MAGNETOHYDRODYNAMICS

Bulletin of exercises n°5: Energy balance in MHD. The energy equation.

1. The kinetic energy equation:

By scalar multiplication of the momentum equation (i.e., Navier-Stokes equation including Lorentz' force) with the velocity field and after integrating in a material region $\Omega_{\rm t}$ we obtain an integral balance equation for the kinetic energy of the plasma contained in $\Omega_{\rm t}$.

Show that the time variation of the total kinetic energy contained in the material region, viz. $\dot{\mathcal{K}}(\Omega_{\rm t})$, is given by

$$\frac{d}{dt}\int_{\Omega_{\rm t}} \frac{1}{2}\rho\,\|\mathbf{v}\|^2 + \int_{\Omega_{\rm t}} \hat{\boldsymbol{\sigma}} : \hat{\boldsymbol{D}} = \int_{\Omega_{\rm t}} \rho\,\mathbf{g}\cdot\mathbf{v} + \oiint_{\partial\Omega_{\rm t}} \mathbf{v}\cdot\hat{\boldsymbol{\sigma}}\mathbf{n} + \int_{\Omega_{\rm t}} \mathbf{v}\cdot\frac{\mathbf{j}\wedge\mathbf{B}}{c} \tag{1.a}$$

or

$$\dot{\mathcal{K}}\,(\Omega_{\rm t}) + \mathcal{W}^{\rm def}\,(\Omega_{\rm t}) = \mathcal{W}^{\rm vol}\,(\Omega_{\rm t}) + \mathcal{W}^{\rm sur}\,(\partial\Omega_{\rm t}) + \int \mathbf{v}\cdot\frac{\mathbf{j}\wedge\mathbf{B}}{c} \tag{1.b}$$

Here $\hat{\boldsymbol{\sigma}}$ is the stress tensor and $\hat{\boldsymbol{D}}$ is the deformation tensor (i.e., the symmetric part of the gradient-of-velocity tensor). Further, $\mathcal{W}^{\rm def}\,(\Omega_{\rm t})$ is the deformation power, $\mathcal{W}^{\rm vol}\,(\Omega_{\rm t})$ is the power exerted by the long-range forces –volume forces– on the plasma inside $\Omega_{\rm t}$ and $\mathcal{W}^{\rm sur}\,(\partial\Omega_{\rm t})$ is the power exerted by the contact forces –surface forces– on the system through its boundary $\partial\Omega_{\rm t}$.

- What is the physical meaning of the integral $\int_{\Omega_{\rm t}} \mathbf{v}\cdot(\mathbf{j}\wedge\mathbf{B})/c$?
- Show that under some restriction (which one?), the power exerted by the long-range forces on the plasma contained in $\Omega_{\rm t}$ can be expressed as minus the rate of change of the gravitational potential energy, viz. $\mathcal{W}^{\rm vol}\,(\Omega_{\rm t}) = -\dot{V}\,(\Omega_{\rm t})$.
- Separate in the integral $\mathcal{W}^{\rm def}\,(\Omega_{\rm t}) = \int_{\Omega_{\rm t}} \hat{\boldsymbol{\sigma}} : \hat{\boldsymbol{D}}$ the 'net deformation power' (i.e., the 'useful power') from the power that irreversibly goes into thermal energy through viscous effects.

2. Balance for the kinetic and (electro)magnetic energy budget of a MHD-plasma.

Combining the integral expression of Poynting's theorem for a material region $\Omega_{\rm t}$ [Eq. (4) in bulletin 3] with the kinetic energy theorem derived in exercise 1 [Eq. (1) in this bulletin] we can obtain an expression for the kinetic plus (electro)magnetic energy budget of a MHD-plasma.

- Express the source/sink term appearing in Poynting's equation as $-\mathbf{v}\cdot(\mathbf{j}\wedge\mathbf{B})/c - \|\mathbf{j}\|^2/\sigma_{\rm e}$.
- Add together the balance equations for the (electro)magnetic and for the kinetic energy of the plasma contained in $\Omega_{\rm t}$.
- Discuss what we can understand as 'dynamo' and 'motor' from the resulting balance equation.

3. The so-called *First Principle of Thermodynamics* for a MHD-plasma.

From the integral expression of the principle of energy conservation in a MHD-plasma (without chemical or nuclear reactions) we can obtain an integral balance equation for the internal –or 'thermal'– energy $U(\Omega_{\rm t})$.

- Use Poynting's theorem (for a material region) along with the theorem for the kinetic energy to show that
$$\frac{d}{d\,t}\int_{\Omega_{\rm t}} \rho\epsilon = \int_{\Omega_{\rm t}} \hat{\boldsymbol{\sigma}} : \hat{\boldsymbol{D}} - \oiint_{\partial\Omega_{\rm t}} \boldsymbol{\mathcal{F}}\cdot\mathbf{n} + \int_{\Omega_{\rm t}} \frac{\|\mathbf{j}\|^2}{\sigma_{\rm e}}\,, \tag{2}$$
where $\epsilon(\mathbf{x},t)$ is the specific internal energy (i.e., per unit mass) and $\boldsymbol{\mathcal{F}}(\mathbf{x},t)$ is the heat flux density vector.

- Obtain a differential equation from the integral expression (2). This equation is the local or differential form of the *First Principle of Thermodynamics* for a MHD-plasma.

4. Energy equation in the (p,T) representation.

Starting from the *First Principle of Thermodynamics* in differential form, show that the energy equation for a MHD-plasma which is a Newtonian fluid can be written in terms of the thermodynamic variables pressure and temperature in the form
$$\rho c_p \frac{DT}{D\,t} - \alpha\,T\frac{D\,p}{D\,t} = \Phi_v + \Phi_m - \mathrm{div}\,\boldsymbol{\mathcal{F}}\,, \tag{3}$$
where Φ_v and Φ_m are, respectively, the viscous and the Ohmic dissipation functions, α is the coefficient of thermal expansion and c_p is the specific heat at constant pressure.