HYDRODYNAMICS

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Bulletin of exercises n° 6: Vorticity.

Circulation and vorticity are two quantities that measure the degree of rotation in a fluid. Circulation, a scalar quantity, is an integral measure of rotation, while vorticity, a vector quantity, yields a local measure of the degree of rotation in the neighborhood of a fluid element.

1. Kinematic identities.

In many cases it is convenient to describe the flow of a fluid in terms of the vorticity field, $\boldsymbol{\omega} = \mathbf{rot}\,\mathbf{v}$. In order to obtain an equation of motion based on vorticity, it is especially useful to make use of a number of *kinematic identities* involving the vorticity field.

Prove the following vector identities:

$$\mathbf{q} \overset{\text{def}}{=} \frac{D\,\mathbf{v}}{D\,t} = \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\omega} \wedge \mathbf{v} + \frac{1}{2}\mathbf{grad}\;||\mathbf{v}||^2\,, \tag{i}$$

$$\mathbf{rot}\,\mathbf{q} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{rot}\,(\boldsymbol{\omega} \wedge \mathbf{v})\,, \tag{ii}$$

$$\mathbf{rot}\,\mathbf{q} = \frac{D\,\boldsymbol{\omega}}{D\,t} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} + \boldsymbol{\omega}\,\mathrm{div}\,\mathbf{v}\,. \tag{iii}$$

Combining identity (iii) with the continuity equation, the so-called *Beltrami identity* is obtained:

$$\frac{D}{D\,t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{v} + \frac{1}{\rho}\mathbf{rot}\,\mathbf{q}\,. \tag{iv}$$

2. Transport theorem for a material curve:

Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\xi, t)$ be the parametric expression of the material curve $\mathrm{C_t}$ and $\xi \in [\xi_1, \xi_2]$ the parameter (chosen in such a way that it *labels* each matter element along the curve in a unique way. Let $\mathbf{Q}(\mathbf{x}, t)$ be a vector quantity defined in the flow region and consider the line integral

$$\int_{\mathrm{C_t}} \mathbf{Q}\cdot d\,\boldsymbol{\alpha} \overset{\text{def}}{=} \int_a^b \mathbf{Q}[\boldsymbol{\alpha}(\xi,t)]\cdot\boldsymbol{\alpha}'(\xi,t)\,d\,\xi\,. \tag{1.1}$$

Show that the time derivative of (1.1) is

$$\frac{d}{dt}\int_{\mathrm{C_t}} \mathbf{Q}\cdot d\,\boldsymbol{\alpha} = \int_{\mathrm{C_t}} \frac{D\,\mathbf{Q}}{D\,t}\cdot d\,\boldsymbol{\alpha} + \int_a^b \mathbf{Q}\cdot[\,(\boldsymbol{\alpha}'\cdot\nabla)\,\mathbf{v}\,]\,d\,\xi\,, \tag{1.2}$$

where $\boldsymbol{\alpha}'$ is the partial derivative of $\boldsymbol{\alpha}$ with respect to the parameter ξ.

This result is the one-dimensional version (i.e., for a material curve) of Reynold's theorem.

3. Circulation of the velocity: The circulation of the velocity along the material circuit $\mathrm{C_t}$ is defined as:

$$\Gamma(t) = \oint_{\mathrm{C_t}} \mathbf{v}\cdot d\,\boldsymbol{\alpha} \overset{\text{def}}{=} \int_{\xi_1}^{\xi_2} \mathbf{v}[\boldsymbol{\alpha}(\xi,t)]\cdot\boldsymbol{\alpha}'(\xi,t)\,d\,\xi\,, \tag{2.1}$$

where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\xi, t)$ is the parametric expression of the curve and $\xi \in [\xi_1, \xi_2]$ is the parameter.

Show that the time derivative of the circulation is given by

$$\dot{\Gamma}(t) = \frac{d}{dt}\oint_{\mathrm{C_t}} \mathbf{v}\cdot d\,\boldsymbol{\alpha} = \oint_{\mathrm{C_t}} \frac{D\,\mathbf{v}}{D\,t}\cdot d\,\boldsymbol{\alpha}\,. \tag{2.2}$$

Bulletin of exercises n° 6 (continuation): Vorticity.

4. Evolution equation for the kinematic helicity density.

Given a velocity field $\mathbf{v}(\mathbf{x}, t)$, the *kinematic helicity density* is $\mathbf{v} \cdot \boldsymbol{\omega}$ and the helicity of a material region Ω_{t} is defined as

$$\mathcal{H}(\Omega_{\mathrm{t}}) = \int_{\Omega_{\mathrm{t}}} \mathbf{v} \cdot \boldsymbol{\omega} \,.$$

Consider a fluid flow in a barotropic medium (i.e., isopycnic surfaces coincide with isobaric surfaces) under conservative long-range forces and in a situation in which the effects of viscosity can be neglected.

Prove that, under these conditions, from the equations of continuity and motion the following equation can be derived, which governs the evolution of the *kinematic helicity density*, viz.

$$\frac{D}{D\,t} \left(\frac{\mathbf{v} \cdot \boldsymbol{\omega}}{\rho} \right) = \left(\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla \right) (\mathbf{v} \cdot \mathbf{v} - \varphi) \,, \tag{3}$$

where $\varphi(\mathbf{x}, t)$ is a scalar funtion that must be determined.

5. Conservation of kinematic helicity.

Under the same conditions for which the theorems of Kelvin and Helmholtz hold, the kinematic helicity of a vortex tube is conserved.

- Let Ω_{t} be a vortex tube. Prove that under the conditions of exercise **n°4** the kinematic helicity of a vortex tube is a constant of motion.
- How does the result depend on the funtion φ (previous exercise)? Does this result hold for *any* material region Ω_{t}? Why?
- Why do we need to assume a *barotropic medium* in order to come to the result? Could we state the assumption "isopycnic surfaces coinciding with isobaric surfaces" in a different way that does not involve pressure and density?