HYDRODYNAMICS

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Bulletin of exercises n° 3: Basic equations in conservation form.

1. Interpretation of the continuity equation.

The continuity equation can be written in the form:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho\,\mathbf{v}) = 0\,. \tag{1}$$

Integrate Eq. (1) in a **fixed region** $\mathcal{R}$ in 3-D space with boundary $\partial\mathcal{R}$ and apply Gauß' theorem. Interpret the result and justify the name *mass flow vector* for the vector quantity $\rho\mathbf{v}$.

Note that this interpretation of the continuity equation is equally valid for any other equation that can be formally written in the form (1), viz.

$$\frac{\partial Q}{\partial t} + \mathrm{div}\,\mathbf{J}_Q = 0\,, \tag{2}$$

where Q is the *density* of a physical quantity (i.e., per cm^3) and $\mathbf{J}_Q$ is its corresponding flux density (i.e., per cm^2).

The momentum equation (Euler's equation) and the energy equation can also be written in this way. An evolution equation expressed in the form (2) is said to be written in *conservation form.* This form is particularly useful in the numerical integration of hydrodynamic and magnetohydrodynamic problems. It is also especially appropriate for the interpretation of energy and linear momentum fluxes in problems of wave propagation and for the derivation of the relations holding across flow discontinuities (in particular, shock fronts).

2. The momentum equation in conservation form.

Show that if the long-range force per unit volume derives from a scalar potential Φ, then the equation of motion for a fluid can be written in the form

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \mathrm{div}\,\left(\hat{\mathbf{\Pi}} + \Phi\,\hat{\boldsymbol{I}}\right) = \mathbf{0}\,, \tag{3}$$

where $\hat{\mathbf{\Pi}} \overset{\text{def}}{=} \rho\mathbf{v}\otimes\mathbf{v} - \hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{I}}$ is the unit tensor.

If the flow is inviscid, then Eq. (3) can be written

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \mathrm{div}\,\left[\hat{\mathbf{\Pi}} + (p+\Phi)\,\hat{\boldsymbol{I}}\right] = \mathbf{0}\,, \tag{3a}$$

where now $\hat{\mathbf{\Pi}} \overset{\text{def}}{=} \rho\mathbf{v}\otimes\mathbf{v}$.

In order to understand the physical meaning of the tensor $\hat{\mathbf{\Pi}}$, integrate Eq. (3) in a fixed region $\mathcal{R}$ bounded by a regular surface $\partial\mathcal{R}$.

3. The energy equation in conservation form.

Under the assumption that the long-range forces per unit mass derive from a time-independent scalar potential Ψ, show that the energy equation (which is the differential expression of the principle of energy conservation) can be written in the form

$$\frac{\partial \mathcal{E}}{\partial t} + \mathrm{div}\,\mathbf{J}_{\mathcal{E}} = 0\,, \tag{4}$$

where $\mathcal{E} = \rho\,(\epsilon + \|\mathbf{v}\|^2/2 + \Psi)$ is the total energy density (i.e, per cm^3) and $\mathbf{J}_{\mathcal{E}}$ is its corresponding flux density (i.e, per cm^2).

In order to understand the physical meaning of the flux vector $\mathbf{J}_{\mathcal{E}}$, integrate Eq. (4) in a fixed region $\mathcal{R}$ bounded by a regular surface $\partial\mathcal{R}$.

Indication: Employ the original expression of Reynolds' transport theorem (i.e., without making use of the continuity equation).