# HYDRODYNAMICS

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**Bulletin of exercises n°1:** Kinematic aspects in Continuum Mechanics. The Jacobian and its geometrical interpretation. Equation of continuity.

The motion of a fluid is given by a function

$$\mathbf{x} = \mathbf{X}(\mathbf{a}, t)\,, \tag{1}$$

where $\mathbf{a}$ is the position vector at $t = 0$ of the fluid element which is at position $\mathbf{x}$ at time $t$: $\mathbf{X}(\mathbf{a}, 0) = \mathbf{a}$.

For each fixed $t$, (1) defines an invertible transformation of the continuum onto itself. The Jacobian determinant

$$J\,(\mathbf{a}, t) \stackrel{\text{def}}{=} \det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{a}}\right)_t = \det D\mathbf{X}(\mathbf{a}, t)$$

is, therefore, always different from zero.

## 1. Streamlines and trajectories.

Prove that for a time-independent velocity field $\mathbf{v}(\mathbf{x})$ the trajectories and the streamlines coincide geometrically.

## 2. Euler's identity.

For the proof of Reynolds' transport theorem use is made of the so-called *Euler's identity*,

$$\left(\frac{\partial J}{\partial t}\right)_{\mathbf{a}} = J \operatorname{div} \mathbf{v}\,,$$

where $J$ is the Jacobian determinant of the transformation $\mathbf{x} = \mathbf{X}(\mathbf{a}, t)$ and $\mathbf{v}(\mathbf{x}, t)$ is the velocity field.

Prove Euler's identity.

## 3. Incompressible flows.

A flow is said to be incompressible if the volume of any arbitrary portion of the fluid remains constant in time; i.e., if for any arbitrary $\Omega_{\mathrm{t}}$ it holds that

$$\frac{d}{dt}\int_{\Omega_{\mathrm{t}}} 1 \;=\; 0\,.$$

- Prove that a flow is incompressible **if and only if** the Jacobian $J$ is equal 1 at all times.
- Prove that a flow is incompressible **if and only if** $D\rho/Dt \equiv 0$.

## 4. Continuity equation in the material representation.

Prove that the assumption of mass conservation yiels the following relation,

$$\rho(\mathbf{a}, t)\, J(\mathbf{a}, t) = \rho(\mathbf{a}, 0)\,, \tag{2}$$

which can be called "the equation of continuity in the material (or *Lagrangian*) representation."

Check that you come to the same result if you start from the *customary form* of the continuity equation and apply Euler's identity.