MAGNETOHYDRODYNAMICS

Bulletin of exercises n°4: The magnetic induction equation.

1. The induction equation: Starting from Faraday's law and Ampere's law (neglecting the displacement current) and making use of Ohm's constitutive relation, eliminate the electric field and the current density to obtain the inducion equation for a plasma in the MHD approximation:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{rot}\,(\mathbf{v} \wedge \mathbf{B}) - \mathbf{rot}\left(\frac{c^2}{4\pi\sigma_{\mathrm{e}}}\,\mathbf{rot}\,\mathbf{B}\right). \tag{1}$$

Show that if the electrical conductivity is uniform, the induction equation can be cast into the form

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{rot}\,(\mathbf{v} \wedge \mathbf{B}) + \eta\,\nabla^2\,\mathbf{B}\,, \tag{2}$$

where $\eta \stackrel{\mathrm{def}}{=} c^2/(4\pi\sigma_{\mathrm{e}})$ is the "magnetic diffusivity."

2. Combined form of the induction and the continuity equations.

Show that the induction equation can be combined with the equation of continuity into one single differential equation which gives the time evolution of $\mathbf{B}/\rho$ following a fluid element, viz:

$$\frac{D}{D\,t}\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho}\cdot\nabla\right)\mathbf{v} \;-\; \frac{1}{\rho}\,\mathbf{rot}\,(\eta\,\mathbf{rot}\,\mathbf{B})\,. \tag{3}$$

In the case of an ideal MHD-plasma, Eq. (3) simplifies to a form known as *Walén's equation*.

3. Strength of a magnetic flux tube.

A *magnetic flux tube* is the region enclosed by the surface determined by all magnetic field lines passing through a given material circuit C_{t} .

The "strength" of a flux tube is defined as the circulation of the potential vector $\mathbf{A}$ along the material circuit C_{t} , viz.

$$\Phi_{\mathrm{m}}(t) \stackrel{\mathrm{def}}{=} \oint_{C_{\mathrm{t}}} \mathbf{A}\cdot d\boldsymbol{\alpha} \stackrel{\mathrm{def}}{=} \int_{\xi_1}^{\xi_2} \mathbf{A}[\boldsymbol{\alpha}(\xi,t)]\cdot\boldsymbol{\alpha}'(\xi,t)\,d\,\xi\,, \tag{4}$$

where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\xi,t)$ is the parametric expression of the circuit C_{t} .

- It is immediate to show that the circulation of $\mathbf{A}$ around the circuit C_{t} must be equal to the flux of $\mathbf{B}$ through any surface Σ_{t} spanned by the the contour C_{t}:

$$\Phi_{\mathrm{m}}(t) = \iint_{\Sigma_{\mathrm{t}}} \mathbf{B}\cdot\mathbf{n}\,. \tag{5}$$

- Prove the following result (which is purely kinematic): *At any instant t, the magnetic flux through any section of a magnetic tube tube is the same.*

- Comment on the fact that the concept of 'tube' is introduced only for solenoidal fields.

4. Walén's theorem on the 'freezing' of magnetic field lines.

THEOREM: Assume a perfectly conducting MHD-plasma. Then, if a material curve is a line of force at an initial time t_0, it will be a magnetic line of force at any later time t.

The above result is expressed in a pictorial way by saying that *the magnetic lines of force are frozen in a perfectly conducting plasma.*

Hints: (a) First pose the problem from a purely geometrical point of view (which condition must be satisfied by a material curve that is initially a magnetic field line in order to keep being a magnetic field line at any later time?). (b) Integrate Walén's equation [i.e., Eq. (3) for the case $\eta = 0$] in the Lagrangian representation. (c) Make use of the identity

$$\frac{\partial X_i}{\partial a_k}\frac{\partial A_k}{\partial x_j} = \delta_{ij}\,, \qquad \text{where} \quad \mathbf{x} = \mathbf{X}\,(\mathbf{a}, t) \quad \text{and} \quad \mathbf{A} \stackrel{\text{def}}{=} \mathbf{X}^{-1}\,. \tag{6}$$

5. Conservation of magnetic helicity.

The quantity $\mathbf{A}\cdot\mathbf{B}$ is called the *magnetic helicity density.* The magnetic helicity of a material region Ω_{t} is defined as

$$\mathcal{H}(\Omega_{\mathrm{t}}) = \int_{\Omega_{\mathrm{t}}} \mathbf{A}\cdot\mathbf{B} \tag{7}$$

and it can be shown to be a *topological quantity* expressing the 'degree of complexity' of the magnetic field lines in the region Ω_{t}. Under some circunstances, the magnetic helicity $\mathcal{H}(\Omega_{\mathrm{t}})$ is a constant of motion.

- Starting from the induction equation obtain the following evolution equation for $\mathbf{A}\cdot\mathbf{B}/\rho$ in the limit of a perfectly conducting MHD-plasma:

$$\frac{D}{D\,t}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho}\cdot\nabla\right)(\mathbf{A}\cdot\mathbf{v} - \phi)\,, \tag{8}$$

where $\phi(\mathbf{x}, t)$ is a differentiable scalar function.

- Integrating Eq. (8) in a material region Ω_{t} such that (permanently) $\mathbf{n}\cdot\mathbf{B} = 0\ \forall\mathbf{x}\in\partial\Omega_{\mathrm{t}}$ show that the magnetic helicity $\mathcal{H}(\Omega_{\mathrm{t}})$ is a conserved quantity, viz. $\dot{\mathcal{H}}(\Omega_{\mathrm{t}}) = 0$.

- Deduce from the above that the magnetic helicity of a flux tube in ideal MHD is a constant of motion.