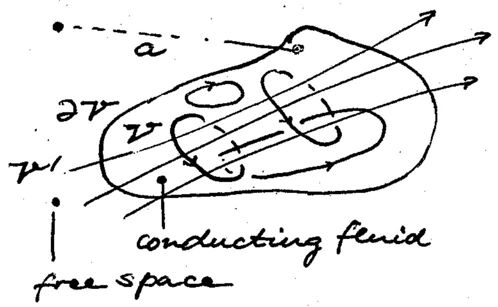
$T=\text{const}$, $r=\text{const}$ defines toroidal field lines

4.

THE KINEMATIC DYNAMO PROBLEM

4.1.

Formulation of a typical problem



magnetic field
flux density B

electric current
current density j

fluid motion
velocity u

(I)
$$\nabla\times E = -\frac{\partial B}{\partial t}$$ Faraday's law

$$\nabla\times H = j$$ Ampere's law

$$\nabla\cdot B = 0$$ initial condition only

$$B = \mu H$$

(everywhere)

$$j = \begin{cases} \sigma(E+u\times B) & \text{in } V \\ 0 & \text{in } V' \end{cases}$$ Ohm's law

$$\left.\begin{array}{l} E = O(a^{-2}) \\ B = O(a^{-3}) \end{array}\right\} \text{ as } a\to\infty$$ no electric charges or currents at infinity

(II)
$$\left.\begin{array}{l} \nabla\times(\eta\nabla\times B - u\times B) + \dfrac{\partial B}{\partial t} = 0 \\ \nabla\cdot B = 0 \end{array}\right\} \text{in } V$$

$\eta = 1/\mu\sigma$

$$\left.\begin{array}{l} \nabla\times B = 0 \\ \nabla\cdot B = 0 \end{array}\right\} \text{in } V'$$

If V' simply connected

$$\left.\begin{array}{l} B = -\nabla\phi \\ \Delta\phi = 0 \end{array}\right\} \text{in } V'$$

$[B] = 0$ across ∂V — no surface currents

$B = O(a^{-3})$ as $a\to\infty$

Initial value problem for $B(x,t)$

(I) and (II) (almost) equivalent.

$\nabla\times B = 0$ in V' and $[B] = 0$ across ∂V imply $(\nabla\times B)_n = 0$ at ∂V

DYNAMO: $B \not\to 0$ as $t\to\infty$

• 4.2.

Some comments

$$R_m = \frac{UL}{\eta_c}$$

∘ 4.2.1

Necessary (but not sufficient) for a dynamo:

$R_m \geq R_{m\,crit} = O(1)$

∘ 4.2.2

»Dynamo« means more than amplification of magnetic fields

$$\nabla \times (\eta \nabla B - u \times B) + \frac{\partial B}{\partial t} = \nabla \times E^{(e)}$$

DYNAMO: non-decaying solution of the homogeneous equation

∘ 4.2.3

A dynamo corresponds to an instability of the non-magnetic state of a system

Consider problem (II) with u independent of t and look for solutions of the form

$B(x,t) = \mathrm{Re}\,(\hat{B}(x) \exp\{pt\})$ $\hat{B}, p$ complex

$$\nabla \times (\eta \nabla \times \hat{B} - R_m u \times \hat{B}) + p\hat{B} = 0$$

etc.

Eigenvalue problem for p

$p = p(R_m)$

$p = \lambda + i\omega$ λ, ω real

DYNAMO : $\underline{\lambda \geq 0}$

$\lambda = 0$ defines »marginal values« of R_m

In general infinite set of solutions: $\hat{B}_i, p_i$

If set of $\hat{B}_i$ complete then general solution of the initial value problem

$B(x,t) = \mathrm{Re}\left(\sum_i c_i \hat{B}_i(x) \exp\{pt\}\right)$

↑ constants determined by $B(x,0)$

o 4.2.4

Reduction of problem (II) (for all space) to an »inner problem« (for the fluid body)

Suppose $\mathcal{V}'$ simply connected so that

$$\left.\begin{array}{l} B = -\nabla\phi \\ \Delta\phi = 0 \end{array}\right\} \text{ in } \mathcal{V}'$$

outer Neumann problem
$$\left[\begin{array}{l} \Delta\phi = 0 \\ -n\cdot\nabla\phi = B_n \text{ on } \partial\mathcal{V} \end{array}\right.$$

limes from inside

$$\phi(x) = \int_{\partial\mathcal{V}} \Lambda(x,x')B_n(x')\,ds'$$

Green's function for Neumann problem

Boundary condition for the » inner problem «

$$B_{tang}(x) = -\nabla_{tang}\int_{\partial\mathcal{V}} \Lambda(x,x')B_n(x')\,ds' \text{ on } \partial\mathcal{V}$$

non-local condition!

o 4.2.5.

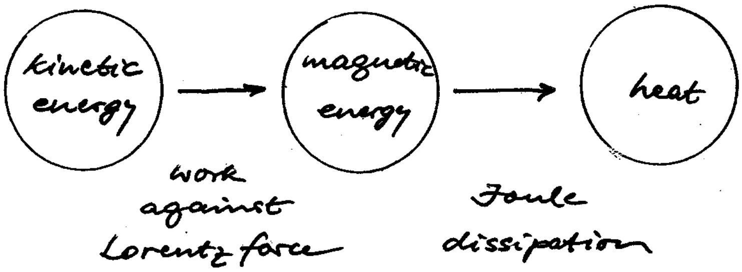
Energy balance

$$\frac{d}{dt}\int_\infty \frac{B^2}{2\mu}\,dv = -\int_V \frac{j^2}{\sigma}\,dv - \int_V u\cdot(j\times B)\,dv$$

- $\int_\infty \frac{B^2}{2\mu}\,dv$: total magnetic energy
- $\int_V \frac{j^2}{\sigma}\,dv$: Joule heat production
- $\int_V u\cdot(j\times B)\,dv$: work done by ($\int>0$) or against ($\int<0$) Lorentz force

DYNAMO:

$$-\int_V u\cdot(j\times B)\,dv \geq \int_V \frac{j^2}{\sigma}\,dv \quad \left[\begin{array}{r} \curvearrowleft \\ Rm \geq Rm\,crit \\ = O(1) \end{array}\right]$$



Dynamo requires
stretching of magnetic field lines

o 4.2.6.

Time scales

$T_\eta = \frac{L^2}{\eta}$ diffusive time scale

$T_U = \frac{L}{U}$ kinetic time scale

$$\frac{T_\eta}{T_U} = Rm$$

λ dynamo growth rate

$\lambda T_U \to 0$ as $Rm \to \infty$

slow dynamo

$\lambda T_U \to$ positive value as $Rm \to \infty$

fast dynamo

● 4.3.

(Anti-) Dynamo theorems

○ 4.3.1

Theorems concerning the intensity of the fluid motion

Lower bounds for (properly defined) Rm

Backus 1958

Childress 1969

○ 4.3.2.

Theorems concerning the geometry (symmetry) of the magnetic field

- Cowling's theorem

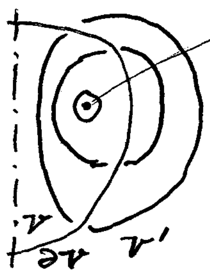
Cowling 1934
Bragins ky 1964
⋮

A magnetic field that is symmetric about any axis can never be maintained by a dynamo.

- » Plane version « of this theorem

A magnetic field that is independent of one Cartesian coordinate (i.e., e.g., $B = B(x,y)$) can never be maintained by a dynamo.

- Cowling's neutral point argument for the steady case



neutral point inside the fluid

$$\nabla \times B = \mu\sigma(E + u \times B)$$

$$\oint_{\partial \mathcal{S}} B \cdot dl = \int_{\mathcal{S}} \mu\sigma(E + u \times B) \cdot ds \qquad E_p = 0$$

in meridional plane

$$\bar{B}^L L \le |\mu\sigma u|_{max} \int_{\mathcal{S}} |B| \cdot ds$$

assume > 0

$$= |\mu\sigma u|_{max} \bar{B}^{\mathcal{S}} S$$

Consider limit $L, S \to 0$

$$\underbrace{\bar{B}^L / \bar{B}^{\mathcal{S}}}_{\to c > 0} \le |\mu\sigma u|_{max} \underbrace{S/L}_{\to 0}$$

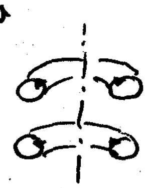
Contradiction!

- Cowling's theorem breakes down if conductivity σ no longer isotropic

Lortz 1989

o 4.3.3.

Theorems concerning the geometry of the fluid flow



- No analogue to Cowling's theorem!
  - Gailitis 1970:
    axisymmetric motion maintains a (non-axisymmetric) magnetic field
  - Roberts 1972:
    motion depending only on x and y (but non-zero u_x, u_y and u_z) maintains a magnetic field (depending on x, y and z)

- Theorem by Bullard & Gellman

Bullard & Gellman 1954

Spherical body

$\eta = \eta(r \text{ alone})$

A magnetic field cannot be maintained by a purely toroidal motion
(i.e. $\nabla \cdot u = 0$ and $u_r = 0$).

  - No dynamo by differential rotation alone!
  - Theorem breakes down for non-radial $\nabla\eta$

- » Plane version « of this theorem

Zeldovich 1956
Zeldovich & Ruzmaikin 1980

$\eta = \eta(z \text{ alone})$

A magnetic field cannot be maintained by a motion in planes z = const
(i.e. $u_x = u_x(x,y,z,t)$, $u_y = u_y(x,y,z,t)$, $u_z = 0$).

• Theorem by Busse Busse 1975

Which deviation from a toroidal motion is necessary for a dynamo?

Spherical body

$\eta = \text{const}$

Dynamo requires $\dfrac{|u \cdot r|_{max}}{\eta} \geq \dfrac{E^P}{E^P + E^T}$

E^P, E^T energies stored in the poloidal and the toroidal parts of the magnetic field

o 4.3.4.

A presumption

In all spherical dynamo models investigated so far there is an interplay between the poloidal and the toroidal part of the magnetic field.

Presumption: no dynamo with purely poloidal
or purely toroidal magnetic field.

A dynamo with a purely toroidal magnetic field would be an »invisible dynamo«,
i.e., no magnetic field at the surface
or outside the fluid body.

Arguments against invisible dynamos
by Kaiser et al. 1994

● 4.4.

Developments of the idea of the dynamo

○ 4.4.1.

1866 Werner von Siemens
Charles Wheatstone
Principle of the self-exciting dynamo

1919 Sir Joseph Larmor
Magnetic fields of the Sun (and the Earth) due to dynamo processes?

1934 T. G. Cowling
Anti-dynamo theorem: No dynamo with axisymmetric magnetic field

1954 E. Bullard and H. Gellman
Anti-dynamo theorem: No dynamo due to toroidal motion

1955 E. N. Parker
The role of cyclonic convection in dynamo processes

1958 E. Herzenberg
Existence proof for homogeneous dynamo

1964 S. I. Braginsky
Theory of the nearly symmetric dynamo („Γ-effect")

1966 M. Steenbeck, F. Krause and K.-H. Rädler
Mean-field dynamo theory (α-effect etc.)

1968 D. Lortz
1973 Ju. B. Ponomarenko
Dynamos in infinite cylinders

1972 G. O. Roberts
Spatially periodic dynamos

1973 C. L. Pekeris, Y. Accad and B. Shkoller
D. Gubbins
1974 S. Kumar and P. H. Roberts
1989 M. L. Dudley and R. W. James
Spherical kinematic dynamo models

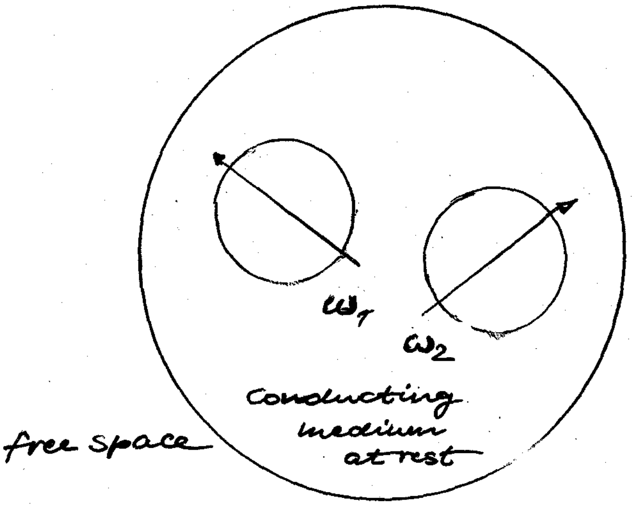
...... *Dynamo models for the Earth, the Sun,*
planets, stellar objects and galaxies
Numerical simulations

1999 *First successful runs of the Riga and Karlsruhe dynamo experiments*

O 4.4.2.

Herzenberg dynamo

Herzenberg 1958



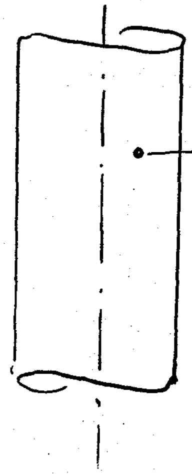
Self-excitation of magnetic fields
for special mutual positions of the rotation axes
and sufficiently large angular velocities ω_1 and ω_2

● 4.5.

Examples of dynamos
working in an infinitely extended conducting medium

○ 4.5.1
Lortz dynamo

Lortz 1968



Growing magnetic field
(with helical structure)
possible

Lortz 1972

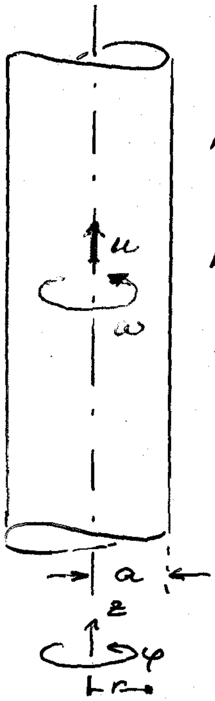
A modification:

Cylinder → torus in free space

An example in which deviations
of the magnetic from axisymmetry
occur only in the fluid body,
not in outer space

o 4.5.2.

Ponomarenko dynamo Ponomarenko 1973



medium at rest

helical motion

$$B = Re\left(\hat{B}(r)\exp(i(m\varphi + kz) + pt)\right)$$

$\hat{B}$, p complex

$$R_\omega = \frac{\omega a^2}{\eta}, \quad R_u = \frac{ua}{\eta}$$

$$R_m = \sqrt{R_\omega^2 + R_u^2}$$

For rigid motion of the cylinder

DYNAMO : $\underline{R_m \geq 17.722}$

Marginal case : $m = 1$

$k/a = -0.3875$

$R_u / R_\omega = 1.3114$

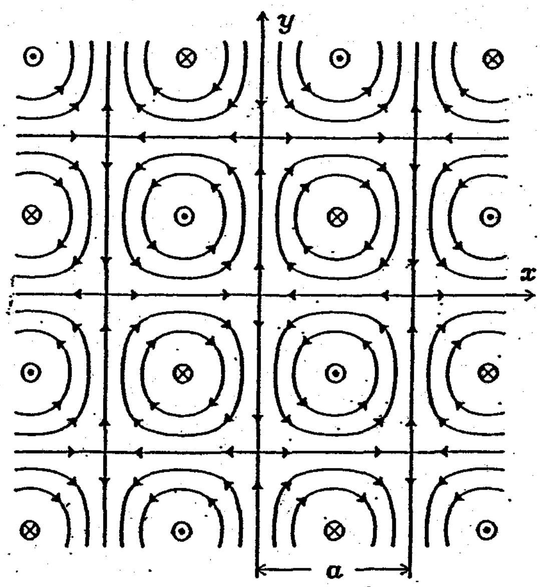
$\longrightarrow$ RIGA dynamo experiment

o 4.5.3.

Roberts dynamo

G.O. Roberts 1972

(precursor: Gailitis 1967)



helicity

$\mathbf{u}\cdot(\nabla\times\mathbf{u}) \geq 0$

Example

$$u_x = -u_\perp \frac{\pi}{2}\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}y\right)$$

$$u_y = u_\perp \frac{\pi}{2}\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{a}y\right)$$

$$u_z = -u_\parallel \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{a}y\right)$$

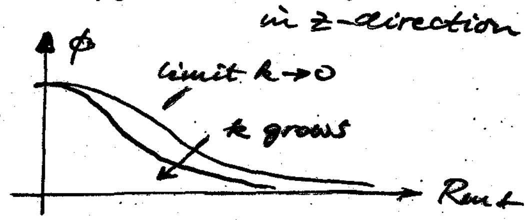
$$\mathbf{B} = \mathrm{Re}\left(\hat{\mathbf{B}}(x,y)\exp(ikz+pt)\right)$$

$$Rm_\perp = \frac{u_\perp a}{2\eta} \qquad Rm_\parallel = \frac{u_\parallel a}{\eta}$$

DYNAMO: $\underline{Rm_\perp Rm_\parallel \phi(Rm_\perp) \geq \frac{32}{\pi}\frac{a}{\ell}}$

$\ell = \frac{2\pi}{k}$ period length of $\mathbf{B}$ in z-direction

These results can be obtained in a mean-field approach!



→ KARLSRUHE dynamo experiment

Roberts dynamo works also with a flow pattern modified such that $\mathbf{u}\cdot(\nabla\times\mathbf{u}) = 0$

if $\mathbf{a}\cdot(\nabla\times\mathbf{a})$ predominantly >0

or predominantly <0,

where $\mathbf{u} = \nabla\times\mathbf{a}$, $\nabla\cdot\mathbf{a} = 0$.

● 4.6.

Dynamos in spherical fluid bodies

○ 4.6.1

Mathematical tools, Bullard-Gellman formalism

Consider

$$\left.\begin{array}{l}\eta\nabla^2 B + \nabla\times \underset{u\times B}{F} - \partial_t B = 0\\ \nabla\cdot B = 0\end{array}\right\} \text{ in } r<R$$

$$\left.\begin{array}{l}\nabla\times B = 0\\ \nabla\cdot B = 0\end{array}\right\} \text{ in } r>R$$

Spherical coordinates r, ϑ, φ

For simplicity $\eta = \text{const}$

$[B] = 0$ across $r=R$

$B \to 0$ as $r\to\infty$

Introduce $B = B^P + B^T$ with $\nabla\cdot B^P = 0$

and $F = F^P + F^T$; $\nabla\cdot F^P$ not necessarily $=0$

$\curvearrowright$

$$\left.\begin{array}{l}\eta\nabla^2 B^P + \nabla\times F^T - \partial_t B^P = 0\\ \eta\nabla^2 B^T + \nabla\times F^P - \partial_t B^T = 0\end{array}\right\} \text{ in } r<R$$

$\nabla\times B^P = 0$, $B^T = 0$ in $r>R$

$[B^P] = 0$ across $r=R$, $B^T = 0$ on $r=R$

$B^P \to 0$ as $r\to\infty$

Introduce $B^P = -\nabla\times(r\times\nabla S)$, $B^T = -r\times\nabla T$,

$\nabla\times F^P = +r\times\nabla F$; $F^T = -r\times\nabla G$

$\curvearrowright$

$$\left.\begin{array}{l}\eta\Delta S - G - \partial_t S = 0\\ \eta\Delta T - \Delta F - \partial_t T = 0\end{array}\right\} \text{ in } r<R$$

$\Delta S = 0$, $T = 0$ in $r>R$

$[S] = [\partial S/\partial r] = 0$ across $r=R$, $T=0$ on $r=R$

$S \to 0$ as $r\to\infty$

if S, T, F and G properly normalized

Expand $\quad S = \sum_{n,m} S_n^m(r,t) Y_n^m(\vartheta,\varphi)$ $\leftarrow$ spherical harmonics

$\curvearrowright$ $\quad T, F, G$ analogously

$$\left.\begin{aligned} \eta D_n S_n^m - G_n^m - \partial_t S_n^m &= 0 \\ \eta D_n T_n^m - D_n^m F_n^m - \partial_t T_n^m &= 0 \end{aligned}\right\} \text{in } r<R \qquad (*)$$

$$D_n f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) - \frac{n(n+1)}{r^2} f$$

$(S_n^m \sim r^{-(n+1)}, \ T_n^m = 0 \quad \text{in } r>R)$

$$\frac{\partial S_n^m}{\partial r} + \frac{n+1}{r} S_n^m = T_n^m = 0 \quad \text{at } r=R \qquad (**)$$

Specify u, express u^P and u^T like B^P and B^T
by spherical harmonics,
calculate F_n^m and G_n^m as functions of $S_{n'}^{m'}, T_{n'}^{m'}$
and their derivatives, e.g., $F_n^m = \sum_{n',m'} (FS)_{nn'}^{mm'} S_{n'}^{m'} + \ldots$

Assume $B \sim \exp(pt)$

$\curvearrowright$ $S, T \sim \exp(pt)$

Equations $(*)$ together with boundary conditions $(**)$,
with F_n^m and G_n^m expressed by S_n^m and T_n^m
and ∂_t replaced by p,
pose an eigenvalue problem
with p as eigenvalue parameter.

o 4.6.2.

Special models

Bullard & Gellman 1954

no acceptable results,

convergency problems

⋮

Pekeris, Accad & Shkoller 1973

Beltrami flows $\nabla \times u = \lambda u$

Gubbins 1973

axisymmetric poloidal and toroidal flows

Kumar & Robert 1975

non-axisymmetric poloidal and toroidal flows

Dudley & James 1989

axisymmetric poloidal and toroidal flows

⋮

5.

THE GENERAL DYNAMO PROBLEM

(DYNAMICALLY CONSISTENT DYNAMO MODELS)

5.1.

Thermally driven dynamos

For the sake of simplicity

Boussinesq-approximation

$\eta, \nu, c_v, \tilde{\kappa}$ constant

$$\partial_t B - \nabla\times(u\times B) = \eta\nabla^2 B$$

$$\nabla\cdot B = 0$$

$$\partial_t u + (u\cdot\nabla)u = -\frac{1}{\varrho}\nabla p + \nu\nabla^2 u - 2\Omega\times u + \underbrace{\frac{1}{\varrho}F}_{\text{buoyancy}} + \frac{1}{\mu\varrho}(\nabla\times B)\times B$$

$$\nabla\cdot u = 0$$

Reference state $B = u = 0$, $\varrho = \varrho_0(x)$, $p = p_0(x)$, $T = T_0(x)$

$$T = T_0 + \Theta$$

$$\varrho = \varrho_0(1 + \underbrace{\alpha}_{\text{volume expansion coefficient}}\Theta)$$

$$F = \varrho_0\alpha\Theta \underbrace{g}_{\text{gravitational acceleration}}$$

$$\partial_t\Theta + u\cdot\nabla\Theta = \underbrace{\kappa}_{=\tilde{\kappa}/\varrho c_v}\Delta\Theta - u\cdot\nabla T_0$$

Choose values of

- $P_m = \frac{\nu}{\eta}$ magnetic Prandtl number
- $E = \frac{\nu}{\Omega L^2}$ Ekman number

  $\left(Ta = (2/E)^2 = \left(\frac{2\Omega L^2}{\nu}\right)^2 \right.$ Taylor number)
- $P = \frac{\nu}{\kappa}$ Prandtl number
- $Ra = \frac{\alpha g (\partial T)_c L^4}{\kappa \nu}$ Rayleigh number

  $(\partial T)_c$ characteristic value of ∇T_0

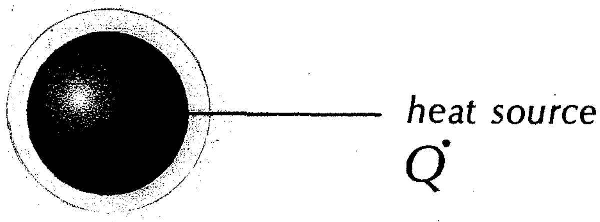
Calculate solutions $\mathbf{B}, \mathbf{u}, \Theta$

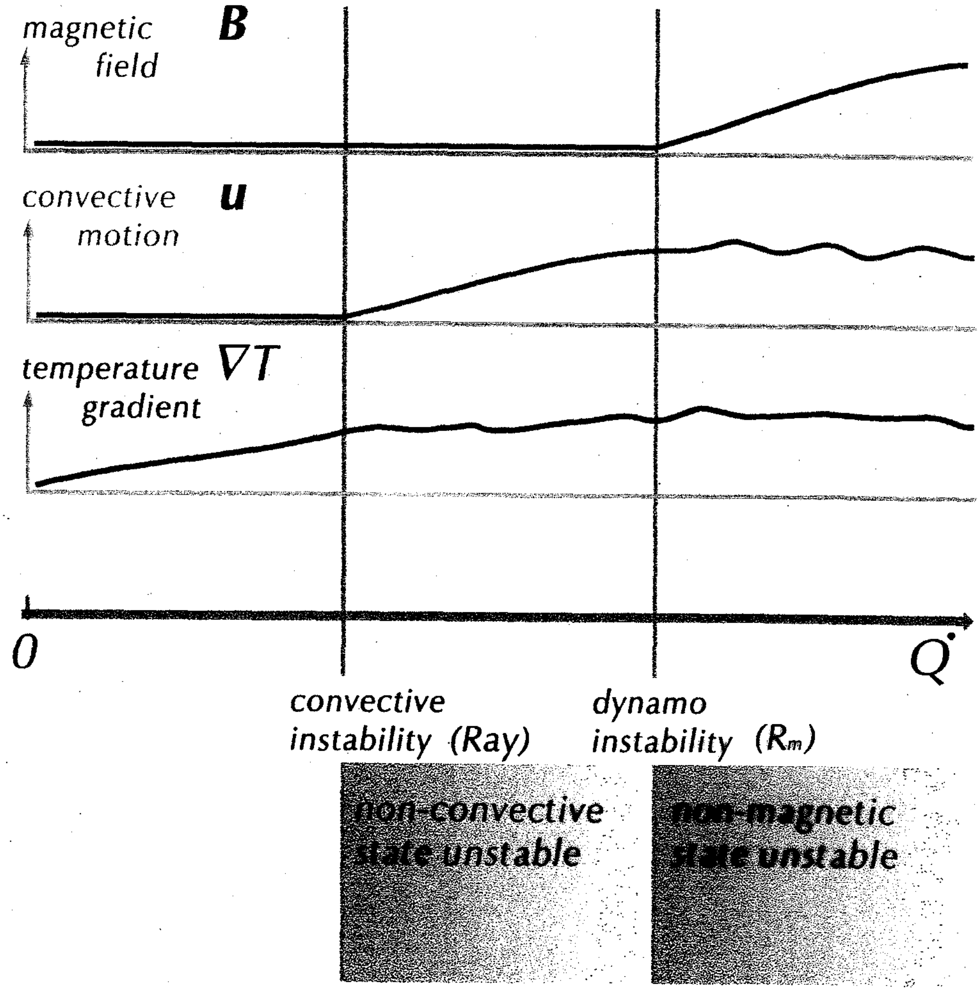
B, U characteristic values of $\mathbf{B}, \mathbf{u}$

Solutions can be interpreted in terms of

- $Rm = \frac{UL}{\eta}$ magnetic Reynolds number
- $Re = \frac{UL}{\nu}$ Reynolds number
- $Ro = \frac{U}{2\Omega L}$ Rossby number
- $N = \frac{\sigma B^2 L}{\rho u}$ Stuart number
- $H = \frac{\sqrt{\sigma} BL}{\sqrt{\rho \nu}}$ Hartmann number
- $\Lambda = \frac{\sigma B^2}{2\rho\Omega}$ Elsasser number

⋮

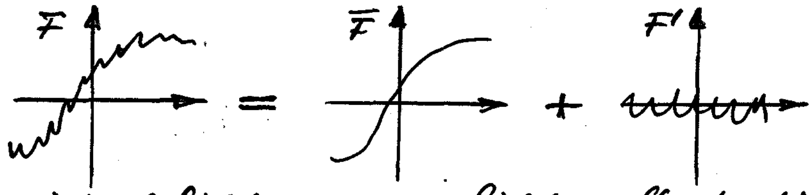

heat source
Q̇


magnetic field
B
convective motion
u
temperature gradient
∇T
0
Q̇
convective instability (Ray)
dynamo instability (Rm)
non-convective state unstable
non-magnetic state unstable

6. MEAN-FIELD ELECTRODYNAMICS

6.1. The general concept

6.1.1. Mean fields and fluctuations



original field $\quad$ mean field $\quad$ fluctuation

$F = \overline{F} + F'$ $\qquad$ $\overline{F}$ defined by a proper averaging procedure

$\mathbf{F} = \overline{\mathbf{F}} + \mathbf{F}'$ $\qquad$ $\mathbf{F} = \mathbf{e}_i F_i$ $\quad$ $\overline{\mathbf{F}} = \mathbf{e}_i \overline{F_i}$

unit vectors w.r.t. Cartesian, spherical, cylindrical, ... coordinate system

6.1.2. Reynolds rules

(R1) $\overline{F+G} = \overline{F} + \overline{G}$

(R2) $\overline{\overline{F}} = \overline{F}$ $\quad \rightsquigarrow \quad$ $\overline{F'} = 0$

(R3) $\overline{\overline{F}\,G} = \overline{F}\,\overline{G}$ $\quad \rightsquigarrow \quad$ $\overline{\overline{F}\,\overline{G}} = \overline{F}\,\overline{G}$, $\overline{\overline{F}G'} = 0$

(R4) $\overline{\partial F/\partial x} = \partial \overline{F}/\partial x$, $\overline{\partial F/\partial t} = \partial \overline{F}/\partial t$

$$\overline{FG} = \overline{F}\,\overline{G} + \overline{F'G'}$$

○ 6.1.3.

Examples of averages

□

Statistical (ensemble) average

$$\overline{F}(x,t) = \int F(x,t;p)f(p)\,dp$$

$$\int f(p)\,dp = 1$$

(R1) … (R4) apply exactly.
BUT: relation of averages to observable quantities?

□

Space averages

$$\overline{F}(x,t) = \int F(x+\xi,t)\,g(\xi)\,d^3\xi$$

$$\int g(\xi)\,d^3\xi = 1$$

(R2) and (R3) in general violated, apply approximately in two-scale situations.

special case

» Braginsky's average «

$$\overline{F}(\dots,t) = \frac{1}{2\pi}\int_0^{2\pi} F(\dots,\varphi,t)\,d\varphi$$

(R1) … (R4) apply exactly.
BUT: mean fields by definition axisymmetric.

□

Time average

$$\overline{F}(x,t) = \int F(x,t+\tau)\,g(\tau)\,d\tau$$

$$\int g(\tau)\,d\tau = 1$$

(R2) and (R3) in general violated, apply approximately in two-scale situations.

□

Averages by » filtering « of spectra,

e.g.,

$$F(x,t) = \int_{\infty} \hat{F}(k,t)\,e^{ik\cdot x}\,d^3k$$

$$\overline{F}(x,t) = \int_{|k|\leq K} \hat{F}(k,t)\,e^{ik\cdot x}\,d^3k$$

(R3) in general violated, applies with special (two-scale) assumptions on the spectrum.

6.2.
Basic equations for the mean electromagnetic fields

6.2.1.
Starting point

$$\nabla\times E = -\partial_t B \qquad B = \mu H$$
$$\nabla\times H = j \qquad j = \sigma(E + u\times B)$$
$$\nabla\cdot B = 0$$

$$\curvearrowright \quad \nabla\times(\eta\nabla\times B - u\times B) + \partial_t B = 0$$
$$\nabla\cdot B = 0$$

6.2.2.
Mean-field equations

Validity of the Reynolds rules assumed

$$\nabla\times\overline{E} = -\partial_t\overline{B} \qquad \overline{B} = \mu\overline{H}$$
$$\nabla\times\overline{H} = \overline{j} \qquad \overline{j} = \sigma(\overline{E} + \overline{u}\times\overline{B} + \underline{\underline{\mathcal{E}}})$$
$$\nabla\cdot\overline{B} = 0$$

$$\nabla\times(\eta\nabla\times\overline{B} - \overline{u}\times\overline{B} - \underline{\underline{\mathcal{E}}}) + \partial_t\overline{B} = 0$$
$$\nabla\cdot\overline{B} = 0$$

$\underline{\underline{\mathcal{E} = \overline{u'\times B'}}}$ mean electromotive force due to fluctuations of the motion and the magnetic field

○ 6.2.3.

Properties of the mean electromotive force

$$\mathcal{E} = \overline{u' \times B'}$$

$$\nabla \times (\eta \nabla \times B' - \bar{u} \times B' \underbrace{- u' \times B' - \overline{u' \times B'}}_{-(u' \times B')'}) + \partial_t B' = \nabla \times (u' \times \bar{B})$$

$$\nabla \cdot B' = 0$$

$$B' = B'^{(0)} + B'^{(\bar{B})}$$

$$\mathcal{E} = \mathcal{E}^{(0)} + \mathcal{E}^{(\bar{B})}$$

$\mathcal{E}^{(\bar{B})}$ — functional of $\bar{u}$, u' and $\bar{B}$, linear and homogeneous in $\bar{B}$

$\mathcal{E}^{(0)}$ — functional of $\bar{u}$ and u'

$\mathcal{E}$ depends only via averaged quantities on u'.

Assume until further notice ← see 6.6.

that B' decays to zero if $\bar{B} = 0$

(i.e. no small-scale dynamo, purely hydrodynamic »background turbulence«).

↷

$\mathcal{E}^{(0)}$ decays to zero

$$\underline{\mathcal{E}_i(x,t) = \int_0^\infty \int_{\infty} \underbrace{K_{ij}(x,t;\xi,\tau)}\, \bar{B}_j(x-\xi, t-\tau)\, d^3\xi\, d\tau} \quad // (*)$$

K_{ij} — depends on $\bar{u}$ and u' only, vanishes for large $|\xi|$ and τ

Assume »nearly local and instantaneous« dependence of $\mathcal{E}$ on $\overline{B}$

$$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\,\partial \overline{B}_j / \partial x_k + \cdots \qquad (**)$$

depend on $\overline{u}$ and u' only

$$a_{ij} = \int_0^\infty \int_\infty K_{ij}(x,t;\xi,\tau)\, d^3\xi\, d\tau$$

$$b_{ijk} = - \int_0^\infty \int_\infty K_{ij}(x,t;\xi,\tau)\, \xi_k\, d^3\xi\, d\tau \qquad (***)$$

Relations (*), (**) and (***) apply in the case of a (given) dependence of $\overline{u}$ and u' on $\overline{B}$, too.

However, this case is discussed only later.

See 6.7.

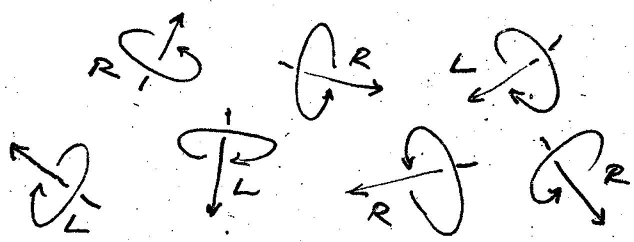
6.3.

Definitions concerning turbulence

Turbulence

- homogeneous
  if all mean quantities depending on u'
  are invariant under translation of u'
- steady
  ... under time shift in u'
- axisymmetric
  ... under rotation of u' about a given axis
- isotropic
  ... under rotation of u' about arbitrary axes
- reflectionally symmetric
  (mirror-symmetric)
  w.r.t. a given plane or a given point
  ... under reflexion of u' about this plane
  or at this point
- reflectionally symmetric
  (mirror-symmetric, gyrotropic)
  ... under reflexion of u' about arbitrary planes
  or at arbitrary points

Under reflexion right-handed structures
turn into left-handed ones
and vice versa.



● 6.4.

A simple (academic) example:

$\bar{u} = 0$

u' homogeneous isotropic turbulence

η constant

○ 6.4.1.

The structure of $\mathcal{E}$

$$\mathcal{E}_i = a_{ij} \bar{B}_j + b_{ijk} \partial \bar{B}_j / \partial x_k$$

u' homogeneous turbulence

$\curvearrowright$ a_{ij}, b_{ijk} independent of position

u' isotropic turbulence

$\curvearrowright$ a_{ij}, b_{ijk} isotropic tensors (i.e., their components do not change under rotations of the coordinate system about arbitrary axes)

$$\underbrace{a_{ij} = \alpha \delta_{ij} , \quad b_{ijk} = \beta \varepsilon_{ijk}}_{\text{determined by } u'}$$

$$\underline{\mathcal{E} = \alpha \bar{B} - \beta \underbrace{\nabla \times \bar{B}}_{\mu \bar{j}}} \ //$$

○ 6.4.2.

Ohm's law for mean fields

$$\bar{j} = \sigma(\bar{E} + \mathcal{E})$$

$$\curvearrowright \underline{\bar{j} = \underbrace{\frac{\sigma}{1 + \mu\sigma\beta}}_{\sigma_M} (\bar{E} + \alpha \bar{B})} \ //$$

$\alpha \bar{B}$: »α-effect«

$\sigma_M \leftarrow$ mean-field conductivity

O 6.4.3.

The dependence of α and β

on symmetry properties of u'

Relation $\mathcal{E} = \overline{u' \times B'} = \alpha \overline{B} - \beta \nabla \times \overline{B}$ (#)

has to be considered as a consequence

of $\nabla \times (\eta \nabla \times B - u \times B) + \partial_t B = 0, \; \nabla \cdot B = 0,$ (##)

with $B = \overline{B} + B'$ and $u = u'$.

Equations (##) remain valid if $\overline{B}, B'$ and u'

are reflected at $x = 0$,

i.e., replaced by $\overline{B}^{rfl}$, B'^{rfl} and u'^{rfl},

where $F^{rfl}(x) = -F(-x)$.

The same has to apply to (#).

Original relation (#) at $x=0$:

$$\overline{u'(0) \times B'(0)} = \alpha(u') \overline{B}(0) - \beta(u')(\nabla \times \overline{B})(0) \qquad (X)$$

Modified relation (#) at $x=0$:

$$\overline{u'^{rfl}(0) \times B'^{rfl}(0)} = \alpha(u'^{rfl}) \overline{B}^{rfl}(0) - \beta(u'^{rfl})(\nabla \times \overline{B}^{rfl})(0)$$

$\curvearrowright$

$$\overline{u'(0) \times B'(0)} = -\alpha(u'^{rfl}) \overline{B}(0) - \beta(u'^{rfl})(\nabla \times \overline{B})(0) \qquad (XX)$$

$$\left\{ \begin{aligned} (\nabla \times \overline{B})_x &= \frac{\partial \overline{B}_z(\vec{x})}{\partial y} - \frac{\partial \overline{B}_y(\vec{x})}{\partial z} \\ (\nabla \times \overline{B}^{rfl})_x &= \frac{\partial \overline{B}_z(-\vec{x})}{\partial(-y)} - \frac{\partial \overline{B}_y(-\vec{x})}{\partial(-z)} \\ &\curvearrowright (\nabla \times \overline{B}^{rfl})(0) = (\nabla \times \overline{B})(0) \end{aligned} \right.$$

Comparison of (X) and (XX)

$\curvearrowright$ $\underline{\alpha(u'^{rfl}) = -\alpha(u') \;, \; \beta(u'^{rfl}) = \beta(u')}$!!

If turbulence reflectionally symmetric

then $\alpha = 0$

6.4.4.

Dynamo action of a homogeneous isotropic not reflectionally symmetric turbulence

$$\eta_M \nabla^2 \bar{B} + \alpha \nabla \times \bar{B} - \frac{\partial \bar{B}}{\partial t} = 0 \qquad \eta_M = \eta + \beta$$

(suppose > 0)

$$\nabla \cdot B = 0$$

Ansatz

$$\bar{B} = \mathrm{Re}\left(\hat{B} \exp\{ i k \cdot x + pt \}\right)$$

$\curvearrowright$

$$(\eta_M k^2 + p)\hat{B} - \alpha i k \times \hat{B} = 0$$

$$k \cdot \hat{B} = 0$$

Specify coordinate system so that $k = (0,0,k)$

$\curvearrowright$

$$(\eta_M k^2 + p)\hat{B}_x + i\alpha k \hat{B}_y = 0$$

$$i\alpha k \hat{B}_x - (\eta_M k^2 + p)\hat{B}_y = 0$$

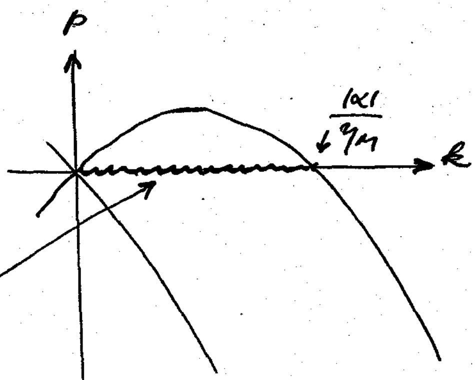
$$\hat{B}_z = 0$$

$\curvearrowright$

$$(\eta_M k^2 + p)^2 = \alpha^2 k^2$$

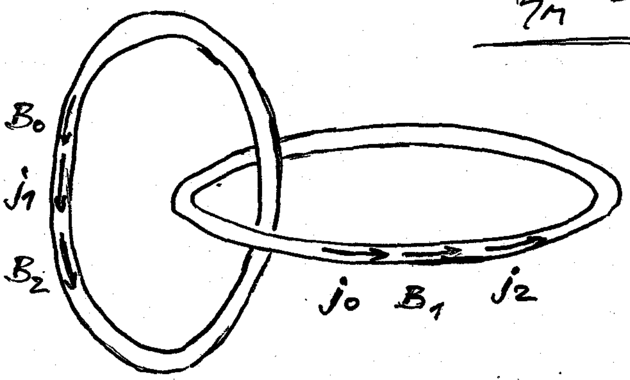
$\curvearrowright$

$$p = -\eta_M k^2 \pm |\alpha| k$$



DYNAMO: $k \leq \frac{|\alpha|}{\eta_M}$

$k = \frac{2\pi}{\ell}$: $\underline{\frac{|\alpha| \ell}{\eta_M} \geq 2\pi}$



$$\nabla \times B = \mu j$$

$$j = \sigma(E + \alpha B)$$

$$\alpha > 0$$

o 6.4.5.

The coefficients α and β as functions of u'

$$\eta \nabla^2 B' - \partial_t B' = -\nabla \times \left(u' \times \overline{B} + \left(u' \times B' - \overline{u' \times B'} \right) \right), \quad \nabla \cdot B' = 0$$

First-order smoothing approximation (FOSA)

= Second-order correlation approximation (SOCA):

assume u' so small that $|B'| \ll |\overline{B}|$

and cancel ◯.

Remember heat conduction equation

$$\eta \Delta T - \partial_t T = -q$$

$$T(x,t) = \int_\infty G(x-x', t-t_0) T(x', t_0) \, d^3x'$$

$$+ \int_{t_0}^{t} \int_\infty G(x-x', t-t') \, q(x', t') \, d^3x' \, dt'$$

$$G(x,t) = (4\pi\eta t)^{-3/2} \exp(-x^2/4\eta t)$$

$$\eta \Delta G - \partial_t G = 0 \quad \text{for } t > 0$$

$$G(x,t) \to \delta^3(x) \quad \text{as } t \to +0$$

$$B'_k(x,t) = \int_\infty G(x-x', t-t_0) B'_k(x', t_0) \, d^3x'$$

$$+ \varepsilon_{klm} \varepsilon_{mnp} \int_{t_0}^{t} \int_\infty G(x-x', t-t') \frac{\partial}{\partial x'_l} \left(u'_n(x', t') \overline{B}_p(x', t') \right) d^3x' \, dt' \qquad (*)$$

$$\mathcal{E}_i(x,t) = \varepsilon_{ijk} \overline{u'_j(x,t) B'_k(x,t)}$$

$$= \int_0^\infty \int_\infty K_{ip}(x,t;\xi,\tau) \overline{B}_p(x-\xi, t-\tau) \, d^3\xi \, d\tau$$

Subsitute

$x' = x - \xi$, $t' = t - \tau$ in $(*)$ and let $t_0 \to -\infty$

$$K_{ip}(x,t;\xi,\tau) = \varepsilon_{ijk} \varepsilon_{klm} \varepsilon_{mnp} \frac{1}{\xi} \frac{\partial G(\xi,\tau)}{\partial \xi} \overline{u'_j(x,t) u'_n(x+\xi, t-\tau)} \, \xi_l$$

$$\left(+ \frac{\partial V_i(x,t;\xi,\tau)}{\partial \xi_p} \right)$$

no x-dependence in this example

V_i arbitrary vector field

|ξ|

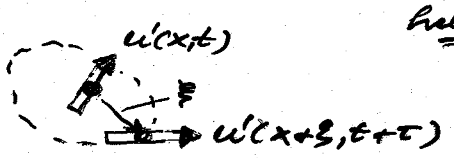
$$\alpha = \frac{1}{3} a_{ii}$$

$$= -\frac{1}{3}\int_0^\infty \int_\infty G(\xi,\tau)\, \overline{u'(x,t)\cdot(\nabla\times u'(x+\xi,t-\tau))}\, d^3\xi\, d\tau$$

|ξ|

$$\overline{u'(x,t)\cdot(\nabla\times u'(x+\xi,t+\tau))} = h(\xi,\tau)$$

helicity



$u'_\xi = (u'\cdot\xi)/\xi$

|ξ|

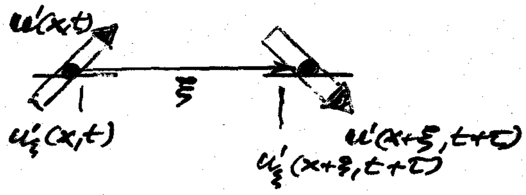
$$\beta = \frac{1}{6}\varepsilon_{ijk} b_{ijk}$$

$$= -\int_0^\infty \int_\infty \frac{\partial G(\xi,\tau)}{\partial \xi}\, \overline{u'_\xi(x,t)\, u'_\xi(x+\xi,t-\tau)}\, \xi\, d^3\xi\, d\tau$$

|ξ|

$$\overline{u'_\xi(x,t)\, u'_\xi(x+\xi,t+\tau)} = \ell(\xi,\tau)$$

longitudinal correlation function



• High-conductivity limit $\eta \to 0$, more precisely $\eta\tau_{cor}/\lambda_{cor}^2 \to 0$

$\left\{ G(\xi,\tau) \to \delta^3(\xi) \right.$ if $\eta \to 0$

$$\alpha = -\frac{1}{3}\int_0^\infty \overline{u'(x,t)\cdot(\nabla\times u'(x,t-\tau))}\,d\tau$$

$$= -\frac{1}{3}\overline{u'\cdot(\nabla\times u')}\,\tilde{\tau}_{cor} \quad /\!/$$

different signs of α and helicity

$$\beta = \frac{1}{3}\int_0^\infty \overline{(u'(x,t)\cdot u'(x,t-\tau))}\,d\tau$$

$$= \frac{1}{3}\overline{u'^2}\,\tau_{cor} \quad /\!/$$

Sufficient (but not necessary) for validity:

$$u'\tau_{cor}/\lambda_{cor} \ll 1$$

Solar convection zone

$$\sigma = 3\cdot 10^3\, S/m$$

$$\left.\begin{aligned} u' &= 2\cdot 10^2\, m/s \\ \tau_{cor} &= 3\cdot 10^2 s \\ \lambda_{cor} &= 2\cdot 10^6 m \end{aligned}\right\} \text{granules}$$

$$\eta\tau_{cor}/\lambda_{cor}^2 = 2\cdot 10^{-5}$$

$$u'\tau_{cor}/\lambda_{cor} = 3\cdot 10^{-2}$$

$$\frac{\sigma_M}{\sigma} = (1+\mu\sigma\beta)^{-1} = 0.7\cdot 10^{-4}$$

- Low-conductivity limit $\eta \to \infty$, more precisely $\eta \tau_{cor}/\lambda_{cor}^2 \to \infty$ $\quad \Big\{ \int_0^\infty G(\xi,\tau)\,d\tau = \frac{1}{4\pi\eta\xi}$

$$\alpha = -\frac{1}{12\pi\eta}\int_\infty \overline{u'(x,t)\cdot(\nabla\times u'(x+\xi,t))}\,\xi^{-1}d^3\xi$$

$$\beta = \frac{1}{12\pi\eta}\int_\infty \overline{u'_\xi(x,t)\,u'_\xi(x+\xi,t)}\,\xi^{-1}d^3\xi$$

For an alternative representation put $u' = \nabla\times a' + \nabla\varphi'$, $\nabla\cdot a' = 0$.

$$\Big\{\ a'(x) = \frac{1}{4\pi}\int_\infty (\nabla\times u'(x+\xi))\,\xi^{-1}d^3\xi$$

$\curvearrowright$

$$\alpha = -\frac{1}{3\eta}\,\overline{a'\cdot(\nabla\times a')}$$

$$\beta = \frac{1}{3\eta}\left(\overline{a'^2} - \overline{\varphi'^2}\right)$$

If $\overline{u'_\xi(x,t)\,u'_\xi(x+\xi,t)} = \frac{1}{3}\,\overline{u'^2}\exp(-\xi^2/2\lambda_{cor}^2)$

then $\beta = \frac{1}{9}\frac{u'^2\lambda_{cor}^2}{\eta}$

$$\frac{\beta}{\eta} = \frac{1}{9}R_m^2, \quad R_m = \frac{u'\lambda_{cor}}{\eta}$$

Sufficient (but not necessary) for validity:

$$R_m = \frac{u'\lambda_{cor}}{\eta} \ll 1$$

● 6.5.

More realistic cases

(i.e. $\overline{u}$ no longer necessarily $=0$,

u' no longer homogeneous isotropic turbulence)

○ 6.5.1.

A simple (but realistic) example

$\overline{u}=0$

u' axisymmetric turbulence

preferred axis defined by a vector λ

a_{ij}, b_{ijk} axisymmetric tensors

$$a_{ij} = a_1 \delta_{ij} + a_2 \varepsilon_{ijl} \lambda_l + a_3 \lambda_i \lambda_j$$

$$b_{ijk} = b_1 \varepsilon_{ijk}$$

because $\partial \overline{B}_j / \partial x_j = 0$ without interest!

$$+ b_2 \delta_{ij} \lambda_k + b_3 \delta_{ik} \lambda_j + b_4 \delta_{jk} \lambda_i$$

$$+ b_5 \varepsilon_{ijl} \lambda_l \lambda_k + b_6 \varepsilon_{ikl} \lambda_l \lambda_j + b_7 \varepsilon_{jkl} \lambda_l \lambda_i$$

$$+ b_8 \lambda_i \lambda_j \lambda_k$$

without loss of generality $b_6 = b_5$,

for $\varepsilon_{ijl} \lambda_l \lambda_k - \varepsilon_{ikl} \lambda_l \lambda_j$

$= \varepsilon_{ijk} \lambda^2 - \varepsilon_{jkl} \lambda_l \lambda_i$

$$\mathcal{E} = a_1 \overline{B} - a_2 \lambda \times \overline{B} + a_3 (\lambda \cdot \overline{B}) \lambda$$

$$- b_1 \nabla \times \overline{B}$$

$$+ b_2 \, \lambda . \nabla \overline{B} + b_3 \, \lambda^{\cdot} \nabla \overline{B}$$

$$- b_5 \lambda \times (\lambda . \nabla \overline{B} + \lambda^{\cdot} \nabla \overline{B}) - b_7 (\lambda \cdot (\nabla \times \overline{B})) \lambda$$

$$+ b_8 (\lambda \cdot (\lambda . \nabla \overline{B})) \lambda$$

Coefficients $a_1, a_2, \ldots b_8$ may depend on $|\lambda|$

$$(\lambda . \nabla \overline{B})_i = \lambda_j \frac{\partial \overline{B}_i}{\partial x_j} = ((\lambda \cdot \nabla) \overline{B})_i$$

$$(\lambda^{\cdot} \nabla \overline{B})_i = \lambda_j \frac{\partial \overline{B}_j}{\partial x_i} \; , \; = (\nabla (\lambda \cdot \overline{B}))_i$$

if λ constant

• Assume that u' reflectionally symmetric
about planes containing λ

$\lambda = g$ (e.g., gradient of turbulence intensity)

$$a_1 = a_3 = b_2 = b_3 = b_8 = 0$$

$$\mathcal{E} = -a_2\, g \times \overline{B} \quad \leftarrow \text{transport of mean magnetic flux}$$
$$-b_1\, \nabla \times \overline{B} \quad \text{»turbulent diamagnetism«}$$

− terms of 2^{nd} order in g » pumping of ... flux«

Rädler 1966

• Assume that u' reflectionally symmetric
about plane perpendicular to λ

$\lambda = \Omega$ (e.g., Coriolis force)

$$a_1 = a_2 = a_3 = 0$$

$$\mathcal{E} = -b_1 \nabla \times \overline{B}$$
$$\underline{+ b_2\, \Omega \cdot \nabla \overline{B} + b_3\, \Omega \cdot \nabla \overline{B}}\ *$$

+ terms of 2^{nd} and 3^{rd} order in Ω

$$\Omega \times (\nabla \times \overline{B}) = -(\Omega \cdot \nabla)\overline{B} + \nabla(\Omega \cdot \overline{B})$$

$$* = -b_2\, \Omega \times (\nabla \times \overline{B}) - (b_2 - b_3)\nabla(\Omega \cdot \overline{B})$$

» $\Omega \times j$-effect «
in combination with
differential rotation
capable of <u>dynamo action</u>

See 8.3.4.

• No α-effect as soon as u' reflectionally symmetric
about any plane !!!

o 6.5.2
Another representation of $\mathcal{E}$

$$\mathcal{E}_i = a_{ij}\,\overline{B}_j + b_{ijk}\,\partial\overline{B}_j/\partial x_k$$

Split a_{ij} into $a_{ij}^{(s)}$ and $a_{ij}^{(a)}$
and put $a_{ij}^{(a)} = \varepsilon_{ijl}\gamma_l$ with some vector γ.
Split $\partial\overline{B}_j/\partial x_k$ into $(\partial\overline{B}_j/\partial x_k)^{(s)}$ and $(\partial\overline{B}_j/\partial x_k)^{(a)}$
and put $(\partial\overline{B}_j/\partial x_k)^{(a)} = \varepsilon_{jkl}R_l$
with $R = \frac{1}{2}\nabla\times\overline{B}$

$$\mathcal{E} = -\alpha\cdot\overline{B} - \gamma\times\overline{B}$$
$$-\beta\cdot\nabla\times\overline{B} - \delta\times(\nabla\times\overline{B}) - \kappa\cdot(\nabla\overline{B})^{(s)}$$

α, β symmetric tensors of 2nd rank
γ, δ vectors
κ tensor of 3rd rank
with some symmetries

$$\alpha_{ij} = -\tfrac{1}{2}(a_{ij}+a_{ji}),\quad \gamma_i = \tfrac{1}{2}\varepsilon_{ijk}a_{jk}$$
$$\beta_{ij} = \tfrac{1}{4}(\varepsilon_{ikl}b_{jkl} + \varepsilon_{jkl}b_{ikl})$$
$$\delta_i = \tfrac{1}{4}(b_{jji} - b_{jij}),\quad \kappa_{ijk} = -\tfrac{1}{2}(b_{ijk}+b_{ikj})$$

o 6.5.3.
Ohm's law for mean fields

$$\overline{j} = \sigma_M\cdot(\overline{E} + (\overline{u}-\gamma)\times\overline{B} - \alpha\cdot\overline{B} - \delta\times(\nabla\times\overline{B}) - \kappa\cdot(\nabla\overline{B})^{(s)})$$

anisotropic α-effect
transport of mean magnetic flux

$$\sigma_{Mij} = \sigma(\delta_{ij} + \mu\sigma\beta_{ij})^{-1}$$ symmetric tensor

or

$$j = \sigma_M\cdot(\overline{E} + (\overline{u}-\gamma)\times\overline{B} - \alpha\cdot\overline{B} - \kappa\cdot(\nabla\overline{B})^{(s)})$$

$$\sigma_{Mij} = \sigma(\delta_{ij} + \mu\sigma(\beta_{ij} - \varepsilon_{ijk}\delta_k))^{-1}$$

no longer symmetric!

o 6.5.4.

Symmetry properties of the basic equations and consequences for the structure of E

$$\left.\begin{array}{ll} \nabla \times E = -\partial_t B & B = \mu H \\ \nabla \cdot B = 0 & j = \sigma(E + u \times B) \\ \nabla \times H = j & \end{array}\right\} (*)$$

If these equations are satisfied with B, H, E, j and u they are, too,

- if B, H, E, j and u are subject to a rotation about an arbitrary axis (ROT)
- if B, H, E, j and u are subject to a reflexion about an arbitrary plane or at any point (RFL) and, in addition, B and H to an inversion of their signs (SGN)

Define an operation RFL* which acts
like RFL on E, j and u
and like RFL plus SGN on B and H.

If the above equations are satisfied with B, H, E, j and u they are, too, if these fields are subject to the operations ROT or RFL*.

Define in addition to the »true« scalars, vectors and tensors, which behave under both rotations and reflexions of the (Cartesian) coordinate system like

$$S' = S, \quad V_i' = a_{ik} V_k, \quad T_{ij}' = a_{ik} a_{jl} T_{kl}, \dots,$$

»pseudo« scalars, vectors and tensors behaving under rotations in the same way but under reflexions like $S' = -S$, $V_i' = -a_{ik} V_k$, $T_{ij}' = -a_{ik} a_{jl} T_{kl}, \dots$.

{ true vector = polar vector
{ pseudo vector = axial vector

- δ_{ij}, defined such that always $\delta_{11} = 1$, is a true tensor
- ε_{ijk}, ... $\varepsilon_{123} = 1$, is a pseudo tensor
- $(V \cdot W)$, with both V and W being either true or pseudo vectors, is a true scalar
  $(V \cdot W)$, with V being a true and W pseudo vector, is a pseudo vector
- $V \times W$, ... both V and W... either true or pseudo vector, is a pseudo vector
  $V \times W$, ... V... true ... W... pseudo vector, is a true vector
- $\nabla \times V$, with V being a true vector, is a pseudo vector
  $\nabla \times V$, ... V ... pseudo vector, is a true vector

System (*)
with E, j and u understood as true
and B and H as pseudo quantities
contains only equalities
between either true or pseudo quantities
(i.e. not between a true and a pseudo quantity).

The same must apply for all consequences of (*),
e.g.,

$$\underline{\mathcal{E}_i} = \underline{\underline{a_{ij}}}\,\underline{\underline{\overline{B}_j}} + \underline{\underline{b_{ijk}}}\,\partial\underline{\underline{\overline{B}_j}}/\partial x_k \qquad - \text{true}$$

or

$$\underline{\mathcal{E}} = -\underline{\underline{\alpha}}\cdot\underline{\underline{\overline{B}}} - \underline{\gamma}\times\underline{\underline{\overline{B}}} \qquad = \text{pseudo}$$

$$- \underline{\beta}\cdot\underline{\nabla\times\overline{B}} - \underline{\underline{\delta}}\times(\underline{\nabla\times\overline{B}}) - \underline{\underline{\kappa}}\cdot\underline{\underline{(\nabla\overline{B})}}^{(s)}$$

Extension of the concept to other basic equations,
e.g.,

$$\underline{\rho}(\underline{\partial_t u} + \underline{(u\cdot\nabla)u}) = -\underline{\nabla p} + \underline{\rho}\,\underline{f} - 2\underline{\rho}\,\underline{\underline{\Omega}}\times\underline{u}$$

o 6.5.5.

Inhomogeneous turbulence
influenced by the Coriolis force and a mean motion

- Consider $\mathcal{E}$ at a given point $x = x_0$ in a rotating frame of reference.
  Note that $\mathcal{E}$, considered as a vector, is the same in other frames, too.
- Specify the frame so that $\overline{u} = 0$ at $x = x_0$, denote the angular velocity responsible for the Coriolis force by Ω and the (tensorial) gradient of $\overline{u}$ by $\nabla\overline{u}$.
- Assume that the inhomogeneity of the turbulence can be described by a vector g.
- Assume that any helicity of the turbulence occurs only as a consequence of Ω, $\nabla\overline{u}$ and g.
- Assume that in the relevant surroundings of the point $x = x_0$ both $\nabla\overline{u}$ and g can be considered as constant.

Construction elements for $\mathcal{E}$

- isotropic tensors δ_{ij} and ε_{ijk}
  Higher-rank isotropic tensors are combinations of them
- Ω, $\nabla\overline{u}$ and g

$$(\nabla\overline{u})_{ij} = (\nabla\overline{u})^{(s)}_{ij} - \tfrac{1}{2}\varepsilon_{ijk}w_k \ , \quad w = \nabla\times\overline{u}$$

No pseudo scalars independent of Ω, $\nabla\overline{u}$ and g !

• Assume (for simplicity only) that the effects of Ω, $\nabla\bar{u}$ and g are weak enough so that $\mathcal{E}$ can be considered as

- linear in Ω and $\nabla\bar{u}$ ← no rapid rotation!
- linear in g ← Contributions to $\mathcal{E}$ bilinear in Ω and g or $\nabla\bar{u}$ and g admitted!

and that $\nabla\cdot\bar{u} = 0$.

$$\mathcal{E} = -\alpha\cdot\bar{B} - \gamma\times\bar{B}$$
$$- \beta\cdot\nabla\times\bar{B} - \delta\times(\nabla\times\bar{B}) - \kappa\cdot(\nabla\bar{B})^{(s)}$$

$$\alpha_{ij} = \alpha_1^{(\Omega)}(g\cdot\Omega)\delta_{ij} + \alpha_2^{(\Omega)}(g_i\Omega_j + g_j\Omega_i) \qquad C1$$
$$+ \alpha_1^{(w)}(g\cdot w)\delta_{ij} + \alpha_2^{(w)}(g_i w_j + g_j w_i) \qquad C2$$
$$+ \alpha^{(us)}(\varepsilon_{ilm}(\nabla\bar{u})^{(s)}_{lj} + \varepsilon_{jlm}(\nabla\bar{u})^{(s)}_{li})g_m$$

$$\gamma_i = \gamma^{(0)} g_i$$
$$+ \gamma^{(\Omega)}\varepsilon_{ilm} g_l \Omega_m + \gamma^{(w)}\varepsilon_{ilm} g_l w_m$$
$$+ \gamma^{(us)}(\nabla\bar{u})^{(s)}_{il} g_l$$

$$\beta_{ij} = \beta^{(0)}\delta_{ij}$$
$$+ \beta^{(us)}(\nabla\bar{u})^{(s)}_{ij}$$

$\alpha_1^{(\Omega)}, \alpha_2^{(\Omega)}, \ldots \kappa^{(us)}$ functions of turbulence intensity, density stratification etc. C3

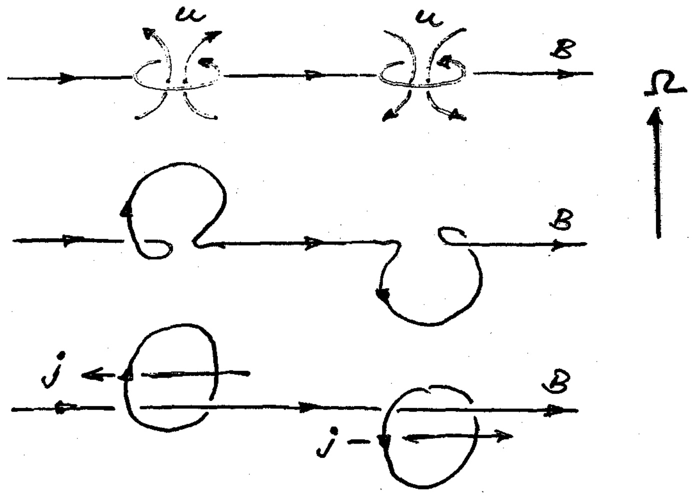
$$\delta_i = \delta^{(\Omega)}\Omega_i + \delta^{(w)} w_i \qquad C4$$

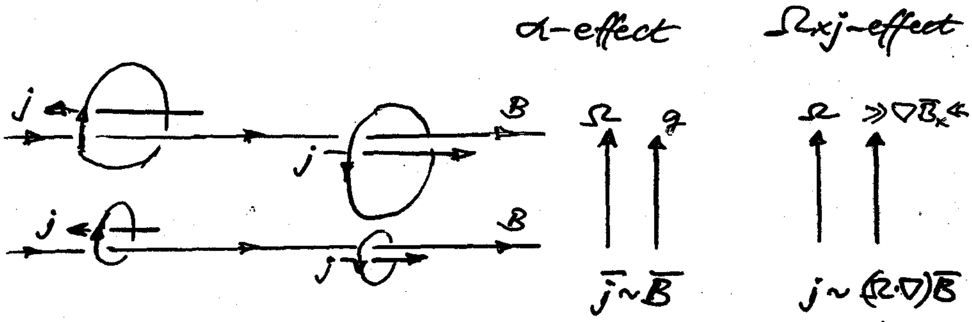
$$\kappa_{ijk} = \kappa^{(\Omega)}(\delta_{ij}\Omega_k + \delta_{ik}\Omega_j) + \kappa^{(w)}(\delta_{ij} w_k + \delta_{ik} w_j)$$
$$+ \kappa^{(us)}(\varepsilon_{ijl}(\nabla\bar{u})^{(s)}_{lk} + \varepsilon_{ikl}(\nabla\bar{u})^{(s)}_{lj})$$

Rädler, Kleeorin & Rogachevskii 2003
Rogachevskii & Kleeorin 2003

C1 & C4

Magnetic field lines and convective motions





$$\Omega\times(\nabla\times\bar{B}) = -(\Omega\cdot\nabla)\bar{B} + \nabla(\Omega\cdot\bar{B})$$

Under realistic conditions an α-effect occurs only as a consequence of both

- an inhomogeneity of the turbulence
- and the influence of a Coriolis force (Ω) or of a mean motion (w, $(\nabla\bar{u})^{(s)}$).

The δ-effect (which in combination with differential rotation is also capable of dynamo action) works already with an homogeneous turbulence under the influence of a Coriolis force (Ω) or of a mean motion (w).

C2 & C3

If the assumption of weak influence of the Coriolis force is relaxed (i.e. rapid rotation is admitted) new terms occur in α_{ij} and β_{ij}, e.g.,

$$\alpha_{ij} = \cdots + \alpha_3^{(\Omega)} (g \cdot \Omega) \Omega_i \Omega_j$$

$$\beta_{ij} = \cdots + \beta_1^{(\Omega)} \Omega_i \Omega_j ,$$

and all coefficients may depend on $|\Omega|$.

● 6.6.

The case of MHD turbulence

○ 6.6.1

Extension of the concept discussed so far

Assume now

that B' does not decay to zero if $\overline{B} = 0$

(i.e., existence of a small-scale dynamo,

» background turbulence « no longer purely hydrodynamic).

Denote u' and B' for $\overline{B} = 0$ by $u'^{(0)}$ and $B'^{(0)}$.

Consider small $\overline{B}$ and put $u' = u'^{(0)} + u'^{(1)}$

$$B' = B'^{(0)} + B'^{(1)}$$

with $u'^{(1)}$ and $B'^{(1)}$ being linear and homogeneous in $\overline{B}$

$\curvearrowright$

$$\mathcal{E} = \mathcal{E}^{(0)} + \mathcal{E}^{(\overline{B})}$$

$$\mathcal{E}^{(0)} = \overline{u'^{(0)} \times B'^{(0)}} \quad \leftarrow \text{no longer necessarily} = 0\,!$$

$$\mathcal{E}^{(\overline{B})} = \overline{u'^{(0)} \times B'^{(1)}} + \overline{u'^{(1)} \times B'^{(0)}}$$

$$\partial_t B' - \ldots = \nabla \times (u' \times \overline{B})$$

$$\partial_t u' + \cdots = \cdots + f' + \frac{1}{\mu\varrho}\left((\nabla \times B') \times \overline{B} + (\nabla \times \overline{B}) \times B'\right)$$

generates fluctuating motions

$\curvearrowright$

$$\partial_t B'^{(1)} - \ldots = \nabla \times (u'^{(0)} \times \overline{B})$$

$$\partial_t u'^{(1)} - \ldots = \cdots + \frac{1}{\mu\varrho}\left((\nabla \times B'^{(0)}) \times \overline{B} + (\nabla \times \overline{B}) \times B'^{(0)}\right)$$

$\curvearrowright$

$$\left.\begin{array}{l} B'^{(1)} \text{ functional of } \overline{u}, u'^{(0)} \text{ and } \overline{B} \\ u'^{(1)} \text{ functional of } \overline{u}, B'^{(0)} \text{ and } \overline{B} \end{array}\right\} \text{both linear and homogeneous in } \overline{B}$$

$\curvearrowright$

$$\underline{\mathcal{E}^{(\overline{B})} = \mathcal{E}^{(\overline{B},u)}(\ldots, u'^{(0)}, \overline{B}) + \mathcal{E}^{(\overline{B},b)}(\ldots, B'^{(0)}, \overline{B})}$$

linear and homogeneous in $\overline{B}$

o 6.6.2

A simple (academic) example

$\overline{u} = 0$

$(u'^{(0)}, B'^{(0)})$ homogeneous isotropic MHD turbulence

$\mathcal{E}^{(0)} = 0$ ← no non-zero mean vector other than $\overline{B}$ available!

$$\mathcal{E} = \alpha \overline{B} - \beta \nabla \times \overline{B}$$

Calculation of α and β in the limit $\eta \to 0$

$$\alpha = -\frac{1}{3}\left(\overline{u'^{(0)} \cdot (\nabla \times u'^{(0)})} - \frac{1}{\mu \varrho} \overline{B'^{(0)} \cdot (\nabla \times B'^{(0)})} \right) \tau_{cor}$$

$$\beta = \frac{1}{3} \overline{u'^{(0)2}} \tau_{cor}$$

$B'^{(0)}$ is only that part of B' which exists independent of $\overline{B}$!

no »magnetic contribution« to β!

Pouquet et al. 1976
Zeldovich et al. 1983
Vainshtein & Kichatinov 1983

o 6.6.3

More realistic cases

Considerations concerning the structure of $\mathcal{E}$ presented so far apply with slight modifications.

The result for $\mathcal{E}$ given under
»Inhomogeneous turbulence
influenced by the Coriolis force and a mean motion«
applies with modified interpretations of g,
e.g.,

$$\alpha_1^{(\Omega)}(g\cdot\Omega) \rightarrow \frac{16}{15}\left(\nabla\left(\overline{u^{(0)2}} - \frac{1}{3\mu\rho}\overline{B^{(0)2}}\right)\cdot\Omega\right)$$

$$\gamma^{(0)} g \rightarrow \frac{1}{6}\nabla\underline{\left(\overline{u^{(0)2}} - \frac{1}{\mu\rho}\overline{B^{(0)2}}\right)}_{*}$$

$$\gamma^{(\Omega)} g\times\Omega \rightarrow -\frac{2}{9}\nabla\left(\overline{u^{(0)2}} + \frac{1}{\mu\rho}\overline{B^{(0)2}}\right)\times\Omega$$

$$\delta^{(\Omega)} \rightarrow -\frac{2}{9}\underline{\left(\overline{u^{(0)2}} - \frac{1}{\mu\rho}\overline{B^{(0)2}}\right)}_{*}$$

$*$ Equipartition $\overline{u^{(0)2}} = \frac{1}{\mu\rho}\overline{B^{(0)2}}$

$$\rightsquigarrow \gamma^{(0)} g = 0\,, \quad \delta^{(\Omega)} = 0\ !$$

Rädler, Kleeorin & Rogachevskii 2003

● 6.7.

Larger mean magnetic fields,
quenching effects

(Both cases, $B' \to 0$ and $B' \not\to 0$ for $\overline{B} = 0$,
again admitted)

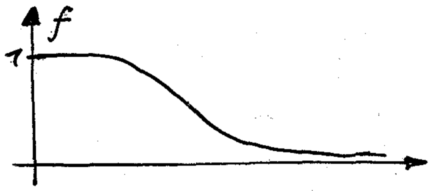
○ 6.7.1.

A heuristic ansatz

$$\mathcal{E} = \alpha \overline{B} - \beta \nabla \times \overline{B}$$

values for the limit $\overline{B} \to 0$

$$\alpha = \alpha_0 f_\alpha(\overline{B}^2)$$
$$\beta = \beta_0 f_\beta(\overline{B}^2)$$



$B_{eq}^2 / \mu\rho = \overline{u'^{(0)2}}$

e.g.

$$f = \frac{1}{1 + \underset{?}{\underline{\underline{c}}}\, \overline{B}^2 / B_{eq}^2}$$

What about

- other contributions to $\mathcal{E}$
  which are nonlinear in $\overline{B}$
- dependencies of α and β
  on derivatives of $\overline{B}$?

o 6.7.2.

A simple (academic) example

$\bar{u} = 0$

u' or (u', B') (originally) homogeneous isotropic turbulence

$\bar{B}$ arbitrarily strong but spatial variations weak so that $\mathcal{E}$ linear in derivatives of $\bar{B}$

a_{ij}, b_{ijk} axisymmetric tensors w.r.t. an axis given by $\bar{B}$, independent on the sign of $\bar{B}$

$\rightsquigarrow$

$$\mathcal{E} = (\alpha - \alpha_1(\bar{B}\cdot(\nabla\times\bar{B})))\bar{B}$$
$$-(\gamma_1 \nabla\bar{B}^2 + \gamma_2(\bar{B}\cdot\nabla)\bar{B})\times\bar{B}$$
$$-\beta\nabla\times\bar{B}$$

with $\alpha, \alpha_1, \gamma_1, \gamma_2$ and β depending on $\bar{B}^2$

Alternative representation

$$\mathcal{E} = \alpha\bar{B}$$
$$-(\gamma_1\nabla\bar{B}^2 + \gamma_2(\bar{B}\cdot\nabla)\bar{B})\times\bar{B}$$
$$\underbrace{-\beta\,\nabla\times\bar{B} - \underset{\alpha_1}{\underset{\shortparallel}{\beta_1}}(\bar{B}\cdot(\nabla\times\bar{B}))\bar{B}}$$
$$-\beta_{ij}(\nabla\times\bar{B})_j \quad \text{with } \beta_{ij} = \beta\delta_{ij} + \beta_1\bar{B}_i\bar{B}_j$$

$$(\tfrac{1}{2}\nabla\bar{B}^2 - (\bar{B}\cdot\nabla)\bar{B})\times B = -\bar{B}^2\,\nabla\times\bar{B} - \bar{B}\cdot(\nabla\times\bar{B})\bar{B}$$

$\rightsquigarrow$ without loss of generality either $\alpha_1 = \beta_1 = 0$ or $\gamma_1 = 0$ or $\gamma_2 = 0$!

o 6.2.3.

More general cases

It seems inconsequent to discuss the nonlinear dependence of $\mathcal{E}$ on $\overline{B}$ (i.e. the dependence of quantities like α or β on $\overline{B}$) without considering how $\overline{u}$ depends on $\overline{B}$.

$\rightarrow$ MEAN-FIELD MAGNETOFLUID DYNAMICS

• 7.

MEAN-FIELD MAGNETOFLUIDDYNAMICS

• 7.1.

Basic equations

o 7.1.1.

Starting point

For simplicity incompressible fluid,
fluctuating motions generated by an external force

$$\partial_t \mathbf{B} = -\nabla\times(\eta\nabla\mathbf{B} - \mathbf{u}\times\mathbf{B})$$

$$\nabla\cdot\mathbf{B} = 0$$

$$\rho(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla p + \rho\nu\nabla^2\mathbf{u} - 2\rho\boldsymbol{\Omega}\times\mathbf{u} + \frac{1}{\mu}(\nabla\times\mathbf{B})\times\mathbf{B} + \mathbf{F}$$

$$\nabla\cdot\mathbf{u} = 0$$

o 7.1.2.

Mean-field equations

$$\partial_t\overline{\mathbf{B}} = -\nabla\times(\eta\nabla\overline{\mathbf{B}} - \overline{\mathbf{u}}\times\overline{\mathbf{B}} - \underline{\boldsymbol{\mathcal{E}}})$$

$$\nabla\cdot\overline{\mathbf{B}} = 0 \qquad \underline{\boldsymbol{\mathcal{E}} = \overline{\mathbf{u}'\times\mathbf{B}'}}$$

$$\rho(\partial_t\overline{\mathbf{u}} + (\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}}) = -\nabla\overline{p} + \rho\nu\nabla^2\overline{\mathbf{u}} - 2\rho\boldsymbol{\Omega}\times\overline{\mathbf{u}} + \frac{1}{\mu}(\nabla\times\overline{\mathbf{B}})\times\overline{\mathbf{B}} + \overline{\mathbf{F}} + \underline{\boldsymbol{\mathcal{F}}}$$

$$\nabla\cdot\overline{\mathbf{u}} = 0$$

$$\underline{\boldsymbol{\mathcal{F}} = -\rho\overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'} + \frac{1}{\mu}\overline{(\nabla\times\mathbf{B}')\times\mathbf{B}'}}$$

$$\mathcal{F}_i = \frac{\partial T_{ij}}{\partial x_j}$$

$$T_{ij} = \underbrace{-\rho\overline{u_i'u_j'}}_{\text{Reynolds stresses}} + \underbrace{\frac{1}{\mu}\left(\overline{B_i'B_j'} - \frac{1}{2}\overline{B'^2}\delta_{ij}\right)}_{\text{Maxwell stresses due to magnetic fluctuations}}$$

o 7.1.3.

Properties of $\mathcal{E}$ and $\mathcal{F}$

$$\partial_t B' = -\nabla\times(\eta\nabla B' - \bar{u}\times B' - u'\times\bar{B} - u'\times B' + \overline{u'\times B'})$$

$$\nabla\cdot B' = 0$$

$$\varrho(\partial_t u' + (\bar{u}\cdot\nabla)u' + (u'\cdot\nabla)\bar{u} + (u'\cdot\nabla)u' - \overline{(u'\cdot\nabla)u'})$$
$$= -\nabla p' + \varrho\nu\nabla^2 u' - 2\varrho\Omega\times u'$$
$$+\frac{1}{\mu}((\nabla\times\bar{B})\times B' + (\nabla\times B')\times\bar{B} + (\nabla\times B')\times B' - \overline{(\nabla\times B')\times B'})$$
$$+F'$$

$$\nabla\cdot u = 0$$

↝

B', u' are functionals of $\bar{B}$, $\bar{u}$ and F'

↝

$\mathcal{E}$, $\mathcal{F}$ are functionals of $\bar{B}$, $\bar{u}$ and F'

Assume that F' independent of $\bar{B}$ and $\bar{u}$.
Denote B' and u' for $\bar{B}=\bar{u}=0$ by $B'^{(0)}$ and $u'^{(0)}$
and express F' by these quantities.

↝

$\mathcal{E}$, $\mathcal{F}$ are functionals of $\bar{B}$, $\bar{u}$, $B'^{(0)}$ and $u'^{(0)}$ //

As for $\mathcal{E}$ see above.

8.

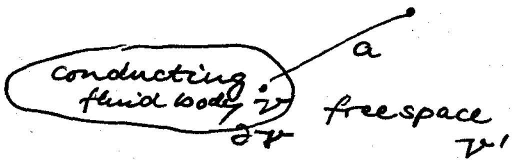
KINEMATIC MEAN-FIELD DYNAMO MODELS

8.1

Basic equations

8.1.1.

Starting point



$$\left.\begin{array}{l}\nabla\times(\eta\nabla\times B - u\times B) + \partial_t B = 0\\ \nabla\cdot B = 0\end{array}\right\}\text{ in }\mathcal{V}$$

$$\left.\begin{array}{l}\nabla\times B = 0\\ \nabla\cdot B = 0\end{array}\right\}\text{ in }\mathcal{V}'$$

$$[B] = 0 \quad \text{across } \partial\mathcal{V}$$

$$B \to 0 \text{ as } a \to \infty$$

8.1.2.

Mean-field equations

$$\left.\begin{array}{l}\nabla\times(\eta\nabla\times\overline{B} - \overline{u}\times\overline{B} - \underline{\mathcal{E}(\overline{B})}) + \partial_t\overline{B} = 0\\ \nabla\cdot\overline{B} = 0\end{array}\right\}\text{ in }\mathcal{V}$$

to be specified

$$\left.\begin{array}{l}\nabla\times\overline{B} = 0\\ \nabla\cdot\overline{B} = 0\end{array}\right\}\text{ in }\mathcal{V}'$$

$$[\overline{B}] = 0 \quad \text{across } \partial\mathcal{V}$$

$$\overline{B} \to 0 \text{ as } a \to \infty$$

»MEAN-FIELD DYNAMO«: $\overline{B} \not\to 0$ as $t \to \infty$

- Existence of a »mean-field dynamo« implies the existence of a dynamo (as discussed before).
- Mean fields are not subject to Cowling's theorem (unless $\mathcal{E}\cdot\overline{B} = 0$).
- Magnetic energy balance

$$\underbrace{\frac{d}{dt}\int_\infty \frac{\overline{B}^2}{2\mu}dv}_{\text{energy stored in the mean magnetic field}} = -\int_\mathcal{V}\frac{\overline{j}^2}{\sigma}dv - \int_\mathcal{V}\overline{u}\cdot(\overline{j}\times\overline{B})dv + \int_\mathcal{V}\overline{j}\cdot\mathcal{E}\,dv$$

● 8.2.

»Traditional« assumptions and consequences

○ 8.2.1.

Assume that

the shape of the fluid body
and the distribution of η
are
- symmetric about the rotation axis
- symmetric about the equatorial plane
- constant in time

Assume also that
all mean quantities depending on u $(= \overline{u} + u')$
are invariant under
- rotation of u about the rotation axis
- reflexion of u about the equatorial plane
- time shift in u

Simple consequences
- $\overline{u} = \overline{u}_{mer} + \overline{u}_{rot}$

  $\overline{u}_{mer}$ and $\overline{u}_{rot}$ symmetric about rotation axis and equatorial plane and steady
- $\overline{u' \cdot (\nabla \times u')}$
  symmetric about rotation axis, antisymmetric about equatorial plane and steady

o 8.2.2.

Then any solution $\overline{B}$ has the form,
or is a superposition of solutions of the form,

$$\overline{B} = Re(\hat{B} \exp(im\varphi + pt))$$

$p = \lambda + i\omega$ λ, ω real

- symmetric about rotation axis
- antisymmetric / or symmetric / about equatorial plane $\left.\begin{matrix} Am \\ Sm \end{matrix}\right\}$ modes
- steady

DYNAMO : $\underline{\lambda \geq 0}$

$\omega = 0$ monotoneous
$\omega \neq 0$ oscillatory   dependence on time

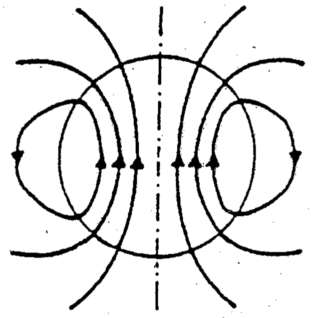
If $\omega \neq 0$

- axisymmetric ($m = 0$) modes
  intrinsically oscillatory
- non-axisymmetric ($m \neq 0$) modes
  waves traveling with angular velocity
  $\dot{\varphi} = -\omega/m$,
  in a proper frame of reference steady

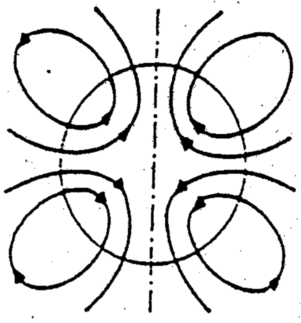
o 8.2.3.

Assume finally

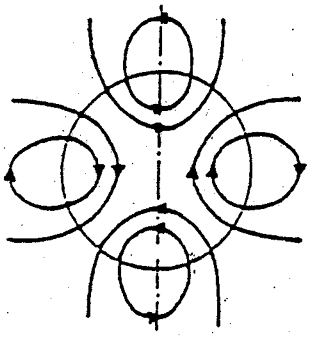
$$\mathcal{E} = -\alpha \cdot \overline{B} - \gamma \times \overline{B}$$
$$- \beta \cdot \nabla \times \overline{B} - \delta \times (\nabla \times \overline{B}) - \kappa \cdot (\nabla \overline{B})^{(s)}$$



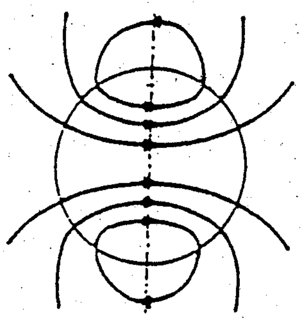
A0



S0



A1



S1

8.3.

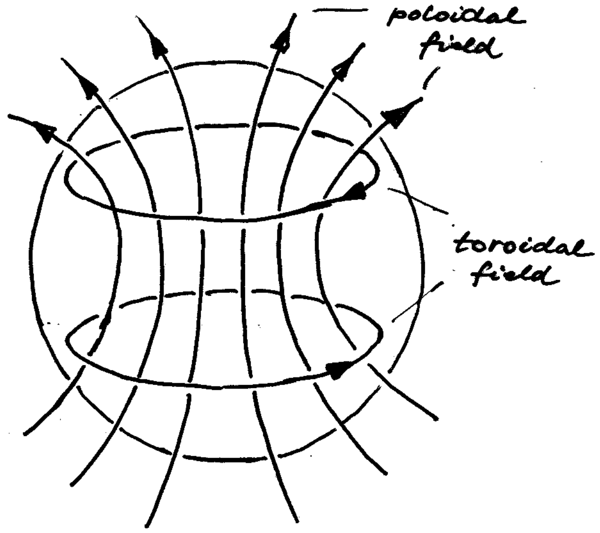
Basic dynamo mechanisms

8.3.1

In all cases investigated so far
interplay
between the poloidal and toroidal parts
of the magnetic field:

$$\begin{tikzcd} \boldsymbol{B}^P \arrow[r, bend left] & \boldsymbol{B}^T \arrow[l, bend left] \end{tikzcd}$$

α-effect dynamo models



$\alpha < 0$

$\alpha > 0$

$$\begin{tikzcd} P \arrow[r, bend left, "\nabla\omega"] & T \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

$\alpha\omega$ – model

$$\begin{tikzcd} P \arrow[r, bend left, "\alpha"] & T \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

α^2–model

○ 8.3.2.

α^2-mechanism

$$\begin{tikzcd} B^P \arrow[r, bend left, "\alpha"] & B^T \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

$$|B^P| \approx |B^T|$$

Steenbeck & Krause 1969
Roberts 1972
Roberts & Stix 1972
Rädler 1975, 1979, 1986
Busse 1979
Busse & Miin 1979
Rüdiger 1980
⋮

Excitation conditions for A0, S0, A1 and S1 modes very close together

A0 and S0 modes in general non-oscillatory

Anisotropies of the α-effect lead to preferences of non-axisymmetric magnetic fields

o 8.3.3.

$\alpha\omega$-mechanism

$$\begin{tikzcd} B^P \arrow[r, bend left, "\nabla\omega"] & B^T \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

$$|B^P| \ll |B^T|$$

Steenbeck & Krause 1969
Deinzer & Stix 1971
Stix 1971, 1973, 1976
Levy 1972
Roberts & Stix 1972
Köhler 1973
Ivanova & Ruzmaikin 1976
Yoshimura 1979
Rädler 1986
Schmidt 1987
Brandenburg 1988
⋮

One of the axisymmetric modes (A0 or S0) clearly preferred over all other modes.

Axisymmetric modes can be oscillatory or non-oscillatory (depending, e.g., on the signs of α and $\partial\omega/\partial r$).

Oscillatory behaviour includes equatorward or poleward migration of toroidal field belts (dynamo waves).

$\alpha^2\omega$-mechanism

$$\begin{tikzcd} B^P \arrow[r, bend left, "\alpha\ \nabla\omega"] & B^T \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

○ 8.3.4.

$\delta\omega$-mechanism

Rädler 1969, 1970, 1976, 1986

Roberts & Stix 1976

Stix 1976

$$\begin{tikzcd} B^P \arrow[r, bend left, "\nabla\omega"] & B^T \arrow[l, bend left, "\delta"] \end{tikzcd}$$

$$\mathcal{E} = \cdots - \delta^{(\Omega)}\, \Omega \times (\nabla \times \overline{B}) - \delta^{(\omega)}\, \omega \times (\nabla \times \overline{B}) \cdots$$

$$\omega = \nabla \times \overline{u}$$

No δ^2-mechanism

— see magnetic energy balance,

$$\overline{j} \cdot (\Omega \times \overline{j}) = \overline{j} \cdot (\omega \times \overline{j}) = 0$$

○ 8.3.5

$\beta\omega$-mechanism

Rädler 1986

$$\begin{tikzcd} B^P \arrow[r, bend left, "\nabla\omega"] & B^T \arrow[l, bend left, "\beta"] \end{tikzcd}$$

β — special components of the β tensor depending on Ω