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Solutions to Exercise Solar Physics (2008)

Chap.1 Introduction

(1) To other fields of science: astrophysics (stellar evolution, cosmic rays); plasma physics (dynamo, turbulence, waves, fusion); particle physics (atoms, molecules, neutrinos, standard model of elementary particles); gravitation (general relativity); Sun-Earth or -planets relation (space weather, climate).

(2) To other stars: The Sun is a main sequence star (age about 4.5×10^9 yr), evolving into a red giant, planetary nebula, and then a white dwarf.

(3) Relations to planets The Sun provides heat and light at various wavelengths (causing expansion of the atmosphere through heating); The interaction with the Sun involves also the solar wind. For magnetized planets (Earth, Jupiter, Saturn, Uranus, Neptune) the interaction with the Sun or the solar wind causes formation of magnetosphere and various magnetospheric or ionospheric activities such as aurora, radiation belt, storms and substorms. For non-magnetized planets (Venus and Mars) the interaction with the Sun and the solar wind ends up with loss (or escape) of the planetary atmosphere into space (called the ion pickup process). For non-atmospheric planets like Mercury or Earth Moon, the solar wind hits directly on the surface and kicks out surface materials (sputtering). The solar wind is a supersonic (and super-Alfvénic) flow and a standing shock wave (bow shock) forms in front of the planets. Comet tails (ion tails) are also caused by the interaction with the solar wind.

Chap.2 Core and interior

2.1 Solar model

(A) Work for shell mass dm and radius r against attraction of the rest mass m

$$\begin{aligned} dE &= -\int_r^\infty \frac{Gmdm}{r'^2} dr' &(1)\\ &= -\frac{Gmdm}{r} &(2) \end{aligned}$$

Integrate over m

$$\begin{aligned} E_G &= -\int_0^{m_\odot} \frac{Gmdm}{r} &(3)\\ &= -\frac{Gm_\odot{}^2}{r_\odot} \int_0^{m_\odot} \frac{r_\odot}{r}\frac{m}{m_\odot}\frac{dm}{m_\odot} &(4) \end{aligned}$$

Approximate by $\int f(x)dx \simeq \sum f(x)\Delta x$,

$$E_G \simeq -\frac{G{m_\odot}^2}{r_\odot} \times 0.125 \times [0 + 1.008 + \cdots + 1] \tag{5}$$
$$\simeq -6.3 \times 10^{41}[\mathrm{J}], \tag{6}$$

where we used the values $G = 6.67 \times 10^{-11}\,[\mathrm{m}^3/\mathrm{kg\,s}^2]$, $m_\odot = 1.99 \times 10^{30}\,[\mathrm{kg}]$, and $r_\odot = 6.96 \times 10^8\,\mathrm{m}$. Hence the Sun's gravitational energy is $|E_G| = 6.3 \times 10^{41}\,[\mathrm{J}]$.

The solar irradiance is $S = 1.336 \times 10^3\,[\mathrm{W/m}^2]$ at 1 AU. Noting that 1 [W] = 1 [J/s] and 1 [AU] = 1.496×10^{11} [m], the total irradiance (integrated over the surface $4\pi r^2$) is $\dot{E}_R = S \times 4\pi r^2 = 3.8 \times 10^{26}\,[\mathrm{J/s}]$. This is the amount of energy that the Sun provides through radiation per second. With the gravitational energy only, the Sun can provide energy for the period $E_G/\dot{E}_R = 6.3 \times 10^{41}/3.8 \times 10^{26}\,[\mathrm{s}] = 1.7 \times 10^{15}\,[\mathrm{s}] = 5.4 \times 10^7\,[\mathrm{yr}]$. In reality, the Sun's age is about 4.5×10^9 [yr]. So one needs to find an alternative energy source (which is nuclear reaction).

(B) Hydrostatic equilibrium is a force balance between pressure gradient and gravity

$$-\nabla P + \rho\vec{g} = 0 \tag{7}$$

where gravity (or gravitational acceleration) is

$$\vec{g} = -\frac{Gm}{r^2}\vec{e}_r. \tag{8}$$

In the radial direction the hydrostatic balance reads

$$\frac{dP}{dr} = -\frac{Gm\rho^2}{r^2} \tag{9}$$

Using mass-radius relation $dm = 4\pi r^2 \rho dr$, the hydrostatic balance equation gives us the pressure-mass relation

$$dP = -\frac{Gm}{4\pi r^4}dm. \tag{10}$$

Multiply the both sides by sphere volume $V = \frac{4\pi}{3}r^3$,

$$(\mathrm{lhs}) = VdP = d(PV) - PdV \tag{11}$$
$$(\mathrm{rhs}) = -\frac{Gm}{4\pi r^4}\frac{4\pi r^3}{3}dm = -\frac{Gm}{3r}dm. \tag{12}$$

Integrate from the Sun center to the surface

$$\int_{cen}^{sur} (\mathrm{lhs}) = \int_c^s d(PV) - \int_c^s PdV \tag{13}$$

Note that the first term on the right hand side vanishes because $P = 0$ at solar surface and $V = 0$ at center. Use the ideal gas pressure $P = \rho\frac{RT}{\mu} = \rho\frac{2C_V}{3}T$,

$$\int_c^s (\mathrm{lhs}) = -\int_c^s PdV \tag{14}$$
$$= -\frac{2}{3}\int_c^s \rho C_V T dV \tag{15}$$
$$= -\frac{2}{3}\int_0^{m_\odot} C_V T dm \tag{16}$$
$$= -\frac{2}{3}U, \tag{17}$$

here the last integral gives us the internal energy U.

The right hand side of the pressure-mass relation yields one third of the gravitational energy when integrated over m

$$\int_c^s (\text{rhs}) = -\int_c^s \frac{Gm}{3r} dm \tag{18}$$

$$= \frac{1}{3} E_G. \tag{19}$$

Hence we have

$$U + 2E_G = 0, \tag{20}$$

and this is called the *virial theorem*.

The mass-weighted mean temperature is

$$\langle T \rangle = \frac{1}{m_\odot} \int_0^{m_\odot} T dm \tag{21}$$

For constant C_V, the virial theorem yields

$$2C_V \int_0^{m_\odot} T dm = -E_G. \tag{22}$$

The left hand side can also be written with the mean temperature as

$$2C_V \int_0^{m_\odot} T dm = 2C_V m_\odot \langle T \rangle. \tag{23}$$

Hence the virial theorem gives us the mean temperature as

$$\langle T \rangle = -\frac{E_G}{2C_V m_\odot} \tag{24}$$

$$= -\frac{E_G}{m_\odot} \frac{\mu}{3R}, \tag{25}$$

where C_V is replaced by the gas constant $R = 8.31\,[\text{J/K mol}]$. If we take the mean molecular weight $\mu = 0.5\,[\text{g/mol}]$, the mean temperature (estimated from the virial equilibrium) is

$$\langle T \rangle = \frac{|E_G| \times \mu}{m_\odot \times 3R} \tag{26}$$

$$= \frac{6.25 \times 10^{41}\,[\text{J}] \times 0.5 \times 10^{-3}\,[\text{kg/mol}]}{3 \times 8.31\,[\text{J/K mol}] \times 1.99 \times 10^{30}\,[\text{kg}]} \tag{27}$$

$$= 6.30 \times 10^6\,[\text{K}]. \tag{28}$$

On the other hand, the tabulated solar model gives us the mean temperature

$$\langle T \rangle_{\text{tab}} = \int_0^{m_\odot} T \frac{dm}{m_\odot} \simeq 7.83 \times 10^6\,[\text{K}], \tag{29}$$

which is close to the virial temperature. The Sun is therefore (roughly speaking) in a hydrostatic and virial equilibrium.

2.2 Nuclear reactions

The ppI reaction releases the energy about 26.732 [MeV] from 4 protons,

$$4\,(\text{protons}) \rightarrow 1\,(\alpha\text{−particle}) + 26.732\,[\text{MeV}]. \tag{30}$$

In other words, the energy release is $\Delta E_{\rm nuc} = 26.732/4\,[\text{MeV}] = 6.683\,[\text{MeV}] = 6.683 \times 10^6 \times 1.602 \times 10^{-19}\,[\text{J}] = 1.071 \times 10^{-12}\,[\text{J}]$ per proton. The number of protons can be simply estimated as $N = m_\odot/m_p = 1.99 \times 10^{30}\,[\text{kg}]/1.67 \times 10^{-27}\,[\text{kg}] = 1.192 \times 10^{57}$ [particles] on the assumption that the Sun entirely consists of protons. The total energy release from the nuclear reaction is $E_{\rm nuc} = N\Delta E_{\rm nuc} = 1.3 \times 10^{45}\,[\text{J}]$. This is about 2000 times larger than the gravitational energy. When compared to the solar irradiance, the nuclear reaction provides the energy for the period $E_{\rm nuc}/\dot{E}_R = 1.3 \times 10^{45}\,[\text{J}]/3.8 \times 10^{26}\,[\text{J/s}] = 3.4 \times 10^{18}\,[\text{s}] = 1.1 \times 10^{11}\,[\text{yr}]$, which can account for the energy source problem of the Sun. (cf. the Sun's age is $4.5 \times 10^9\,[\text{yr}]$) Compared to the internal energy, $E_{\rm nuc}/U = E_{\rm nuc}/(1/2E_G) \sim 4000$.