Bulletin of exercises n°2: Kinematic aspects in Continuum Mechanics. Transport theorem for a material region and for a material circuit.

1. Reynolds' transport theorem.

Reynolds' theorem in Continuum Mechanics is basically a re-statement of the theorem of change of variable in a volume integral.

Let $\Omega_{\rm t}$ be a "material region" (i.e., a region in 3-D space occupied at time t by a portion of continuum). We require from $\Omega_{\rm t}$ to be an open, connected region. (For some applications we may also require that the boundary $\partial\Omega_{\rm t}$ be a piecewise regular surface).

Let $F(\mathbf{x}, t)$ be a function of position and time defined for $\mathbf{x} \in \Omega_{\rm t}$ and $t \in (t_1, t_2)$.

In Hydrodynamics/MHD one often comes across time derivatives of the form

$$\frac{d}{dt} \int_{\Omega_{\rm t}} F(\mathbf{x}, t) \ ,$$

where not only the integrand depends on time but also the region of integration.

1.1. By using *Euler's identity* (see bulletin 1) show that

$$\frac{d}{dt} \int_{\Omega_{\rm t}} F(\mathbf{x}, t) \ = \int_{\Omega_{\rm t}} \left(\frac{DF}{Dt} + F \,\mathrm{div}\, \mathbf{v} \right) = \int_{\Omega_{\rm t}} \left\{ \left. \frac{\partial F}{\partial t} \right|_{\mathbf{x}} + \mathrm{div}\,(F\,\mathbf{v}) \right\} \ , \tag{1}$$

where D/Dt is the material derivative and $\mathbf{v}(\mathbf{x}, t)$ is the velocity of each $\mathbf{x} \in \Omega_{\rm t}$.

1.2. Combining Reynolds' theorem (2.1) with the equation of continuity, prove the following result:

$$\frac{d}{dt} \int_{\Omega_{\rm t}} \rho\, F(\mathbf{x}, t) \ = \int_{\Omega_{\rm t}} \rho \,\frac{DF}{Dt} \ . \tag{2}$$

The above result is usually called *Reynolds' transport theorem*; it is no a longer a purely mathematical result, since use has been made of the continuity equation, which expresses mass conservation.

2. Transport theorem for a material curve:

Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\xi, t)$ be the parametric expression of the material contour or circuit $C_{\rm t}$ (a simple, closed curve made up at all times of the same material elements). Let $\mathbf{Q}(\mathbf{x}, t)$ be a vector quantity defined in the flow region. The circulation of $\mathbf{Q}$ around the circuit $C_{\rm t}$ is defined as

$$\Gamma(C_{\rm t}) = \oint_{C_{\rm t}} \mathbf{Q} \cdot \mathrm{d}\boldsymbol{\alpha} \stackrel{\rm def}{=} \int_{\xi_1}^{\xi_2} \mathbf{Q}[\boldsymbol{\alpha}(\xi, \mathrm{t})] \cdot \boldsymbol{\alpha}'(\xi, \mathrm{t})\, \mathrm{d}\xi \, , \tag{3}$$

where $\xi \in (\xi_1, \xi_2)$ and $\boldsymbol{\alpha}'(\xi, t)$ is the short-hand notation for the tangent vector at (ξ, t). Prove that

$$\frac{d}{dt} \oint_{C_{\rm t}} \mathbf{Q} \cdot d\boldsymbol{\alpha} = \oint_{C_{\rm t}} \left[\frac{\partial \mathbf{Q}}{\partial t} + (\mathbf{rot}\, \mathbf{Q}) \wedge \mathbf{v} \right] \cdot d\boldsymbol{\alpha} \, . \tag{4}$$