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Solutions to Exercise Solar Physics (2008)

Chap.3 Oscillations

Part 1. Continuity equation:

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0. \tag{1}$$

Replace the time derivative by $\delta\rho/\delta t$, multiply the equation by δt,

$$\delta\rho + \nabla \cdot (\rho \vec{v} \delta t) = 0. \tag{2}$$

Use displacement $\vec{\xi} = \delta\vec{x} = \vec{v}\delta t$, and we obtain Eq. (2) in the sheet:

$$\delta\rho = -\nabla \cdot (\rho_0 \vec{\xi}). \tag{3}$$

Pressure variation is derived from the definition of the sound speed:

$$c_s^2 = \frac{P}{\rho}, \tag{4}$$

where we set the polytropic index (or ratio of specific heat) $\gamma = 1$. The fluctuation of the pressure is

$$\delta P = c_s^2 \delta\rho \text{ (adiabatic)}, \tag{5}$$

and we replace $\delta\rho \to \delta\rho + \vec{\xi} \cdot \nabla\rho_0$, where the first term denotes the fluctuation (oscillation or wave field) of density and the second term the change of the background. We also replace the pressure as $\delta P \to \delta P + \vec{\xi} \cdot \nabla P_0$, and here again the first term is the fluctuation of the pressure and the second term the change of the background. The pressure variation is written in the form

$$\delta P + \vec{\xi} \cdot \nabla P_0 = c_s^2(\delta\rho + \vec{\xi} \cdot \nabla\rho_0), \tag{6}$$

which gives Eq. (3) in the sheet:

$$\delta P = c_s^2(\delta\rho + \vec{\xi} \cdot \nabla\rho_0) - \vec{\xi} \cdot \nabla P_0. \tag{7}$$

Incompressibility means $\nabla \cdot \vec{v} = 0$. Multiply by δt and we obtain $\nabla \cdot \vec{\xi} = 0$. The continuity equation becomes

$$\delta\rho + \vec{\xi} \cdot \nabla\rho_0 = 0. \tag{8}$$

Hydrostatic equilibrium in the z (vertical) direction is

$$-\nabla P_0 - \rho_0 g \vec{e}_z = 0. \tag{9}$$

In the pressure variation equation the first term with the bracket vanishes because of the continuity equation under incompressibility $\delta\rho + \vec{\xi} \cdot \nabla\rho_0 = 0$. The pressure variation is hence

$$\begin{aligned} \delta P &= -\vec{\xi} \cdot \nabla P_0 && (10) \\ &= \xi_z \rho g, && (11) \end{aligned}$$

where the ∇-part is replaced by the density fluctuation using the hydrostatic balance.

Now we use the pressure variation

$$\delta P = \xi_z \rho_0 g \quad (12)$$

and the density variation

$$\delta\rho = -\vec{\xi} \cdot \nabla\rho_0 \quad (13)$$

in the momentum equation (for waves),

$$\begin{aligned} \rho_0 \partial_t^2 \vec{\xi} &= -\nabla(\delta P) - \delta\rho g \vec{e}_z && (14) \\ &= -\nabla(\xi_z \rho_0(z) g) + \vec{\xi} \cdot \nabla\rho_0(z) g && (15) \\ &= -\nabla(\xi_z \rho_0(z) g) + \xi_z \partial_z \rho_0(z) g. && (16) \end{aligned}$$

In the z component the right hand side of the momentum equation is written as

$$\begin{aligned} (\text{rhs}) &= -\partial_z(\xi_z \rho_0(z) g) + \xi_z \partial_z(\rho_0(z) g) && (17) \\ &= -\rho_0 g \partial_z \xi_z. && (18) \end{aligned}$$

Here we take $g = \text{const}$. The density ρ_0 in the momentum equation is canceled out and we obtain the momentum equation in the form

$$\partial_t^2 \xi_z = -g \partial_z \xi_z. \quad (19)$$

For the x component,

$$\begin{aligned} (\text{rhs}) &= -(\partial_x \xi_z) \rho_0(z) g && (20) \\ (\text{lhs}) &= \rho_0(z) \partial_t^2 \xi_x, && (21) \end{aligned}$$

hence

$$\partial_t^2 \xi_x = -g \partial_x \xi_z. \quad (22)$$

Pretty.

We use the ansatz

$$\begin{aligned} \xi_z &= \exp[-i(\omega t - k_x x) + k_z z] && (23) \\ \xi_x &= i \exp[-i(\omega t - k_x x) + k_z z]. && (24) \\ & && (25) \end{aligned}$$

This means that we have a plane wave propagation in the x direction with the amplitude unity at base ($z = 0$). In the z direction the wave amplitude grows exponentially. The displacement in the x direction (ξ_x) has a phase shift by $\pi/2$ from that of ξ_z, such that the oscillation of fluid element forms a circular motion in the xz plane.

When we use the ansatz, we obtain the dispersion relation as a solution of the equations, and that is

$$\omega^2 = gk_x = gk_z. \tag{26}$$

The wave propagates solely in the x direction, and the phase speed is

$$v_{ph} = \frac{\omega k_x}{|k^2|} = \frac{\omega}{k_x} = \frac{g}{\omega}. \tag{27}$$

The group speed is

$$v_{gr} = \frac{\omega}{k_x} = \frac{g}{2\omega} = \frac{1}{2}v_{ph}. \tag{28}$$

Part 2. Hydrostatic equilibrium is given as

$$-\nabla P_0 - \rho_0 \vec{g} = 0. \tag{29}$$

We use the ideal gas for the pressure,

$$P_0 = n_0 kT = \rho_0 \frac{kT}{m}, \tag{30}$$

where m is the mean molecular weight. Combining the two equations, we obtain

$$\nabla \rho_0 = -\frac{m\vec{g}}{kT}\rho_0, \tag{31}$$

where we assumed an isothermal gas ($T = \text{const}$). The gravity is in the z direction only, and

$$\frac{d\rho_0}{dz} = -\frac{mg}{kT}\rho_0, \tag{32}$$

which can easily be solved and the solution is an exponential decay of the density:

$$\rho_0(z) = \rho_0(0)e^{-z/H}, \tag{33}$$

where H is called the scale height

$$H = \frac{mg}{kT}. \tag{34}$$

The velocity of the medium associated with the wave oscillation is

$$\vec{v} = i\omega\vec{\xi}. \tag{35}$$

We use the dispersion relation

$$\omega^2 = gk_x, \tag{36}$$

which gives the squared velocity

$$\begin{aligned} |\vec{v}|^2 &= \omega^2 |\vec{\xi}|^2 && (37)\\ &= gk_x(|\xi_x|^2 + |\xi_z|^2) && (38)\\ &= 2gk_x e^{2k_z z}. && (39) \end{aligned}$$

The kinetic energy of the oscillation is hence

$$\frac{1}{2}\rho_0 v^2 = \frac{1}{2}(\rho_0(0)e^{-z/H})(2gk_x e^{2k_z z}) \quad (40)$$

$$= \rho_0(0)e^{(-1/H+2k_z)z}. \quad (41)$$

If $(-1/H + 2k_z) > 0$, the kinetic energy grows in the z direction. If $(-1/H + 2k_z) < 0$, the energy decays in the z direction. (See discussion in Airy wave theory in fluid dynamics.)

The wave mode is called "deep" because the scale height H is large enough for wave energy to grow vertically, $(-1/H + 2k_z) > 0$.

In deriving the deep ocean wave mode we assumed that the gravitational acceleration is constant, but in reality the gravity should be a function of the radius, $g = \text{const} \rightarrow g = -Gm/r^2$. Also we have linearized the equations, neglected all the nonlinear terms (for example the advection term in the momentum equation). This is valid when the wave amplitude is small compared to the background field. We also used the adiabatic change of gas (eq. of state), but in reality we have convection which violates the adiabatic change.

The waves interact with convection motion when the wavelength is close to the convection cell size. If the wavelength is smaller than the cell size the wave can no longer propagate, but becomes distorted, convected, or broken by the cells.

lifetime?

Chap.4 Rotation

See winding of magnetic field (Ω-effect, on the next page). The winding develops until turbulence or convection starts to twist the toroidal magnetic field into the poloidal field (α-effect). The combination of these two effects makes the Sun's magnetic field oscillatory at 11 year cycle. Theoretically the magnetic field may grow until all the kinetic energy is converted to the magnetic field energy (magnetic braking).

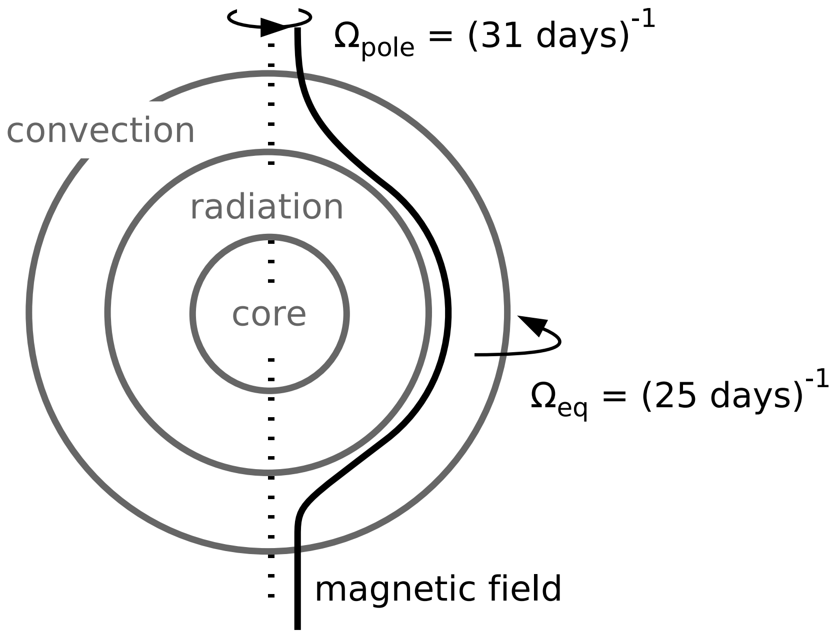

Figure 1: Before winding of magnetic field.

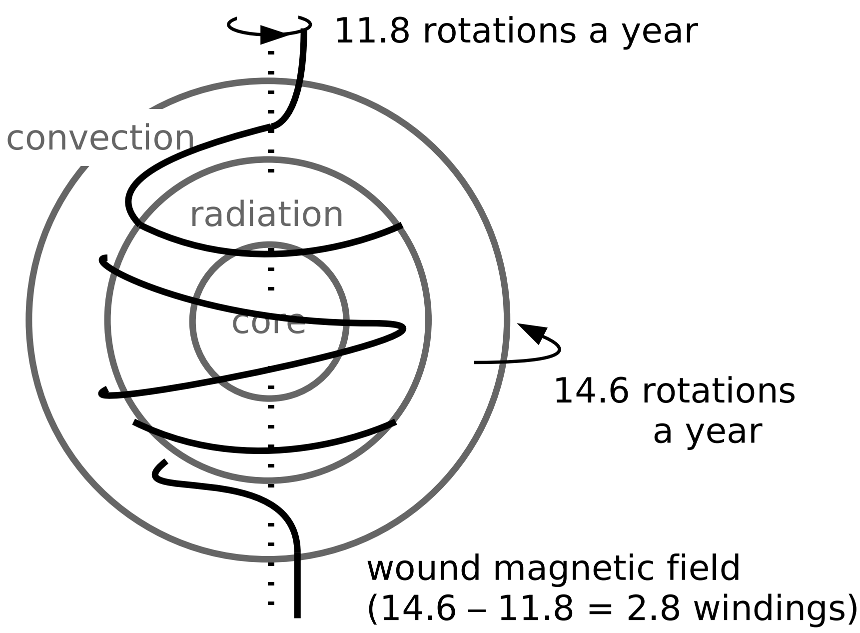

Figure 2: Magnetic field one year later.