HYDRODYNAMICS

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Bulletin of exercises n° 4: Energy equation for a compressible flow.

1. Energy equation in the representation $\mathbb{G}(p, T)$: Assume that the thermodynamic state of a fluid of constant chemical composition can be specified through the variables (p, T). Without assuming any further thermodynamic restriction (such as considering an *ideal gas*), show that the differential expression of the first principle of Thermodynamics is given by

$$\rho\, c_p \frac{D\,T}{D\,t} - \alpha T \frac{D\,p}{D\,t} = -p \operatorname{div} \mathbf{v} + \Phi_{\mathrm{V}} - \operatorname{div} \boldsymbol{\mathcal{F}}\,, \tag{1}$$

where Φ_{V} is the viscous dissipation function, α is the thermal expansion coefficient and c_p is the specific heat at constant pressure.

In many cases, the thermodynamic behaviour of a gas can be quite well approximated by the *ideal gas model*. For instance, air (except when changes of state occur) or the stellar plasma in most main-sequence stars. Which particular form does equation (1) take if the fluid is assumed to be an ideal gas? [Note: You may need to refresh your notions of elementary Thermodynamics. As a suggestion, *Thermodynamics and an Introduction to Thermostatistics*, H. B. Callen, Wiley, 2nd edition, 1985].

2. Adiabatic processes: Consider the local expression of the first principle of Thermodynamics in terms of the variables pressure and temperature [Eq. (1)] and assume that the heat flux vector satisfies *Fourier's law* for heat conduction.

Call L the characteristic length of variation of temperature, τ the characteristic time for the evolution of T and κ_T the coefficient of thermal conductivity.

Show that, in orders of magnitude, the ratio between the heat diffusion term and the term expressing temperature variation is equal to the ratio between the characteristic evolution time for T and the characteristic time for heat conduction (call it τ_c).

Justify why if $\tau \ll \tau_c$ we can say that the process is adiabatic.

3. Physical interpretation of the quantity enthalpy: The physical quantity *enthalpy* is useful in a variety of situations in compressible Hydrodynamics. It appears in a natural way, for instance, in studying energy propagation by waves or in the Rankine-Hugoniot jump relations across a shock front.

In order to better understand its physical meaning, write the first principle of Thermodynamics replacing the specific internal energy ϵ by the specific enthalpy h (defined as $h \stackrel{\text{def}}{=} \epsilon + p/\rho$) and show that it is equivalent to

$$\rho \frac{D\,h}{D\,t} - \frac{D\,p}{D\,t} = \Phi_{\mathrm{V}} - \operatorname{div} \boldsymbol{\mathcal{F}}\,. \tag{2}$$

From Eq. (2) show that if a mass of fluid is undergoing a process at uniform pressure p, then the rate of change of the total enthalpy of that mass of fluid satisfies the inequality

$$\dot{H}(\Omega_t) \geq Q\,(\partial \Omega_t)\,, \tag{3}$$

where Ω_t is the region occupied by the mass of fluid at time t and $Q(\partial \Omega_t)$ is the heat per unit time being exchanged between the fluid mass and its surrounding through its boundary.

Under which conditions would the equal sign hold in inequality (3)? From this result, explain the physical meaning of the thermodynamic quantity "enthalpy."