Bulletin of exercises n°6: Force-free magnetic fields.

If the electric currents flow along the magnetic field lines, the equation of magnetostatic equilibrium is

$$\mathbf{j} \wedge \mathbf{B} = 0\,, \tag{1}$$

in which case the Lorentz force vanishes everywhere and hydrostatic equilibrium is independent of the magnetic field.

From Eq. (1) it is immediate that the curl of $\mathbf{B}$ is everywhere parallel to the field $\mathbf{B}$ itself: $\mathbf{rot}\,\mathbf{B} = \alpha\,\mathbf{B}$, where $\alpha(\mathbf{x})$ is a function of position.

Prove the following general properties of force-free fields:

1. Function $\alpha(\mathbf{x})$. Topological properties.

1.1. The function $\alpha(\mathbf{x})$ is constant along every magnetic field line (but differs, in general, from line to line).

1.2. The integral lines of the fields $\mathbf{B}$ and $\mathbf{j}$ lie on surfaces $\alpha(\mathbf{x}) = \text{const.}$

1.3.* If a surface $\alpha(\mathbf{x}) = \text{const.}$ is closed, then it cannot be simply connected.

2. Magnetic energy:

2. Magnetic energy: The following results are consequences of the restriction imposed on the magnetic energy by the condition that the magnetic field be force-free.

Theorem: If $\mathbf{B}$ is force-free in a region $\mathcal{R}$ with boundary $\partial\mathcal{R}$, the total magnetic energy contained in $\mathcal{R}$ is uniquely determined by the values taken by $\mathbf{B}$ on the boundary $\partial\mathcal{R}$.

Corollary I: It is not possible to have a force-free magnetic field inside a bounded region $\mathcal{R}$ that identically vanishes on the boundary $\partial\mathcal{R}$.

Corollary II: It is not possible to have a force-free magnetic field inside a bounded region $\mathcal{R}$ and such that it is entirely maintained by electric currents confined within $\mathcal{R}$.

3. 'Linear' force-free fields.

In general, $\alpha(\mathbf{x})$ takes different values on different lines of force. In the particular case when $\alpha(\mathbf{x})$ takes the same value everywhere, show that the magnetic field $\mathbf{B}$ satisfies Helmholtz' differential equation, viz.

$$(\nabla^2 + \alpha^2)\,\mathbf{B} = 0\,. \tag{2}$$

The exercise indicated with * is difficult and will not be discussed in the resolution of the exercises.