November 2008

Max Planck Institute for Solar System Research
Prof. S. Solanki with assistants R. Cameron, J. Graham, Y. Narita, and A. Pietarila

Exercise for Solar Physics (2008) - Part 2

Chapter 3: Oscillations

The f-mode corresponds closely to (deep) ocean waves. In these two questions we will try to get some insight into their behaviour. The first part shows you the mathematics behind this type of solution. It is worth doing, and the answer is at the bottom in case you get stuck. The second part asks you to think about the mathematical solution and the actual Sun.

3.1 Local helioseismology & the f-mode (1)

We begin with the momentum equation

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\nabla P' - \rho' g \hat{\mathbf{z}} \tag{1}$$

where ξ is the displacement caused by the waves, and P' and ρ' are the pressure and density pertubations associated with the waves. The continuity equation is

$$\rho' = -\nabla \cdot (\rho_0 \xi) \tag{2}$$

and the equation of state and energy equation (here adiababitc) can be combined to give

$$P' = c_0^2(\rho' + \xi \cdot \nabla \rho_0) - \xi \cdot \nabla P_0 \tag{3}$$

The subscript 0 denotes the background solar stratification which we will assume varies only in the z direction.

To keep life simple we will only consider a small patch of the solar surface (so we can use a cartesian coordinate system x, y, z). To not lose contact with waves on the surface of the ocean we look for solutions which work for both the Sun and the ocean. We keep the full equations, but look for solutions which *additionally* do not compress the gas, i.e., $\nabla \cdot \xi = 0$. Note carefully that we are using the full equations and only looking for solutions obeying this additional constraint. Such solutions need not exist, and hence we need to remember to check the solutions at the end for consistency.

Using equation 2, show $\nabla \cdot \xi = 0$ implies $\rho' + \xi \cdot \nabla \rho_0 = 0$.

Then assume that the atmosphere is hydrostatic ($\nabla P_0 = -\rho_0 g \hat{\mathbf{z}}$) to obtain (from equation 3) $P' = \xi_z \rho_0 g$.

This and $\rho' = -\xi \cdot \nabla\rho_0$ can be used to eliminate P' and ρ' from the momentum equation.

Remembering that the background atmosphere ρ_0 varies only in the z direction, expand what remains until you have an equation for $\partial^2\xi/\partial t^2$ where none of P', ρ', ρ_0, P_0 and c_0 appear.

Write down the z-component of this equation. Write down the x-component of this equation.

At this stage you should have something like

$$\frac{\partial^2 \xi_z}{\partial t^2} = -g\frac{\partial \xi_z}{\partial z}$$
$$\frac{\partial^2 \xi_x}{\partial t^2} = -g\frac{\partial \xi_z}{\partial x}$$

Pretty.

Find some solutions with $\xi_z = e^{-i\omega t} \times e^{ik_x x} \times e^{k_z z}$, $\xi_x = ie^{-i\omega t} \times e^{ik_x x} \times e^{k_z z}$. You will also need to use $\nabla \cdot \xi = 0$ (which means for our solutions $ik_x\xi_x = -k_z\xi_z$). [At home check the solution satisfies all the equations.]

Find the dispersion relationship $\omega = f(k_x)$.

Bonus question: What are the group and phase speeds (both for the ocean and the Sun)?

[Answer: $k_x = k_z$, and $\omega = \sqrt{gk_x}$, i.e.,

$\xi_z = e^{\pm i\omega t} \times e^{ik_x x} \times e^{k_x z}$,

$\xi_x = ie^{\pm i\omega t} \times e^{ik_x x} \times e^{k_x z}$]

3.2 Local helioseismology & the f-mode (2)

[Starting from the answer to question 1 and noting that the velocity associated with the wave is just $i\omega\xi$.]

Draw some of these solutions as a function of z. Where is most of the kinetic energy? [hint: draw a rough sketch of the density and of the behaviour of the veloicty as a function of height. What are the different possibilities depending on k_x?]

Why are the solutions for the deep ocean only? (What does "deep" mean here?)

How bad are the approximations we used to get these results (when are we safe with our assumptions)?

How do you think convection affects the waves? [Hint: Ocean waves can propagate a long way on a smooth sea.] What about the chromosphere? Roughly sketch how you expect the lifetimes of the modes to depend upon k_z. [Hint: The chromosphere is a very violent place for the modes we are discussing.]

Chapter 4: Rotation

In many models the 11 year solar cycle is thought to be bound up with the solar differential rotation, converting “poloidal” flux to “toroidal” flux.

1) Draw a cut through the sun showing the core, the radiative zone and the convection zone.

2) Draw a magnetic field line passing from the just to the right of the north pole to just to the right of the south pole. Note that the field line doesn't have time to penetrate the radiative core and hence makes a detour.

3) What is the rotation rate at the pole? How many times does the field-line rotate about the axis per year? (Assume the field line is “frozen” into the plasma.)

4) What is the rotation rate at the equator? How many times does it get wound up here?

5) If there was no differential rotation what would the answer to 3 and 4 be?

6) If there was no differential rotation how would the fieldline look after a year?

7) How does it look with the solar differential rotation?

8) What if a star had more differential rotation?