MAGNETOHYDRODYNAMICS

Bulletin of exercises n°3: Some results in Electromagnetism.

1. Continuity equation for the electric charge.

- Starting from Maxwell's equations obtain a 'continuity equation' for magnetic charge (i.e., a differential equation expressing the conservation of magnetic charge).

- Interpret the continuity equation for the electric charge by integrating it in a fixed region $\mathcal{R}$ with boundary $\partial\mathcal{R}$. Draw a parallelism with the continuity equation of Hydrodynamics (i.e., the differential equation expressing the conservation of mass).

2. Poynting's vector and Poynting's theorem.

If we have a continuous distribution of charges and currents, the total work per unit time (i.e., *power*) exerted by the electromagnetic fields on the matter in a fixed region $\mathcal{R}$ in space is:

$$\mathcal{W}^{\,\mathrm{em}} = \int_{\mathcal{R}} \mathbf{j}\cdot\mathbf{E}\,. \tag{1}$$

- Starting from Maxwell's equations (Faraday's and Ampere's laws), obtain a differential equation relating this power $\mathcal{W}^{\,\mathrm{em}}$ with the rate of change of the total electromagnetic energy contained in $\mathcal{R}$ and with the energy flux through the boundary $\partial\mathcal{R}$.

- Introducing the following definitions,

$$\begin{cases} \mathcal{E}_{\mathrm{em}} = \dfrac{\|\mathbf{B}\|^2}{8\pi} + \dfrac{\|\mathbf{E}\|^2}{8\pi} & \text{[electromagnetic energy density]} \\ \mathbf{S} = \dfrac{c}{4\pi}\,\mathbf{E}\wedge\mathbf{B}\,, & \text{[Poynting's vector]} \end{cases} \tag{2}$$

write the differential equation in the form

$$\frac{\partial\mathcal{E}_{\mathrm{em}}}{\partial t} + \operatorname{div}\mathbf{S} = -\mathbf{j}\cdot\mathbf{E}\,. \tag{3}$$

- Interpret Eq. (3) [Poynting's theorem] by integrating it in a fixed region $\mathcal{R}$ with boundary $\partial\mathcal{R}$. Draw a parallelism with the continuity equation of Hydrodynamics in the case of chemical reactions.

3. Poynting's theorem for a material region Ω_{t}.

Poynting's theorem [eq. (3)] has been derived for a fixed region $\mathcal{R}$. Now we extend it to the case of a material region Ω_{t} by using Reynolds' theorem.

Show that in the case of a material region Ω_{t}, the integral expression of Poynting's theorem is

$$\dot{E}_{\mathrm{em}}(\Omega_{\mathrm{t}}) = \frac{d}{dt}\int_{\Omega_{\mathrm{t}}} \mathcal{E}_{\mathrm{em}} = -\int_{\Omega_{\mathrm{t}}} \mathbf{j}\cdot\mathbf{E} - \oiint_{\partial\Omega_{\mathrm{t}}} \mathbf{S}\cdot\mathbf{n} + \oiint_{\partial\Omega_{\mathrm{t}}} \mathcal{E}_{\mathrm{em}}\,\mathbf{v}\cdot\mathbf{n}\,. \tag{4}$$

where $E_{\mathrm{em}}(\Omega_{\mathrm{t}})$ is the total electromagnetic energy contained in the material region Ω_{t}, $\mathcal{E}_{\mathrm{em}}$ is the corresponding density and $\mathbf{v}(\mathbf{x},t)$ is the velocity at each point $\mathbf{x}\in\partial\Omega_{\mathrm{t}}$ of the boundary.