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Solutions to Exercise Solar Physics (2008)

Chap.5 Radiation

Optical depth

The optical depth τ is defined in a differential form as

$$d\tau_\nu = -\kappa_\nu dz, \tag{1}$$

where κ is the absorption coefficient in units of [m^{-1}] and z is the length in the line-of-sight direction in units of [m]. The minus sign on the rhs means that we look in the direction toward the cloud. The absorption coefficient is a measure of the inverse mean free path of photons in the cloud, so the optical depth compares the mean free path with the length dz at frequency ν or wavelength λ. The optical depth is dimensionless and can be interpreted as the number of mean free paths through the length dz in the cloud.

We can also argue with the radiative transfer equation. Assuming no emission in the cloud and fixing the direction $\theta = 180°$, the equation has a simple form

$$-\frac{dI}{d\tau} = I, \tag{2}$$

which can be easily solved to

$$I = I_0 e^{-\tau}. \tag{3}$$

Therefore at $\tau = 1$ the intensity I (or the number of photons) is diminished by $1/e$ (about 37%).

Limb darkening

(A) Around the disk center one sees the atmosphere almost vertically. The atmospheric layer becomes hotter as one goes deep into the atmosphere, and therefore at the layer $\tau = 1$ one sees a bright surface (Stefan-Boltzmann). Near the limb, on the other hand, one sees the atmosphere almost in the horizontal direction and the layer of $\tau = 1$ is higher than that of the disk center. At higher layers the temperature is lower, therefore the surface looks darker. The decrease of intensity from the disk center to the limb scales with $\cos\theta$, where θ is the angle from the vertical (or radial) direction (see Figure).

(B) Radiative transfer equation is

$$\cos\theta \frac{dI}{d\tau} = I - S. \tag{4}$$

Multiply the equation by factor $\exp[-\tau/\cos\theta] = \exp[-u]$ (where $u = \tau/\cos\theta$ and $\theta = \text{const}$),

$$\left[\frac{dI}{du} - I\right] e^{-u} = -S e^{-u} \tag{5}$$

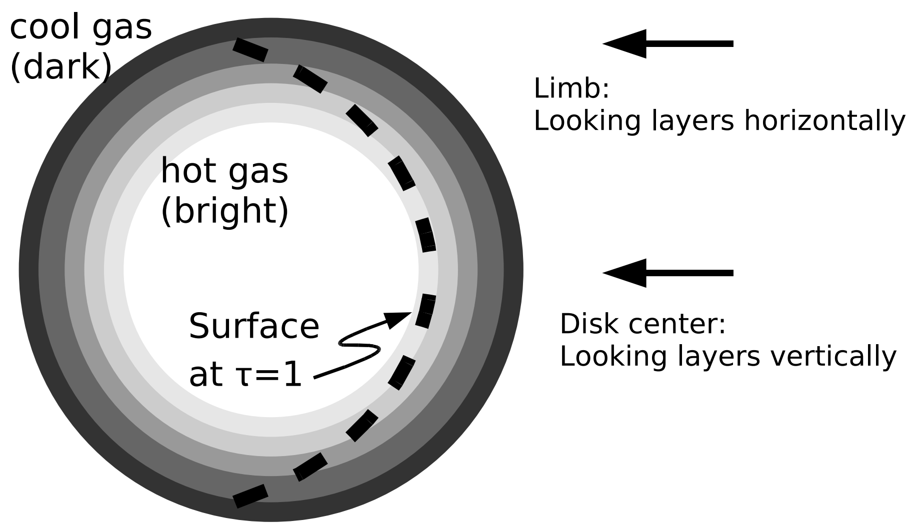

Figure 1: Limb darkening.

The left hand side can become simpler:

$$\frac{d}{du}\left(Ie^{-u}\right) = -Se^{-u}. \tag{6}$$

Integrate from $u = 0$ to ∞,

$$(\text{lhs}) = \left[Ie^{-u}\right]_0^\infty = -I_{\tau=0} = -I_0 \tag{7}$$

$$(\text{rhs}) = -\int_0^\infty Se^{-u}du, \tag{8}$$

here I_0 is the intensity at $\tau = 0$.

Hence we have

$$I_0 = \int_0^\infty Se^{-u}du. \tag{9}$$

Now assume that the source function is linear to the optical depth

$$S = a + b\tau, \tag{10}$$

which means that the source function decreases linearly to the height z,

$$S = a - b\kappa z. \tag{11}$$

The intensity can be obtained by integration

$$I_0 = \int_0^\infty ae^{-u}du + \int_0^\infty b\tau e^{-u}du \tag{12}$$

The first and the second term yields, respectively,

$$(\text{1st}) = a \int_0^\infty e^{-u} du = a \tag{13}$$

$$(\text{2nd}) = b \cos\theta \int_0^\infty u e^{-u} du = b\cos\theta, \tag{14}$$

here we used the formula

$$\int_0^\infty u^n e^{-u} du = n!. \tag{15}$$

The intensity therefore scales to the cosine function,

$$I_0 = a + b\cos\theta. \tag{16}$$

Chap.6 Convection

Onset of convection

We define the entropy as

$$s = \frac{P}{\rho^\gamma}. \tag{17}$$

Entropy actually measures the logarithm of this quantity, $\log(P/\rho^\gamma)$, but we do not use the log-function in the following calculation for simplicity. The derivative of s with respect to the radial distance r is

$$\frac{ds}{dr} = \frac{dP}{dr}\frac{1}{\rho^\gamma} + P(-\gamma)\frac{1}{\rho^{\gamma+1}}\frac{d\rho}{dr} \tag{18}$$

$$= \frac{1}{\rho^\gamma}\left(\frac{dP}{dr} - \frac{\gamma P}{\rho}\frac{d\rho}{dr}\right) \tag{19}$$

$$= \frac{P}{\rho^\gamma}\left(\frac{1}{P}\frac{dP}{dr} - \gamma\frac{1}{\rho}\frac{d\rho}{dr}\right) \tag{20}$$

$$= \frac{P}{\rho^\gamma}\left(\frac{d(\log P)}{dr} - \gamma\frac{d(\log\rho)}{dr}\right). \tag{21}$$

Hence the condition $ds < 0$ means

$$\frac{d(\log P)}{dr} - \gamma\frac{d(\log\rho)}{dr} < 0 \text{ (radially outward).} \tag{22}$$

Now use the ideal gas pressure

$$P = nkT \tag{23}$$

$$= \rho\frac{kT}{\mu}, \tag{24}$$

where μ is the mean molecular mass. The mass density is therefore

$$\rho = \frac{\mu P}{kT}. \tag{25}$$

The log-derivative is

$$\frac{d}{dr}(\log\rho) = \frac{d}{dr}\left[\log\left(\frac{\mu P}{kT}\right)\right] \tag{26}$$

$$= \frac{d}{dr}(\log P - \log T). \tag{27}$$

Substitution into the convection condition:

$$\frac{d}{dr}(\log P) - \gamma\frac{d}{dr}(\log\rho) < 0. \tag{28}$$

The left hand side can be written as

$$(\text{lhs}) = \frac{d}{dr}(\log P) - \gamma\frac{d}{dr}(\log P - \log T) \tag{29}$$

$$= (1-\gamma)\frac{d(\log P)}{dr} + \gamma\frac{d(\log T)}{dr}. \tag{30}$$

Drop dr, and we obtain

$$(1-\gamma)d(\log P) + \gamma d(\log T) < 0. \tag{31}$$

or after an arrangement

$$\frac{d(\log T)}{d(\log P)} < \frac{\gamma - 1}{\gamma}\ \text{(radially outward)}. \tag{32}$$

In solar physics (and stellar physics, too) the direction is often taken as radially inward, so replacing as $r \to -r$, and we obtain

$$\frac{d(\log T)}{d(\log P)} > \frac{\gamma - 1}{\gamma}\ \text{(radially inward)}. \tag{33}$$

The left hand side represents the temperature gradient associated with radiative energy transport, ∇_{rad}, whereas the right hand side is the adiabatic gradient, ∇_{ad}. Hence the condition for onset of convection is

$$\nabla_{\text{rad}} > \nabla_{\text{ad}}\ \text{(radially inward)}. \tag{34}$$

Granulation model

Continuity equation under stationary condition is

$$\nabla \cdot (\rho\vec{v}) = 0. \tag{35}$$

The horizontal flow is periodic, $v_h \sim \sin(kx)$, and the left hand side of the continuity equation is for the horizontal scale

$$|\nabla \cdot (\rho\vec{v})| \sim k\rho v_h. \tag{36}$$

For the vertical scale we have

$$|\nabla \cdot (\rho\vec{v})| \sim \frac{\rho v_r}{H}. \tag{37}$$

Balancing the two mass flux scales gives

$$\frac{\rho v_r}{H} \sim k\rho v_h, \tag{38}$$

which results in

$$\frac{v_h}{v_r} \sim \frac{1}{kH}. \tag{39}$$

Chap.7 Atmosphere

Doppler shift relation

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} \tag{40}$$

and thermal speed

$$v_{th}^2 = \frac{2kT}{m} \tag{41}$$

give the relation

$$\left(\frac{\Delta\lambda}{\lambda}\right)^2 = \frac{2kT}{mc^2}, \tag{42}$$

which can be arranged to

$$kT = \frac{1}{2}\left(\frac{\Delta\lambda}{\lambda}\right)^2 mc^2. \tag{43}$$

Use the electron mass $mc^2 = 511\,[\mathrm{keV}]$, line width $\Delta\lambda = 20\,[\mathrm{nm}]$, and wavelength $\lambda = 400\,[\mathrm{nm}]$. We obtain $kT = 6.39\,[\mathrm{eV}] = 7.4 \times 10^6\,[\mathrm{K}]$.