Spectral Analysis and Time Series

Andreas Lagg

Part I: fundamentals on time series

- classification
- prob. density func.
- auto-correlation
- power spectral density
- cross-correlation
- applications
- pre-processing
- sampling
- trend removal

Part II: Fourier series

- definition
- method
- properties
- convolution
- correlations
- leakage / windowing
- irregular grid
- noise removal

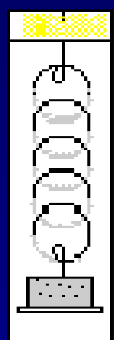
Part III: Wavelets

- why wavelet transforms?
- fundamentals: FT, STFT and resolution problems
- multiresolution analysis: CWT
- DWT

Exercises

Basic description of physical data

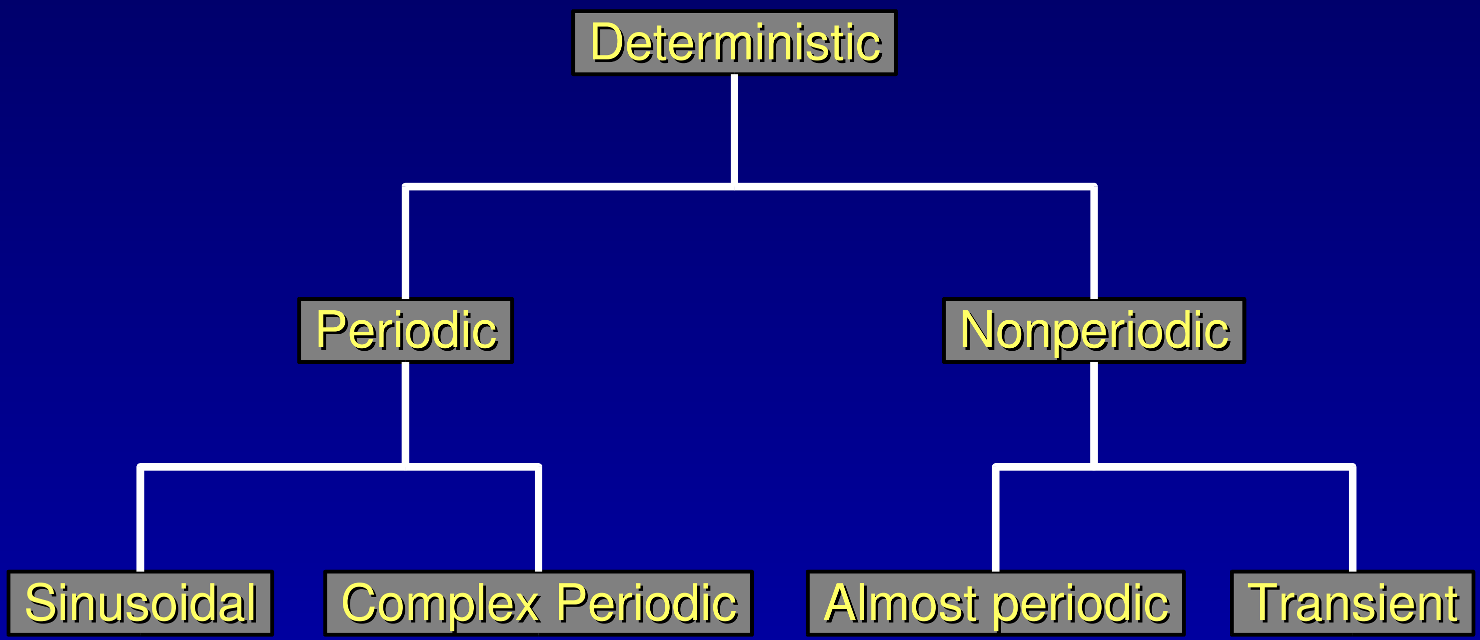
deterministic: described by explicit mathematical relation



$$x(t) = X \cos\left(\sqrt{\frac{k}{t}}\, t\right)$$

non deterministic: no way to predict an exact value at a future instant of time

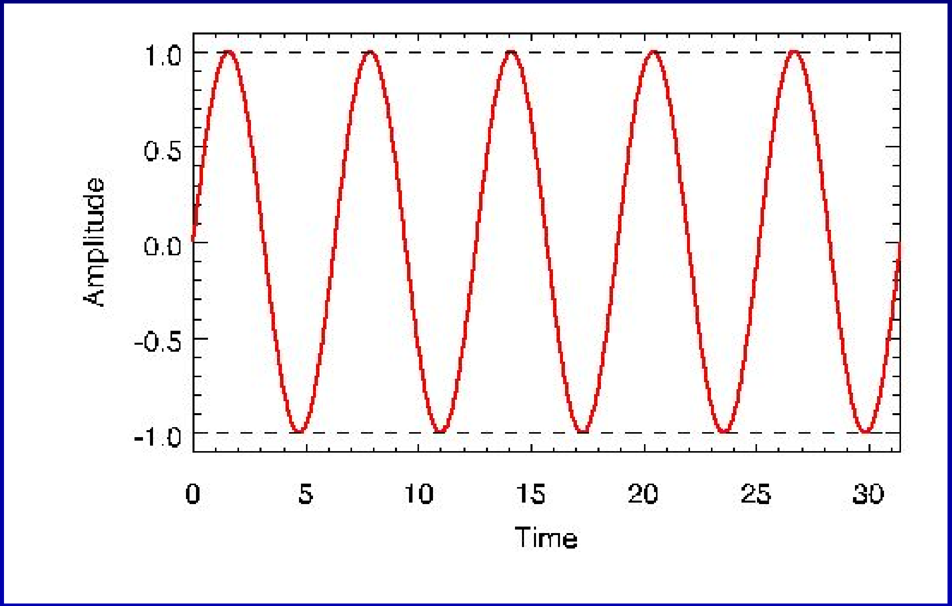
Classifications of deterministic data



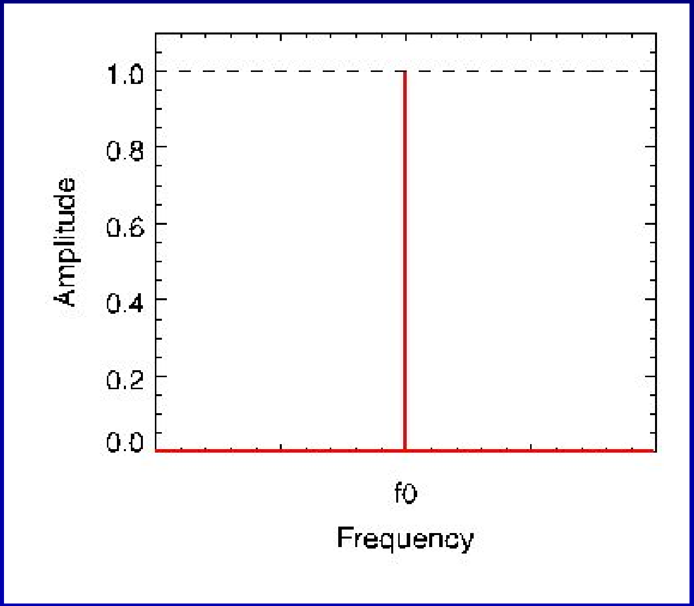
Sinusoidal data

$$x(t)=X\sin(2\pi f_0 t+\Theta)$$
$$T=1/f_0$$

time history



frequency spectrogram



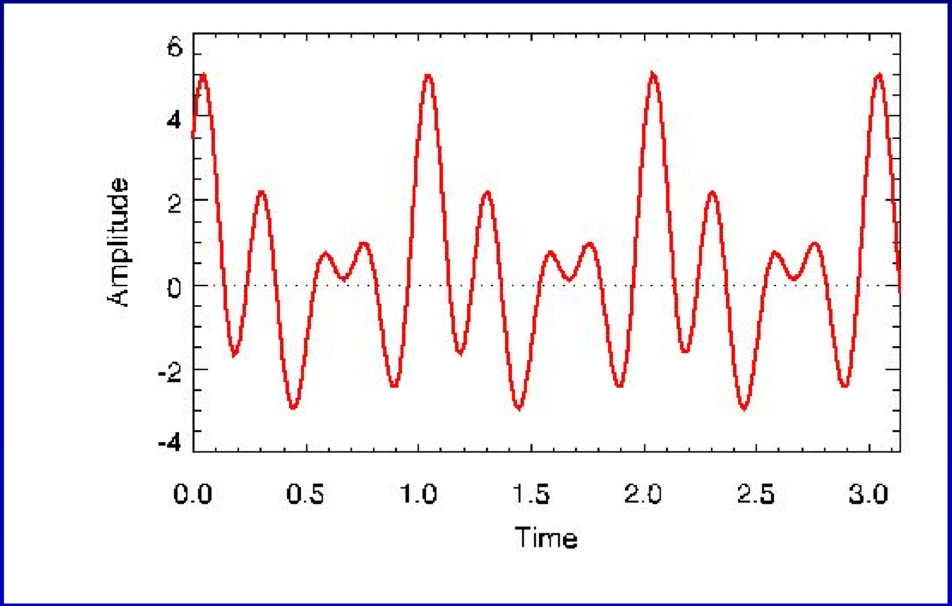
Complex periodic data

$$x(t)=x(t\pm nT) \quad n=1,2,3,\ldots$$

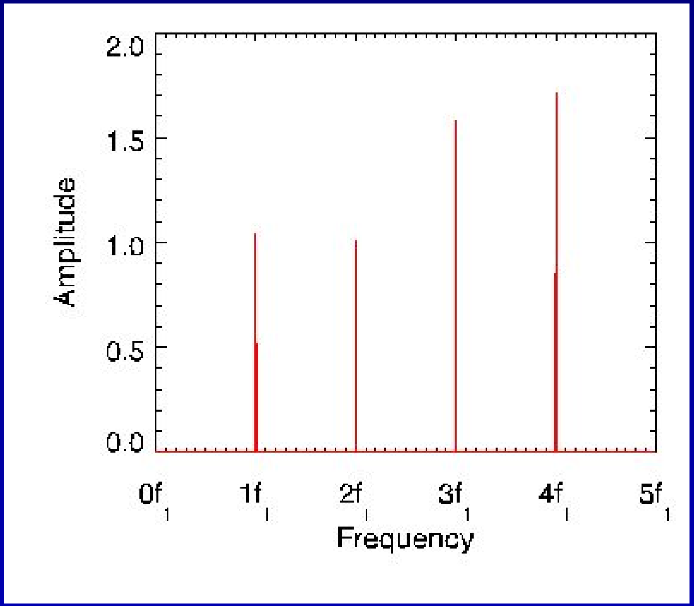
$$x(t) = \frac{a_0}{2} + \sum (a_n \cos 2\pi n f_1 t + b_n \sin 2\pi n f_1 t)$$

(T = fundamental period)

time history



frequency spectrogram

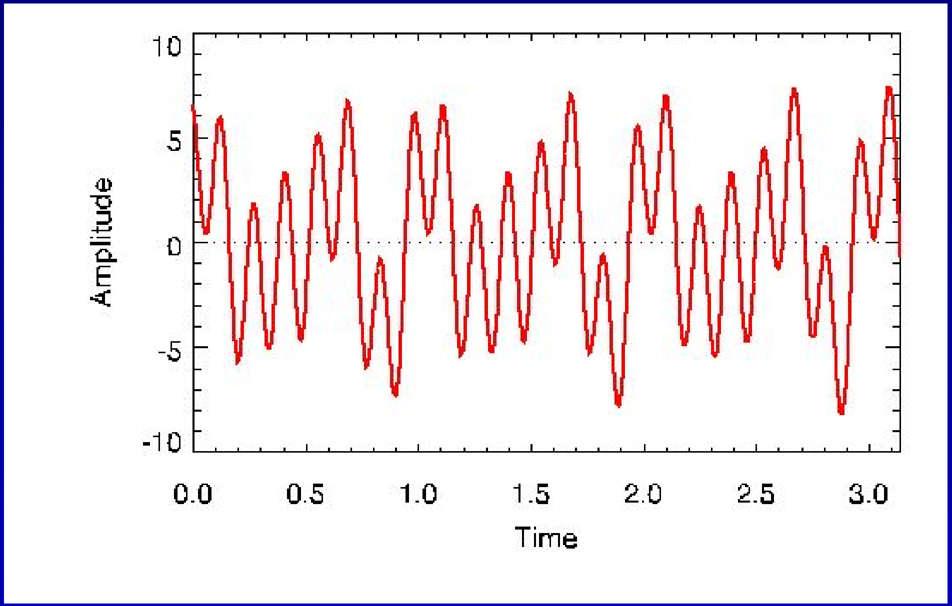


Almost periodic data

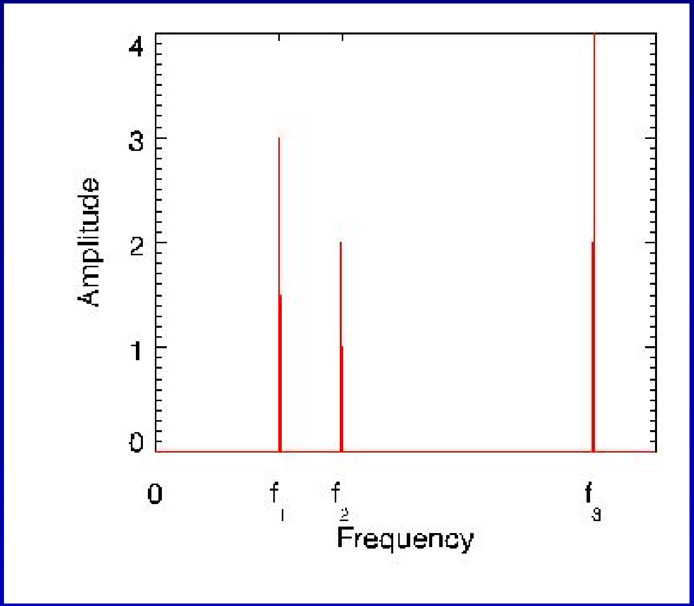
$$x(t) = X_1 \sin(2t + \Theta_1) + X_2 \sin(3t + \Theta_2) + X_3 \sin(\sqrt{50}\, t + \Theta_3)$$

no highest common divisor -> infinitely long period T

time history



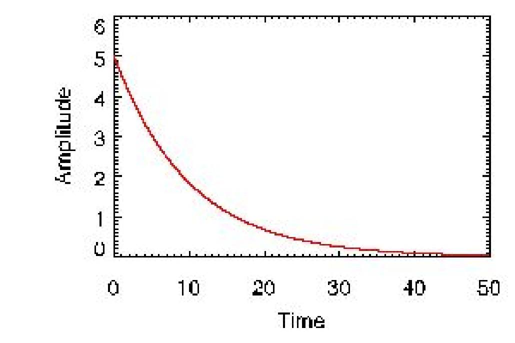
frequency spectrogram

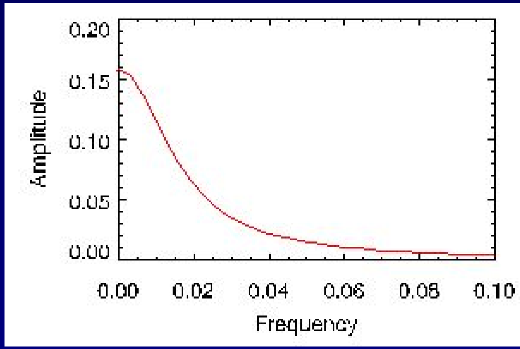


Transient non-periodic data

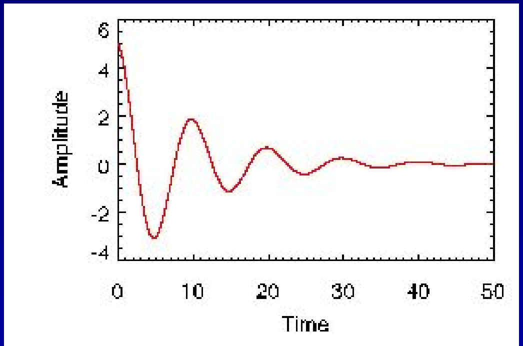
all non-periodic data other than almost periodic data

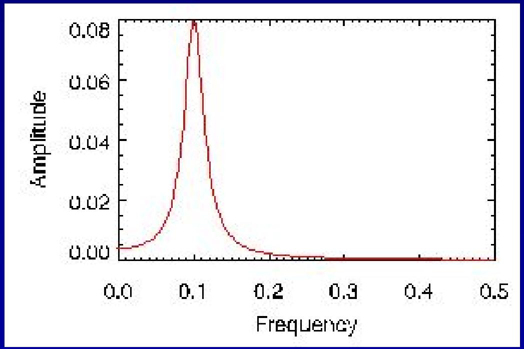
$$x(t)=\begin{cases} A\,e^{-at} & t\geq 0 \\ 0 & t<0 \end{cases}$$



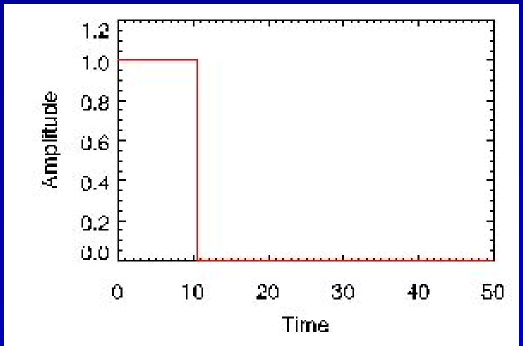


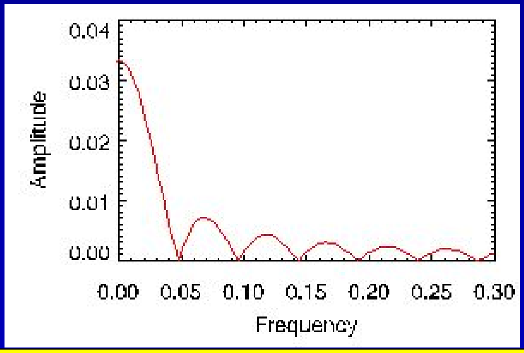
$$x(t)=\begin{cases} A\,e^{-at}\cos bt & t\geq 0 \\ 0 & t<0 \end{cases}$$



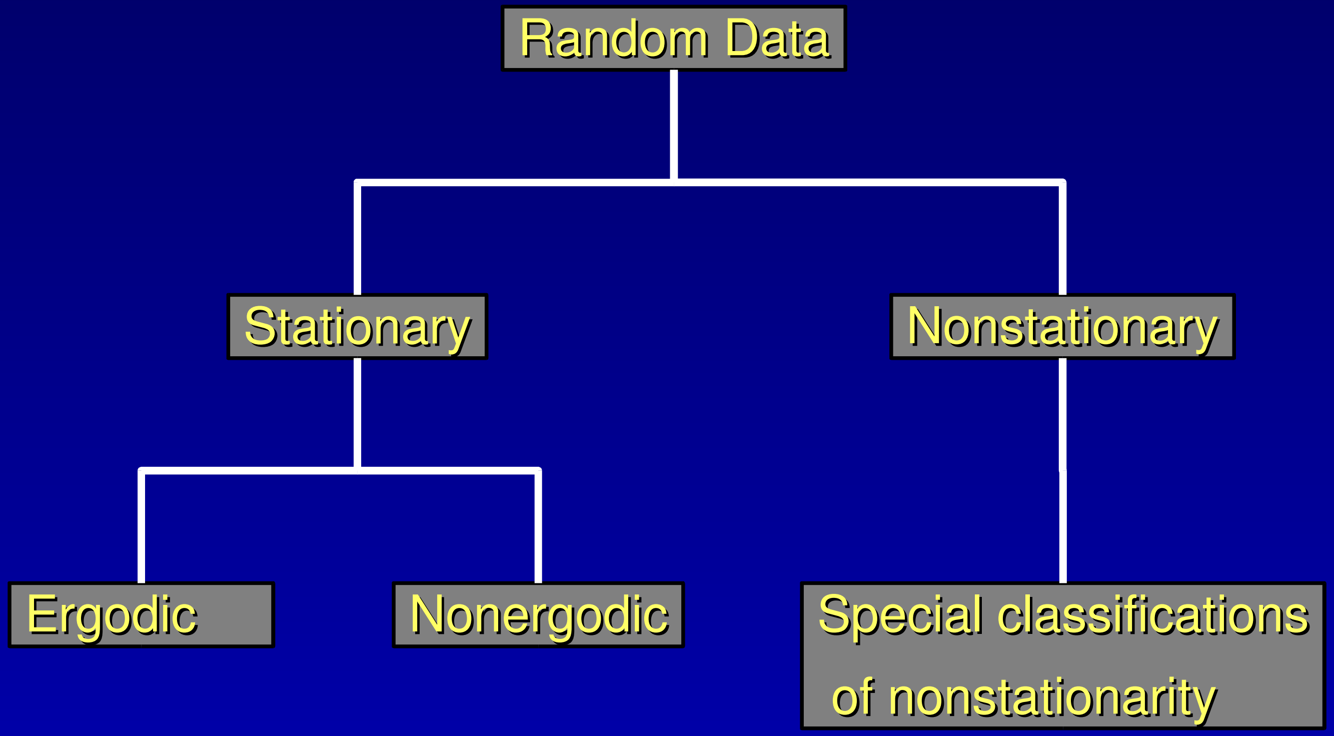


$$x(t)=\begin{cases} A & c\geq t\geq 0 \\ 0 & c<t<0 \end{cases}$$





Classification of random data



stationary / non stationary

collection of sample functions = ensemble

data can be (hypothetically) described by computing ensemble averages (averaging over multiple measurements / sample functions)

mean value (first moment):

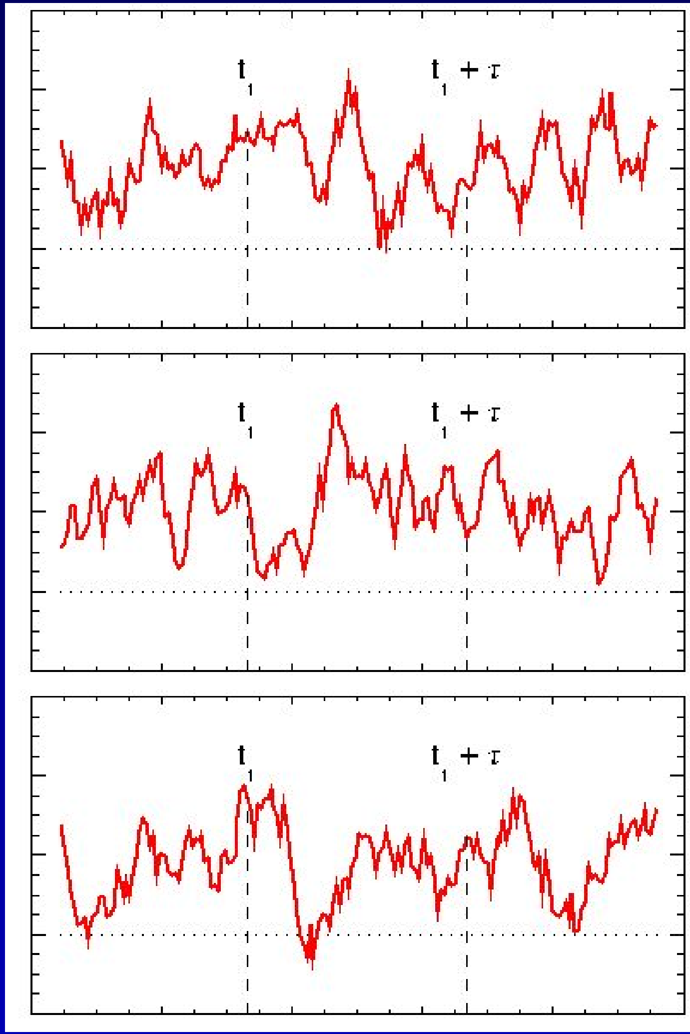
$$\mu_x(t_1) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t_1)$$

autocorrelation function (joint moment):

$$R_x(t_1, t_1+\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t_1)\, x_k(t_1+\tau)$$

stationary: $\mu_x(t_1) = \mu_x, \quad R_x(t_1, t_1+\tau) = R_x$

weakly stationary: $\mu_x(t_1) = \mu_x, \quad R_x(t_1, t_1+\tau) = R_x(\tau)$



ergodic / non ergodic

Ergodic random process:
properties of a stationary random process described by computing averages over only ***one single sample function*** in the ensemble

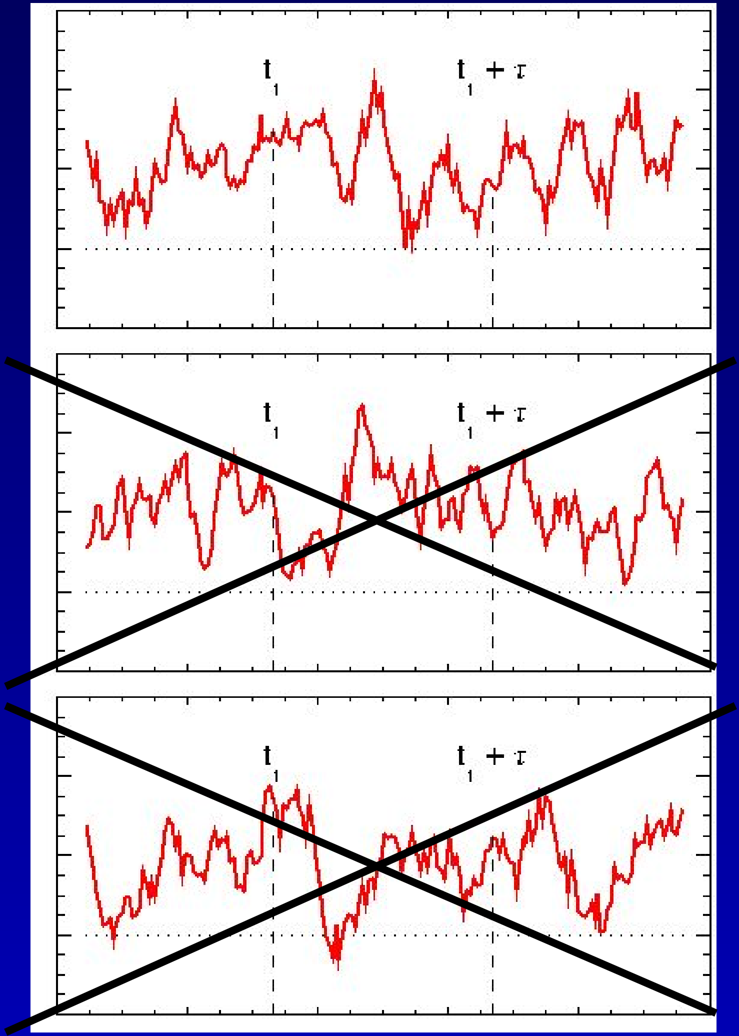
mean value of k-th sample function:

$$\mu_x(k) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x_k(t)\,dt$$

autocorrelation function (joint moment):

$$R_x(\tau,k) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x_k(t)\,x_k(t+\tau)\,dt$$

ergodic: $\mu_x(k) = \mu_x, \quad R_x(\tau,k) = R_x(\tau)$



Basic descriptive properties of random data

- mean square values
- propability density function
- autocorrelation functions
- power spectral density functions

(from now on: assume random data to be stationary and ergodic)

Mean square values

(mean values and variances)

describes general intensity of random data: $$\Psi_x^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T x^2(t)\,dt$$

rout mean square value: $$\Psi_x^{rms} = \sqrt{\Psi_x^2}$$

often convenient:

- static component described by mean value: $$\mu_x = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)\,dt$$
- dynamic component described by variance: $$\sigma_x^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T [x(t)-\mu_x]^2\,dt$$ $$= \Psi_x^2 - \mu_x^2$$

standard deviation: $$\sigma_x = \sqrt{\sigma_x^2}$$

Probability density functions

describes the probability that the data will assume a value within some defined range at any instant of time

$$\text{Prob}[x < x(t) \le x + \Delta x] = \lim_{T \to \infty} \frac{T_x}{T}, \quad T_x = \sum_{i=1}^{k} \Delta t_i$$

for small Δx : $\text{Prob}[x < x(t) \le x + \Delta x] \approx p(x) \Delta x$

→ probability density function

$$p(x) = \lim_{\Delta x \to 0} \frac{\text{Prob}[x < x(t) \le x + \Delta x]}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\lim_{T \to \infty} \frac{T_x}{T}\right]$$

→ probability distribution function

$$P(x) = \text{Prob}[x(t) \le x] = \int_{-\infty}^{x} p(\xi)\, d\xi$$

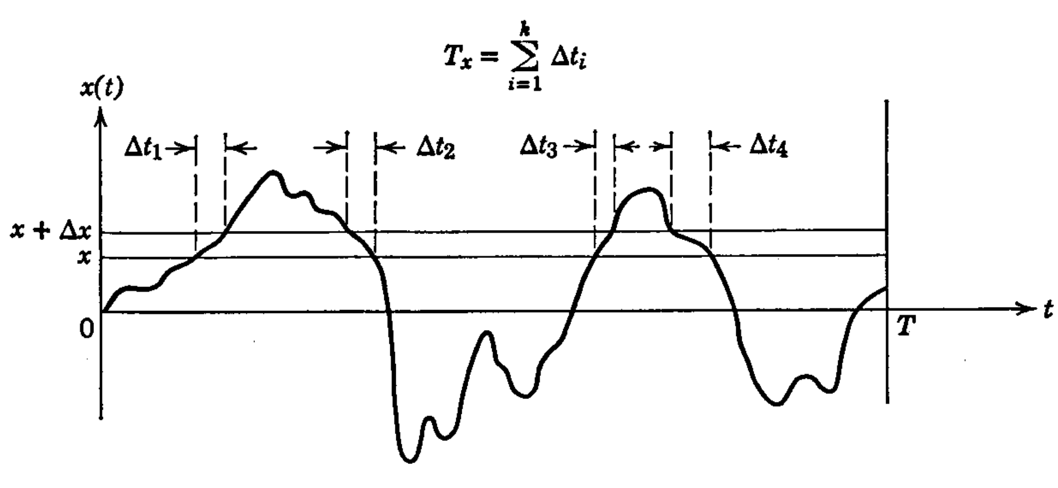


Illustration: probability density function

sample time histories:

- sine wave (a)
- sine wave + random noise
- narrow-band random noise
- wide-band random noise

all 4 cases: mean value $\mu_x = 0$

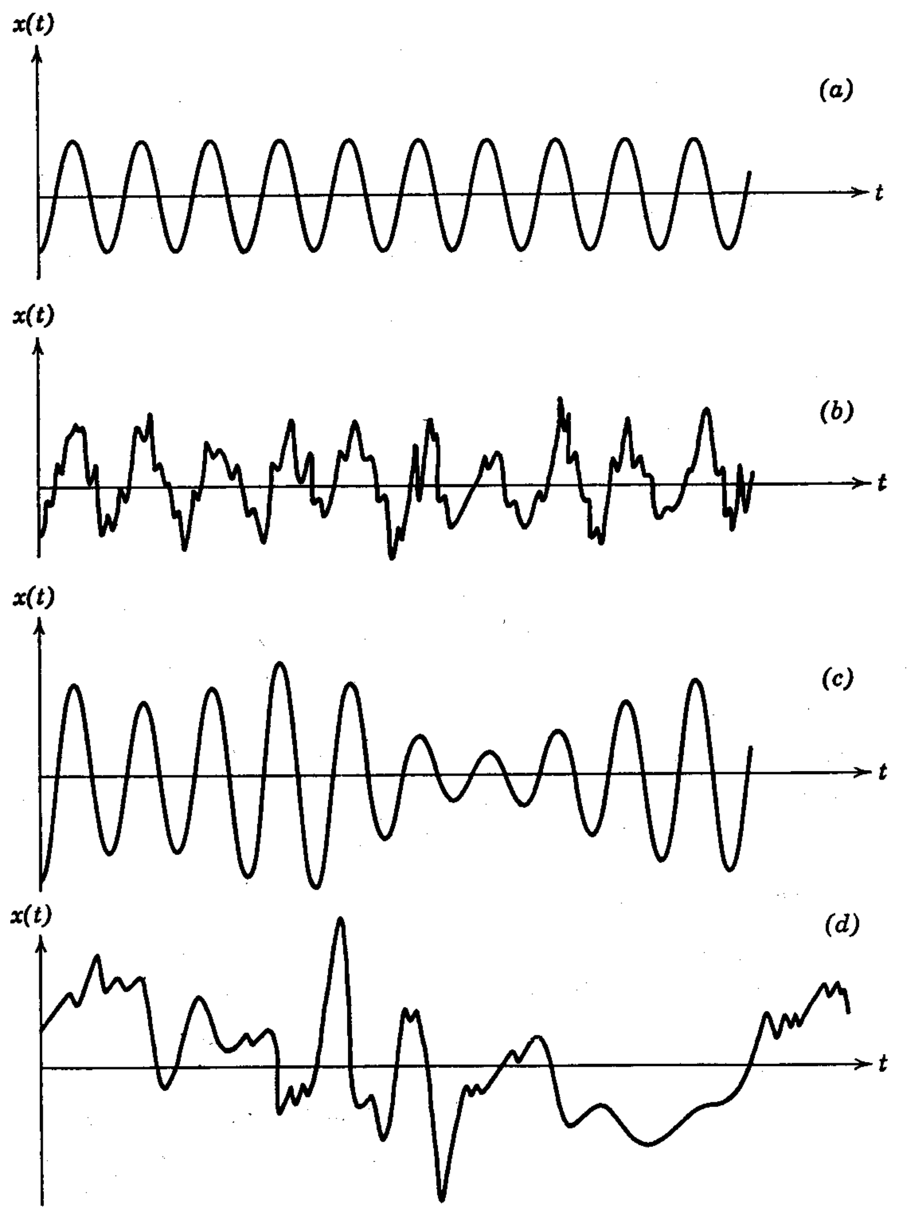
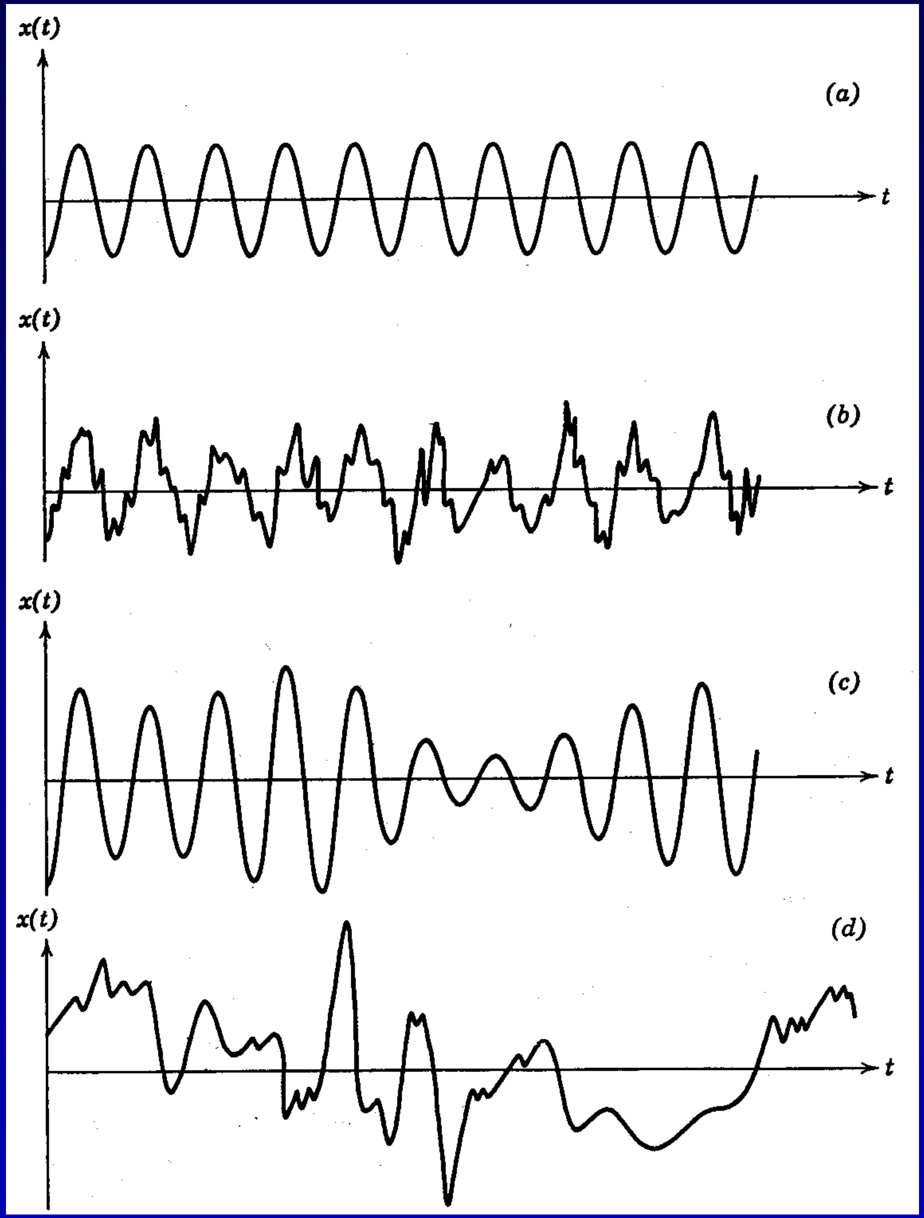
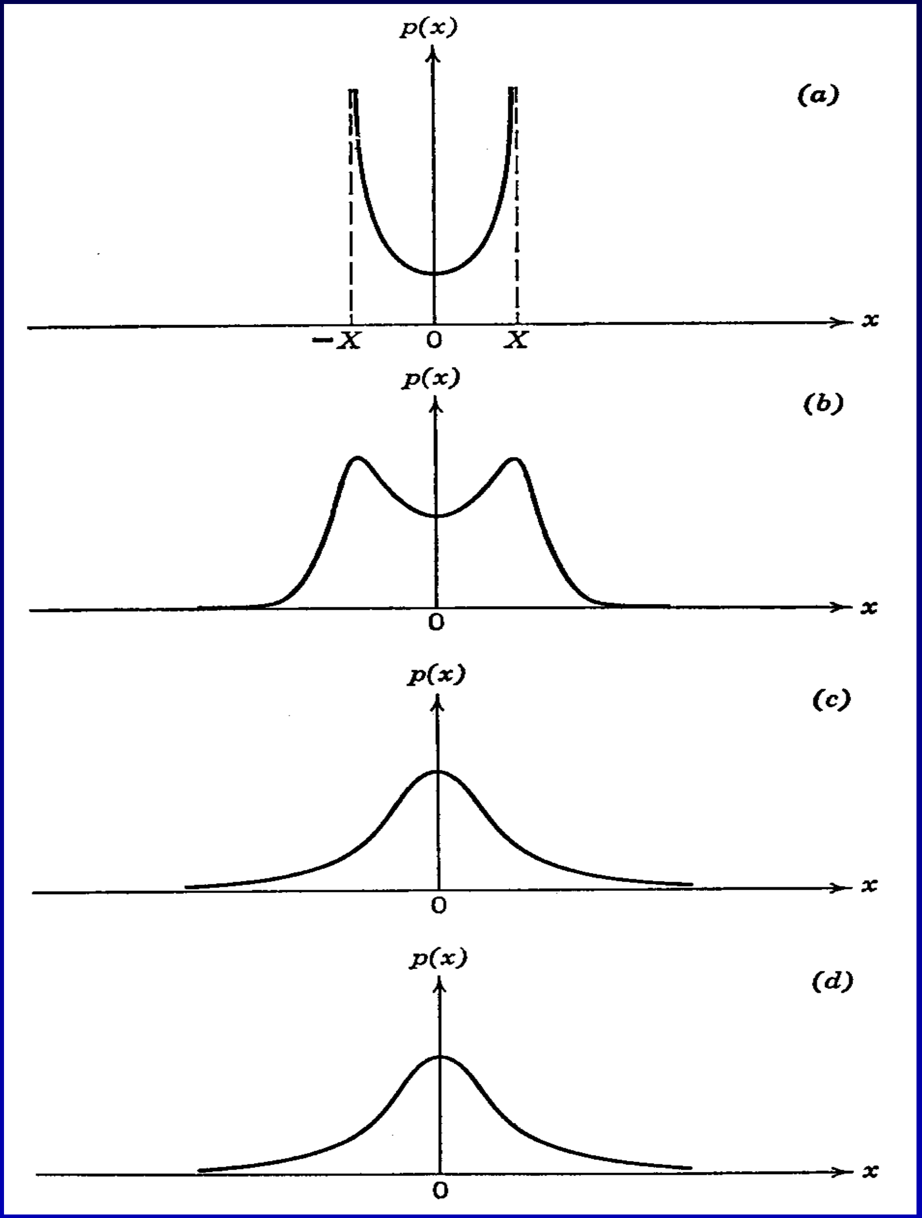


Illustration: probability density function

probability density function

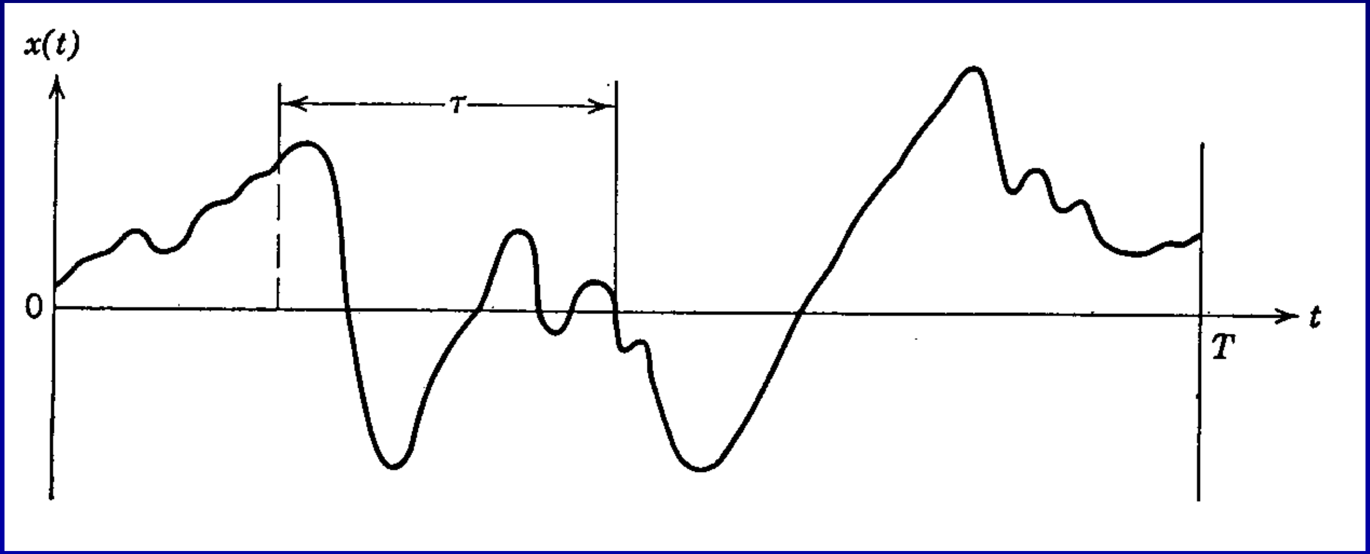




Autocorrelation functions

describes the general dependence of the data values at one time on the values at another time.

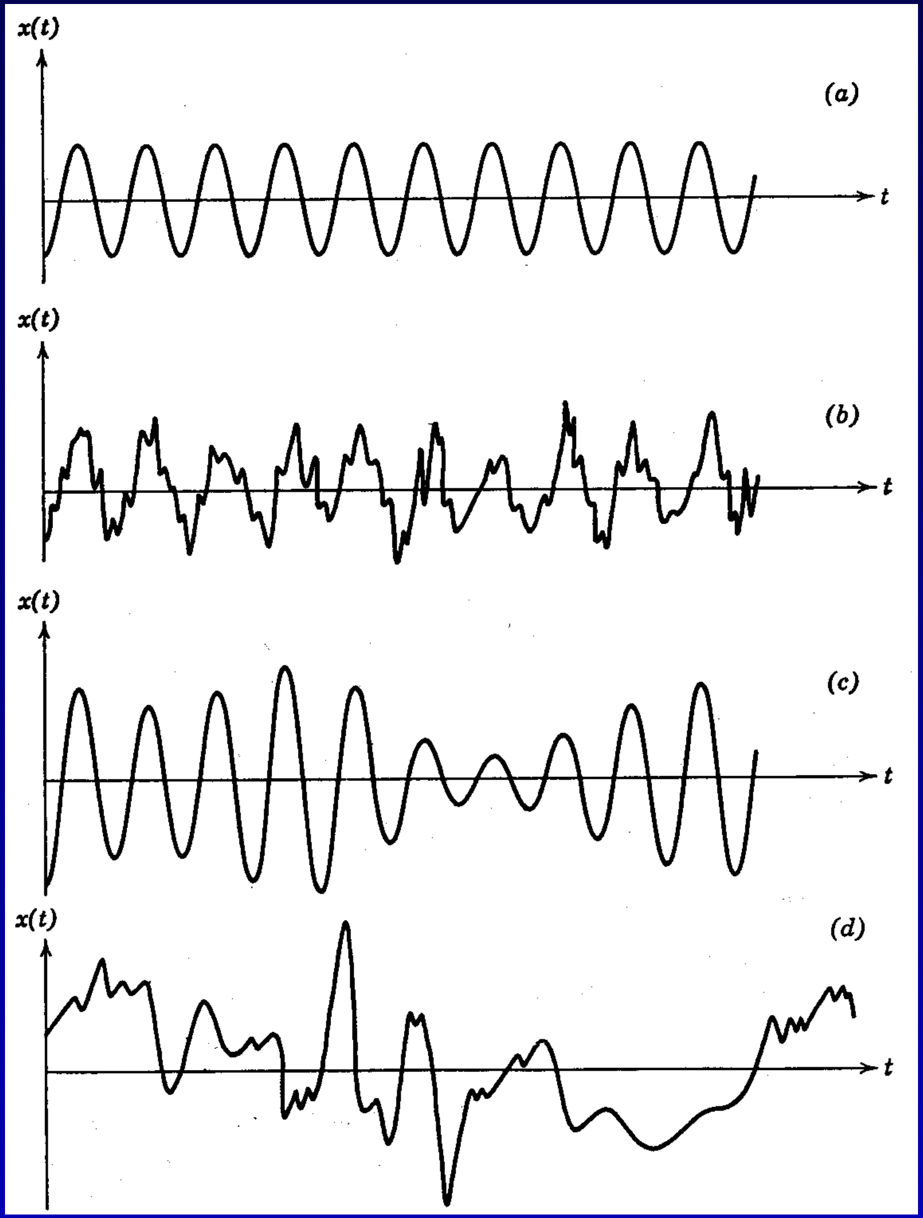
$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, x(t+\tau)\, dt$$

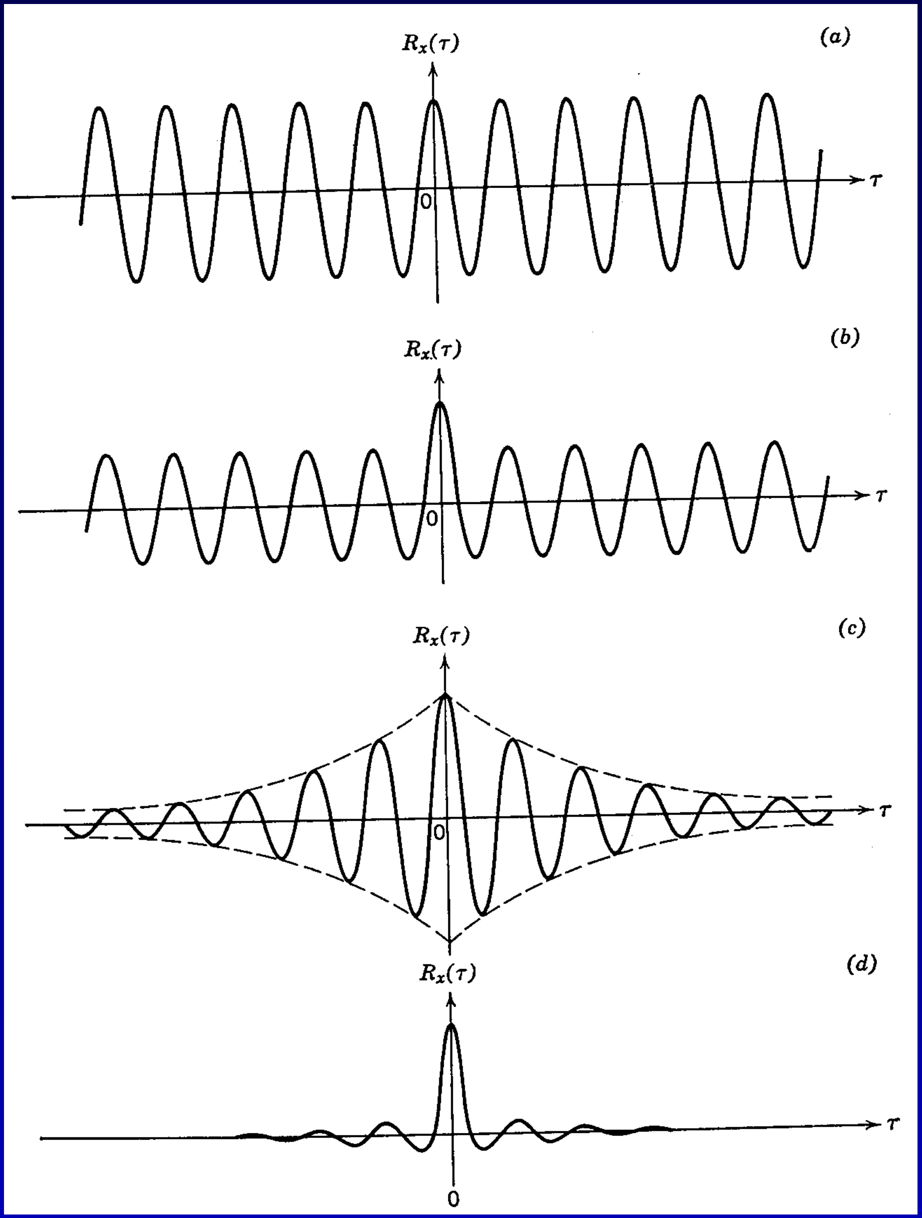


$$\mu_x = \sqrt{R_x(\infty)} \qquad \Psi_x^2 = R_x(0)$$ (not for special cases like sine waves)

Illustration: ACF

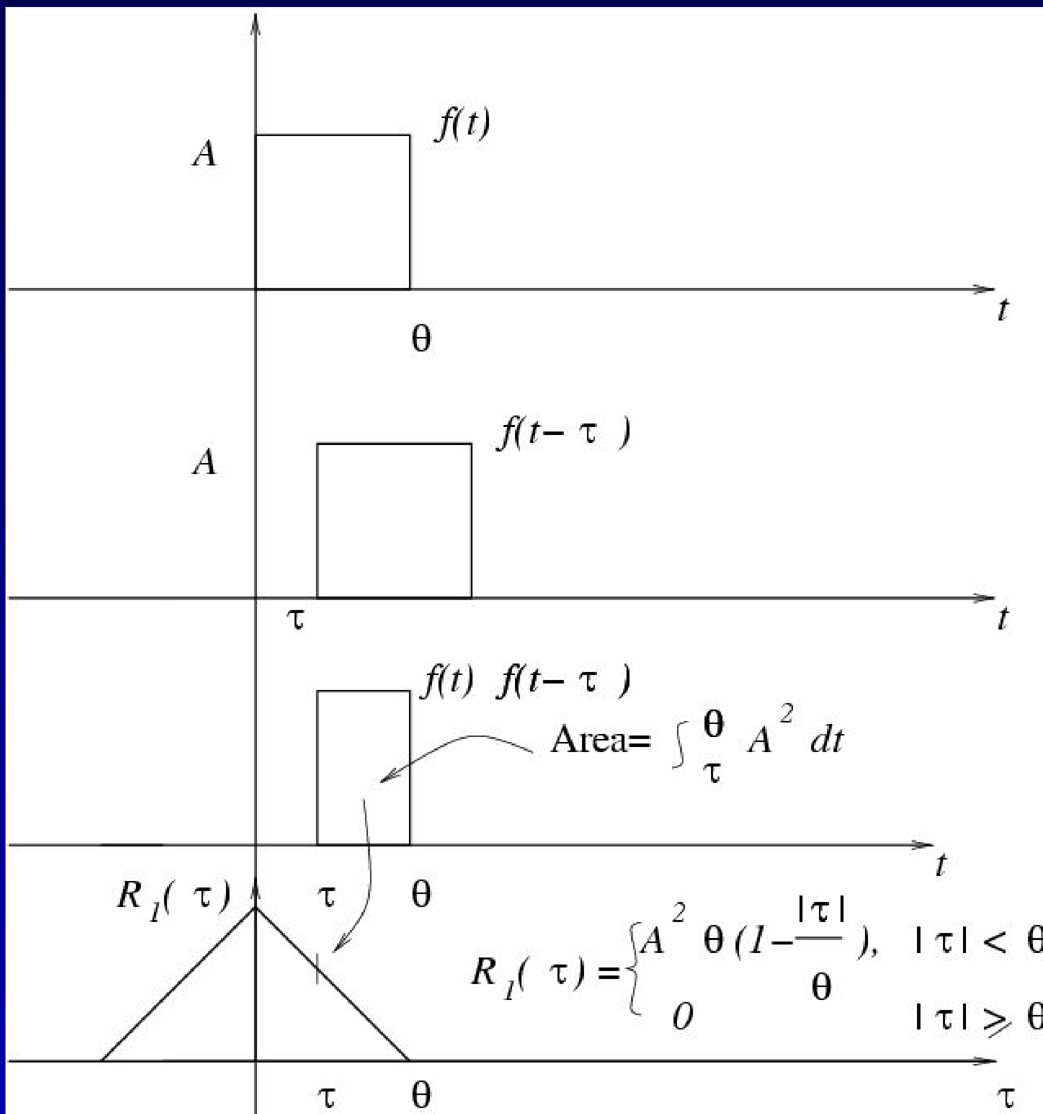
autocorrelation functions (autocorrelogram)

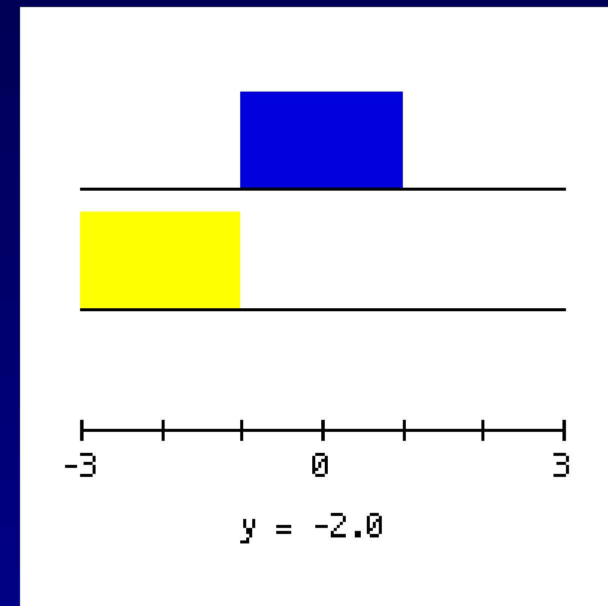




Illustrations

autocorrelation function of a rectangular pulse





$x(t)\,x(t-\tau)$

autocorrelation function

Power spectral density functions

(also called autospectral density functions)

describe the general frequency composition of the data in terms of the spectral density of its mean square value

mean square value in frequency range $(f, \; f+\Delta f)$:

$$\Psi_x^2(f,\Delta f) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \underbrace{x(t,f,\Delta f)^2}_{\text{portion of x(t) in (f,f+}\Delta\text{f)}} dt$$

definition of power spectral density function: $\Psi_x^2(f,\Delta f) \approx G_x(f)\Delta f$

$$G_x(f) = \lim_{\Delta f\to 0} \frac{\Psi_x^2(f,\Delta f)}{\Delta f} = \lim_{\Delta f\to 0} \frac{1}{\Delta f}\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T x(t,f,\Delta f)^2\, dt\right]$$

important property: spectral density function is related to the autocorrelation function by a Fourier transform:

$$G_x(f) = 2\int_{-\infty}^{\infty} R_x(\tau)e^{-i2\pi f\tau}\,d\tau = 4\int_0^{\infty} R_x(\tau)\cos 2\pi f\tau \; d\tau$$

Illustration: PSD

power spectral density functions

Dirac delta function at $f=f_0$

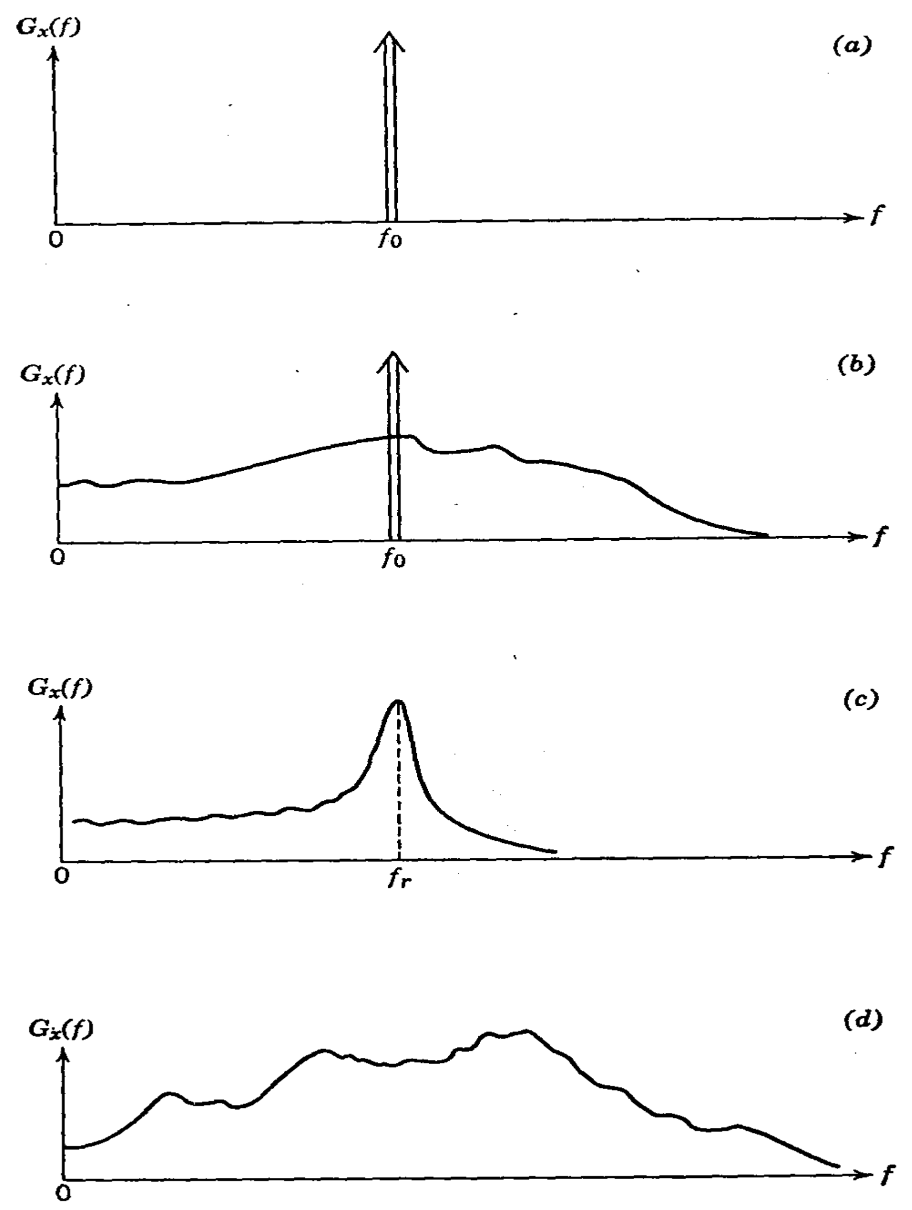
sine wave

sine wave
+ random noise

narrowband
noise

"white" noise:
spectrum is uniform over
all frequencies

broadband
noise



Joint properties of random data

until now: described properties of an individual random process

Joint probability density functions

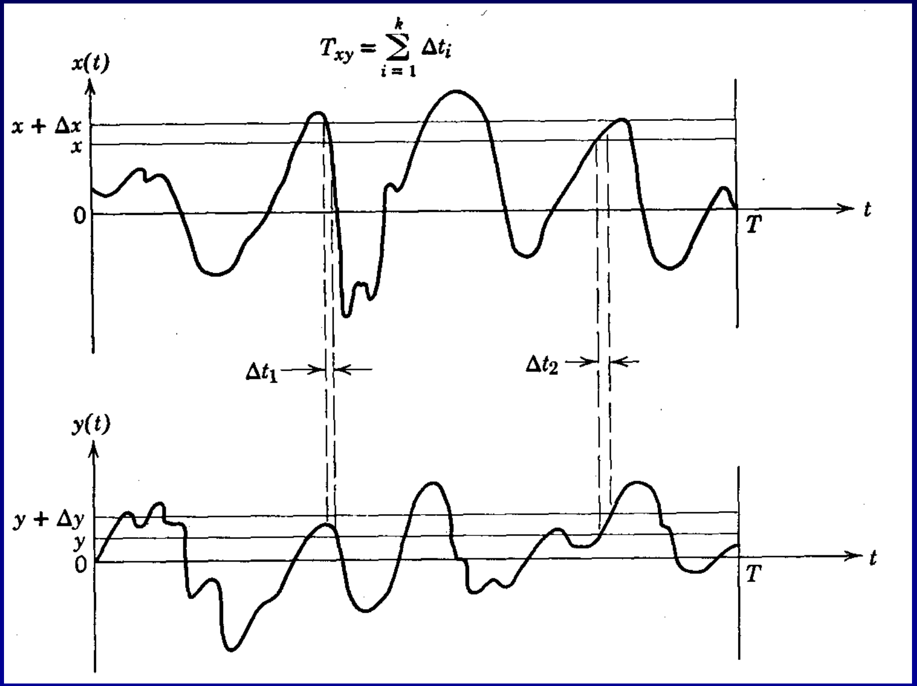
- joint properties in the amplitude domain

Cross-correlation functions

- joint properties in the time domain

Cross-spectral density functions

- joint properties in the frequency domain



joint probability measurement

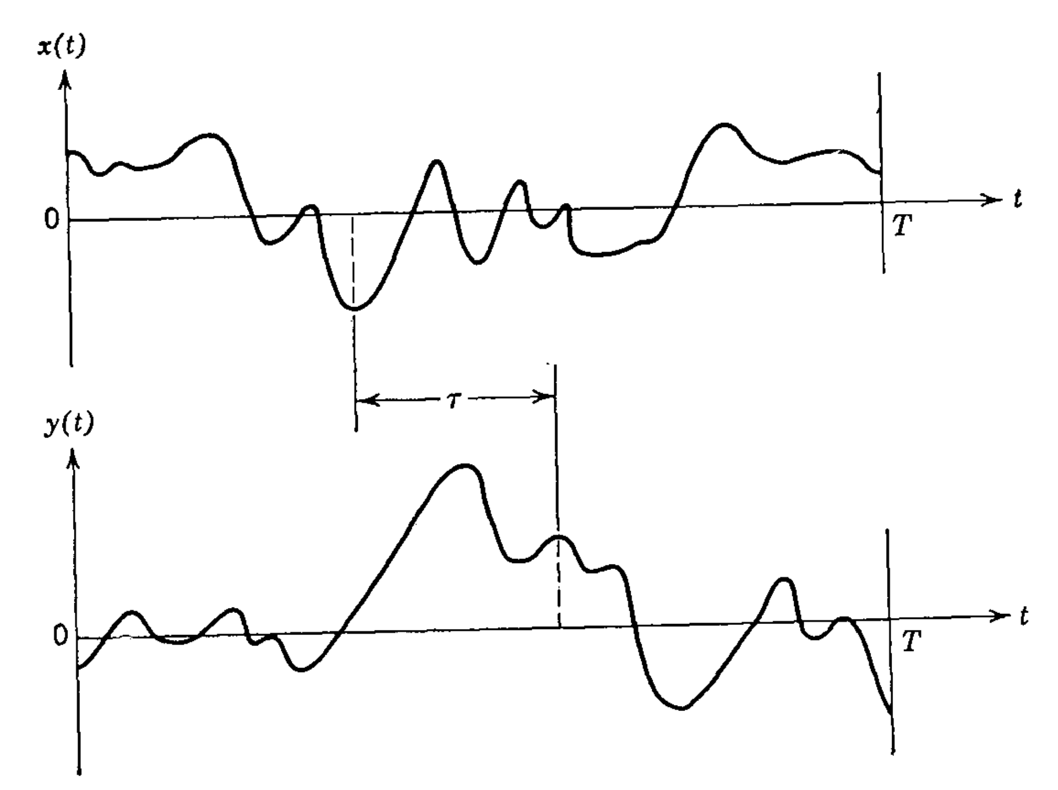
Cross-correlation function

describes the general dependence of one data set to another

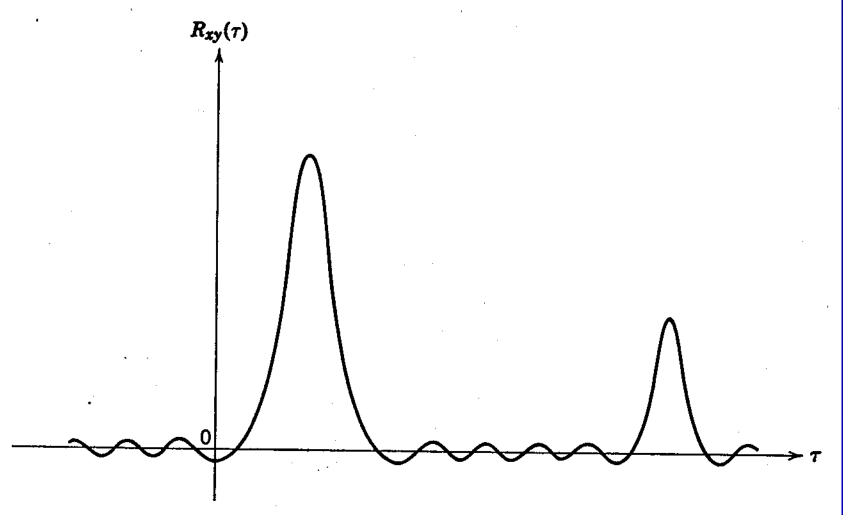
$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)\, y(t+\tau)\, dt$$

similar to autocorrelation function

$R_{xy}(\tau) = 0$ functions are uncorrelated

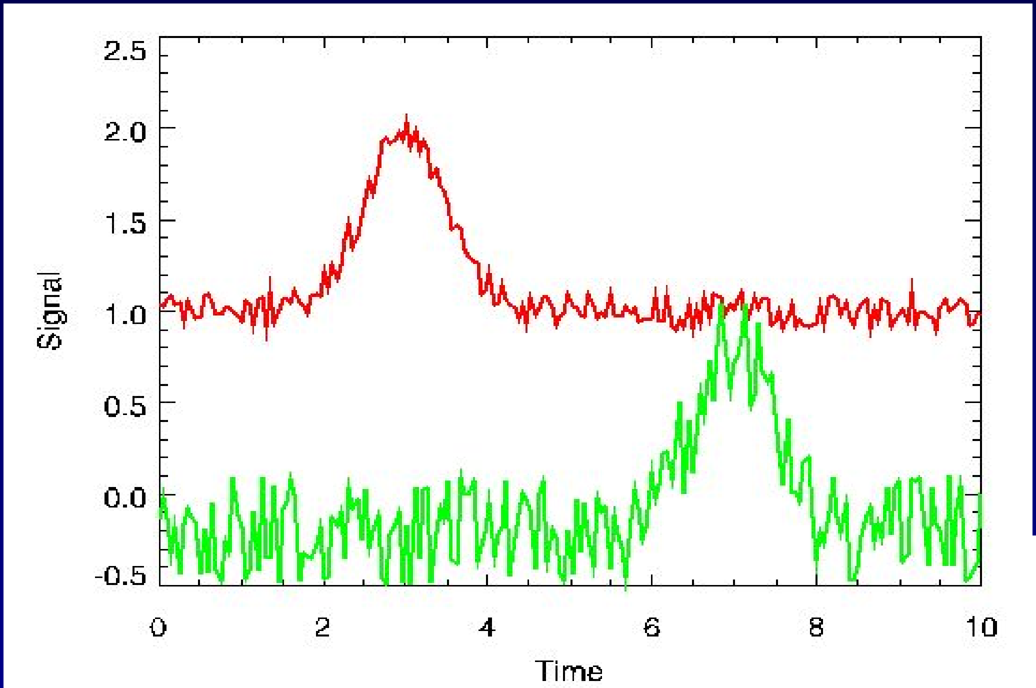


cross-correlation measurement



typical cross-correlation plot (cross-correlogram): sharp peaks indicate the existence of a correlation between x(t) and y(t) for specific time displacements

Applications



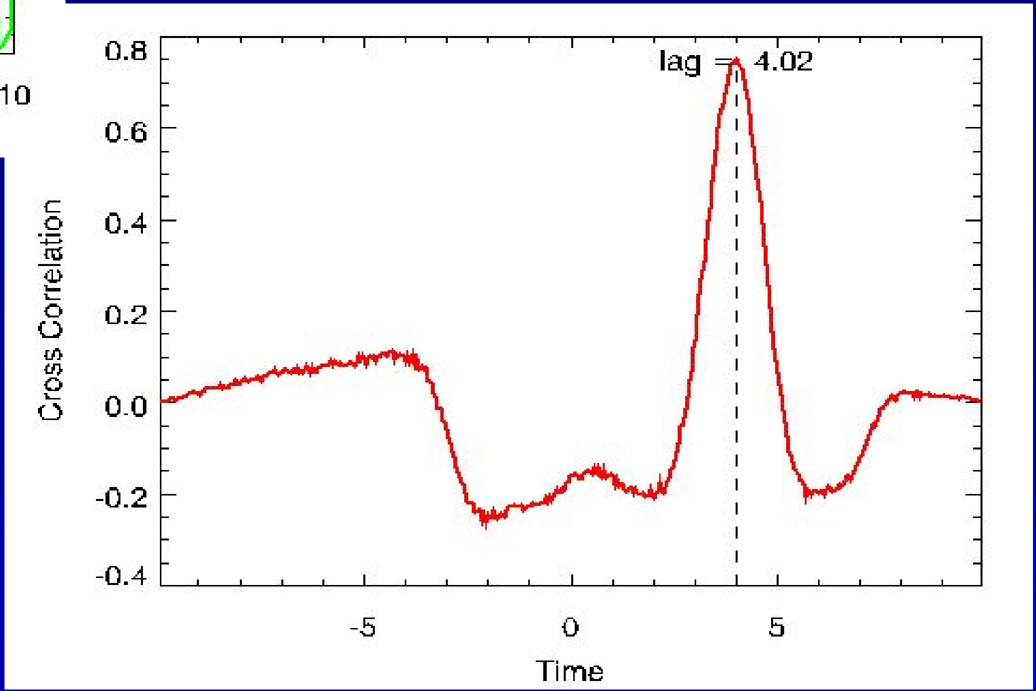
Measurement of time delays

2 signals:

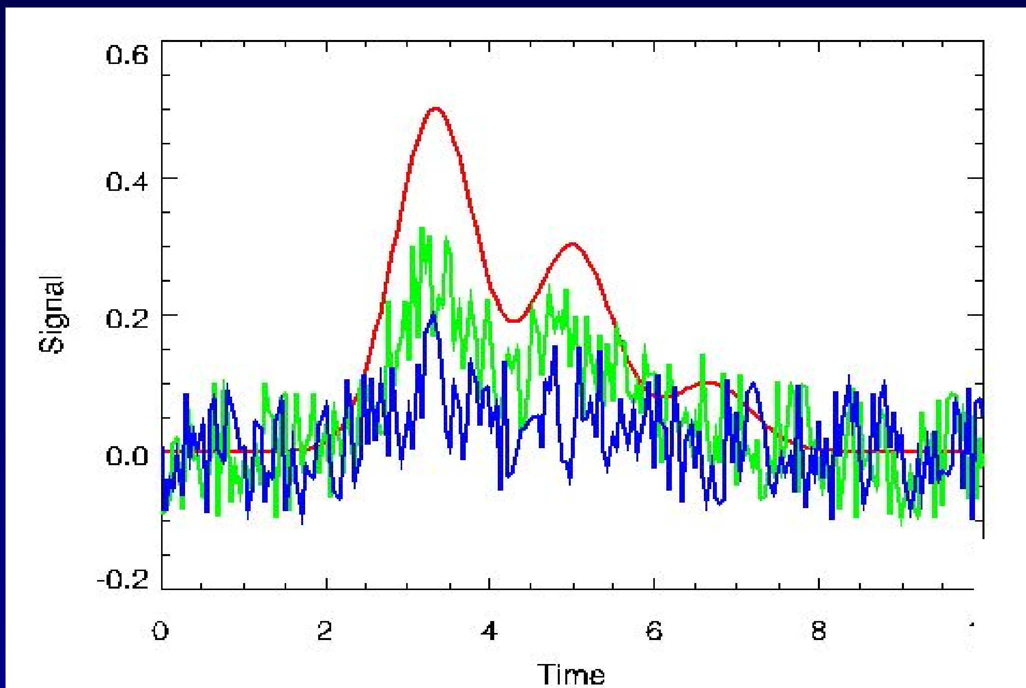
- different offset
- different S/N
- time delay 4s

often used:
'discrete' cross correlation coefficient
lag =l, for l >=0:

$$R_{xy}(l) = \frac{\sum_{k=1}^{N-l}(x_k - \bar{x})(y_{k+l} - \bar{y})}{\sqrt{\sum_{k=1}^{N}(x_k - \bar{x})^2 \sum_{k=1}^{N}(y_k - \bar{y})^2}}$$



Applications

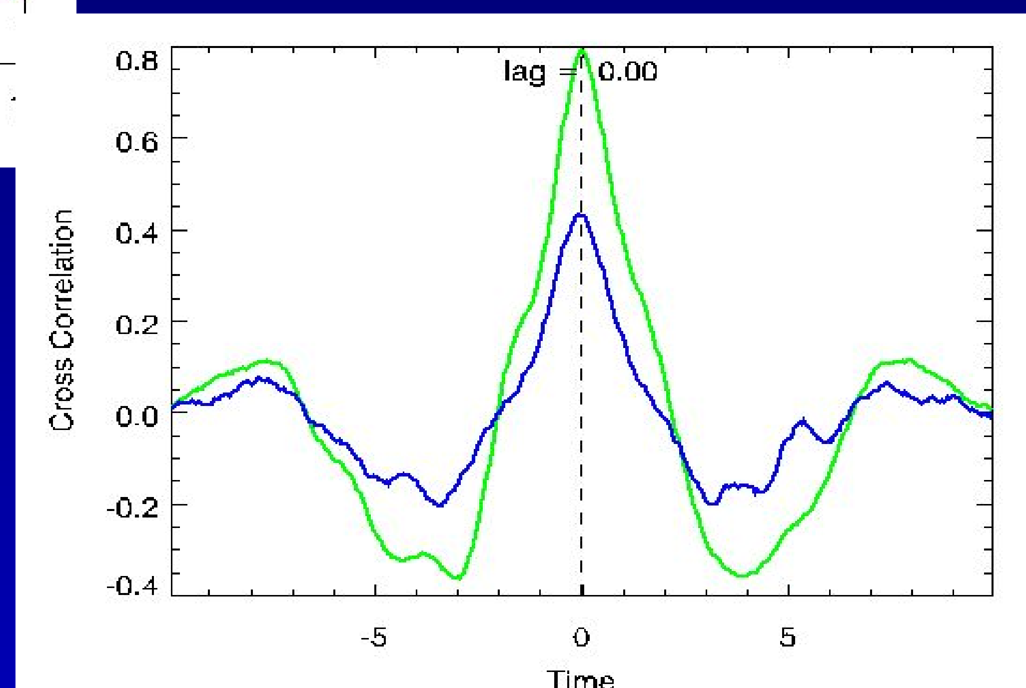


Detection and recovery from signals in noise

3 signals:

- noise free replica of the signal (e.g. model)
- 2 noisy signals

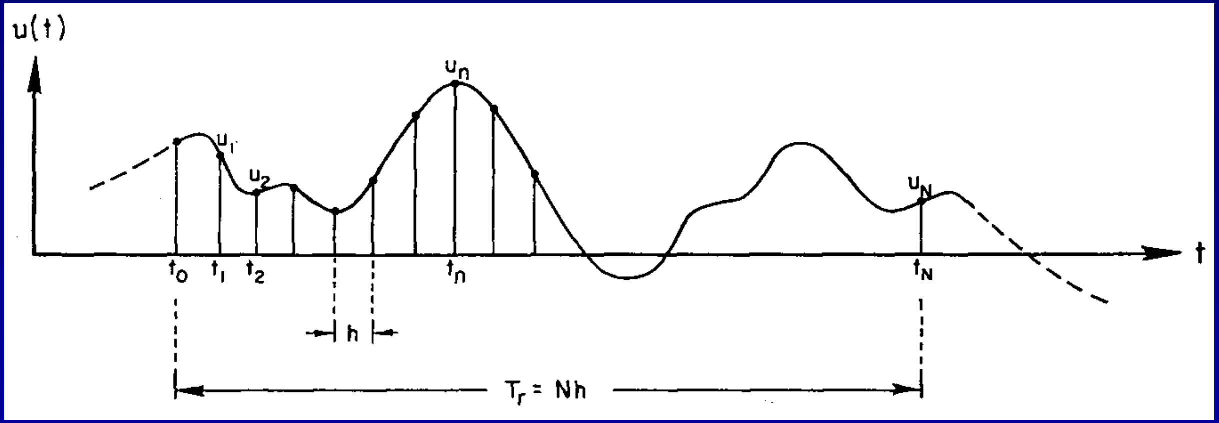
cross correlation can be used to determine if theoretical signal is present in data



Pre-processing Operations

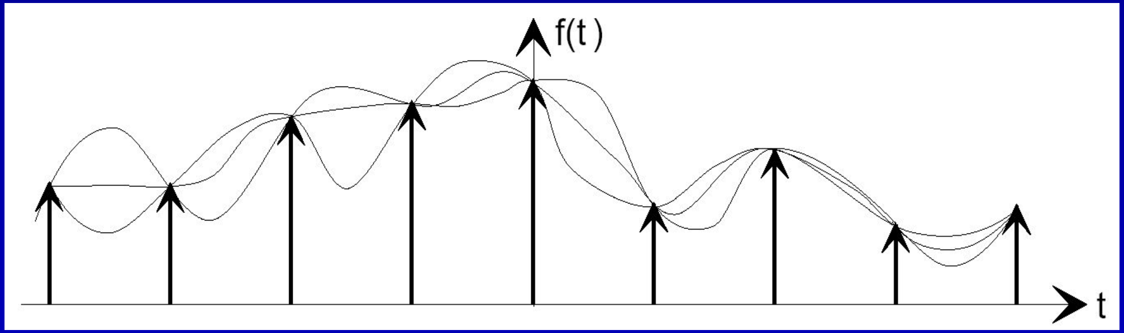
- sampling considerations
- trend removal
- filtering methods

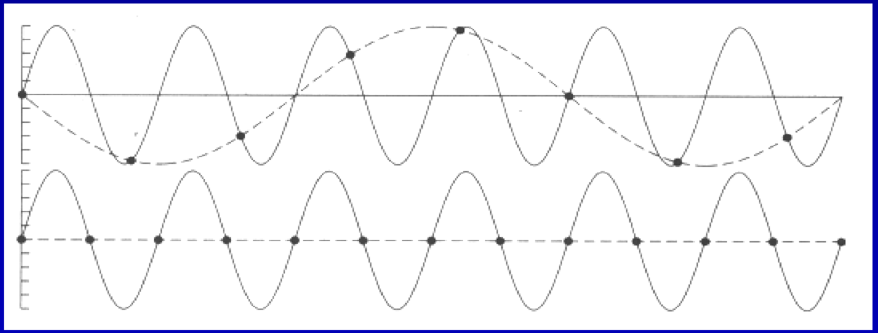
sampling



cutoff frequency (=Nyquist frequency or folding frequency)

$$f_c = \frac{1}{2h}$$





Trend removal

often desirable before performing a spectral analysis

Least-square method:

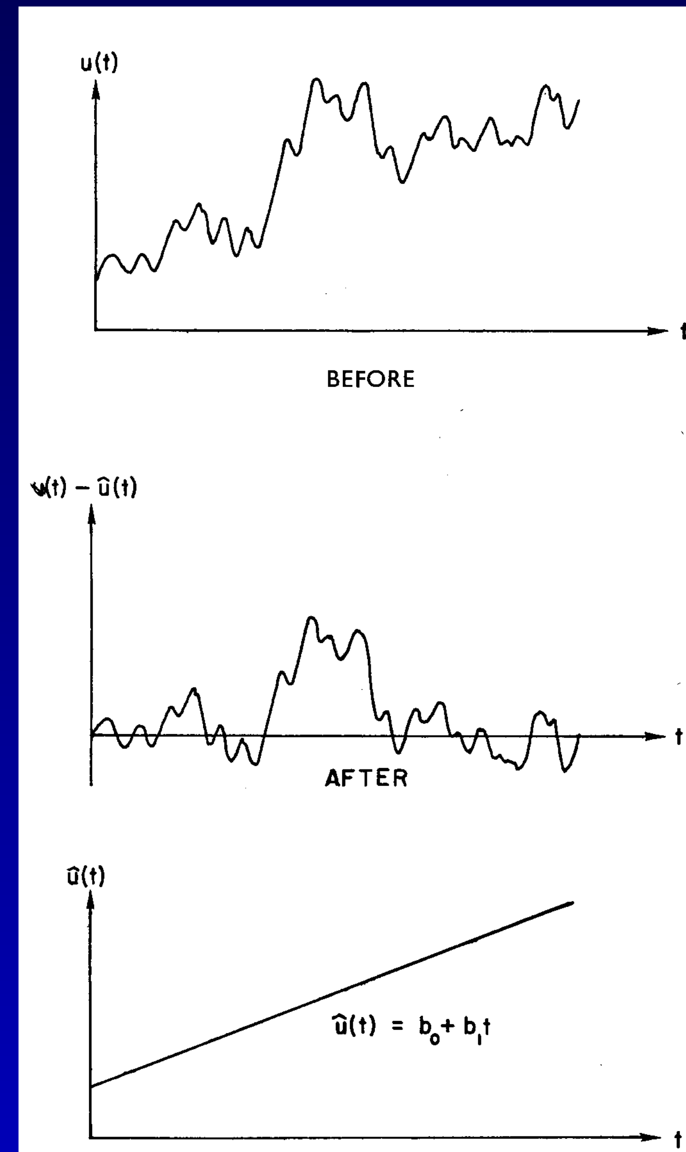
time series: $u(t)$

desired fit
(e.g. polynomial): $\hat{u} = \sum_{k=0}^{K} bk(nh)^k \quad n = 1,2,\ldots,N$

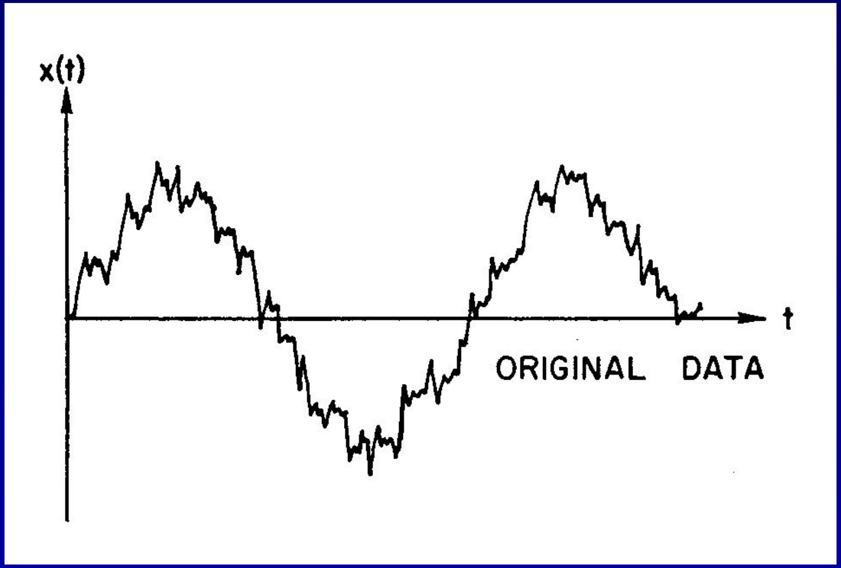
Lsq-Fit: minimize $Q(b) = \sum_{n=1}^{N} (u_n - \hat{u}_n)^2$

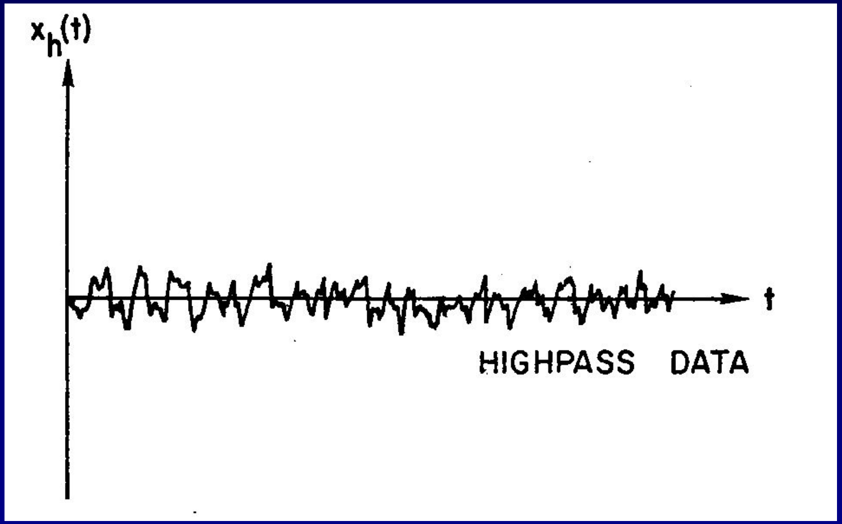
→ set partial derivatives to 0: $\frac{\partial Q}{\partial b_l} = \sum_{n=1}^{N} 2(u_n - \hat{u}_n)[-(nh)^l]$

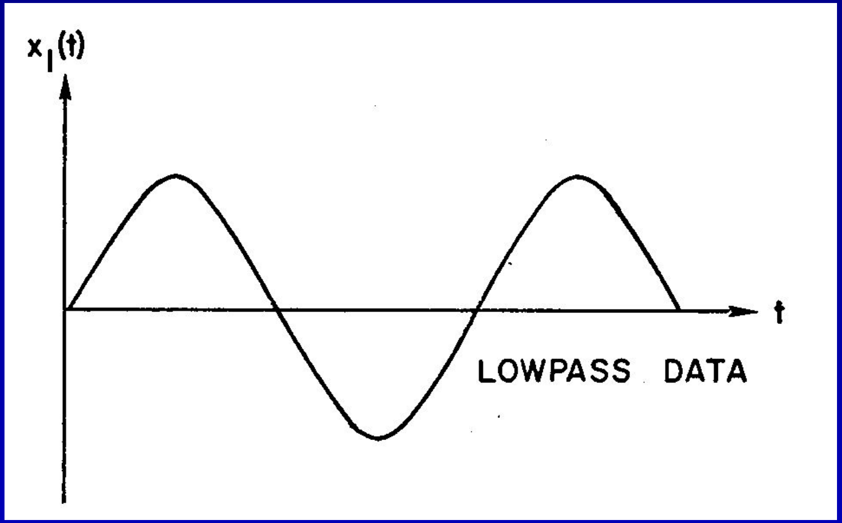
→ K+1 equations: $\sum_{k=0}^{K} b_k \sum_{n=1}^{N} (nh)^{k+l} = \sum_{n=1}^{N} u_n (nh)^l$



Digital filtering







end of part I ...

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Exercises

Fourier Series and Fast Fourier Transforms

Standard Fourier series procedure:

if a transformed sample record x(t) is periodic with a period T_p (fundamental frequency $f_1=1/T_p$), then x(t) can be represented by the Fourier series:

$$x(t) = \frac{a_0}{2} + \sum_{q=1}^{\infty} \left(a_q \cos 2\pi q f_1 t + b_q \sin 2\pi q f_1 t \right)$$

where

$$a_q = \frac{2}{T} \int_0^T x(t) \cos 2\pi q f_1 t \, dt \quad q = 0,1,2,\ldots$$

$$b_q = \frac{2}{T} \int_0^T x(t) \sin 2\pi q f_1 t \, dt \quad q = 1,2,3,\ldots$$

Fourier series procedure - method

sample record of finite length, equally spaced sampled:

$$x_n = x(nh) \quad n = 1,2,\ldots,$$

Fourier series passing through these N data values:

$$x(t) = A_0 + \sum_{q=1}^{N/2} A_q \cos\left(\frac{2\pi q t}{T_p}\right) + \sum_{q=1}^{N/2-1} B_q \sin\left(\frac{2\pi q t}{T_p}\right)$$

Fill in particular points: $t = nh, \quad n = 1,2,\ldots,N, \quad T_p = Nh, \quad x_n = x(nh) = \ldots$

- coefficients A_q and B_q:

$$A_0 = \frac{1}{N}\sum_{n=1}^{N} x_n = \bar{x} \qquad A_{N/2} = \frac{1}{N}\sum_{n=1}^{N} x_n \cos n\pi$$

$$A_q = \frac{2}{N}\sum_{n=1}^{N} x_n \cos\frac{2\pi q n}{N} \qquad q = 1,2,\ldots,\frac{N}{2}-1$$

$$B_q = \frac{2}{N}\sum_{n=1}^{N} x_n \sin\frac{2\pi q n}{N} \qquad q = 1,2,\ldots,\frac{N}{2}-1$$

inefficient & slow => Fast Fourier Trafos developed

Fourier Transforms - Properties

Linearity

$$\{x_n\} \overset{DFT}{\Leftrightarrow} \{X_k\}$$

$$\{y_n\} \overset{DFT}{\Leftrightarrow} \{Y_k\}$$

$$a\{x_n\}+b\{y_n\} \overset{DFT}{\Leftrightarrow} a\{X_k\}+b\{Y_k\}$$

Symmetry

$$\{X_k\} = \{X^*_{-k}\}$$

$$\Re\{X_k\} \text{ is even} \qquad \Im\{X_k\} \text{ is odd}$$

Circular time shift

$$\{x_{n-n_0}\} \overset{DFT}{\Leftrightarrow} \{e^{-ikn_0} X_k\}$$

$$\{e^{ik_0 n} y_n\} \overset{DFT}{\Leftrightarrow} \{Y_{k-k_0}\}$$

Using FFT for Convolution

$$r*s \equiv \int_{-\infty}^{\infty} r(\tau)s(t-\tau)\,d\tau$$

Convolution Theorem:

$$r*s \overset{FT}{\Leftrightarrow} R(f)S(f)$$

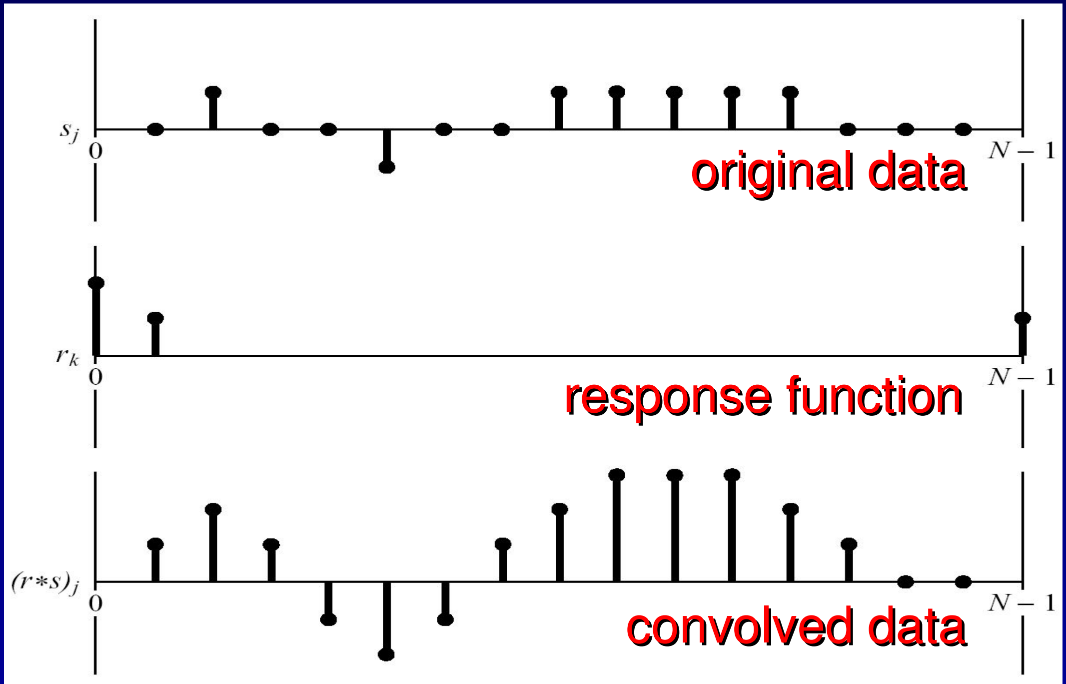
Fourier transform of the convolution is product of the individual Fourier transforms

discrete case:

$$(r*s)_j \equiv \sum_{k=-N/2+1}^{N/2} s_{j-k} r_k$$

Convolution Theorem:

$$\sum_{k=-N/2+1}^{N/2} s_{j-k} r_k \overset{FT}{\Leftrightarrow} R_n S_n$$



(note how the response function for negative times is wrapped around and stored at the extreme right end of the array)

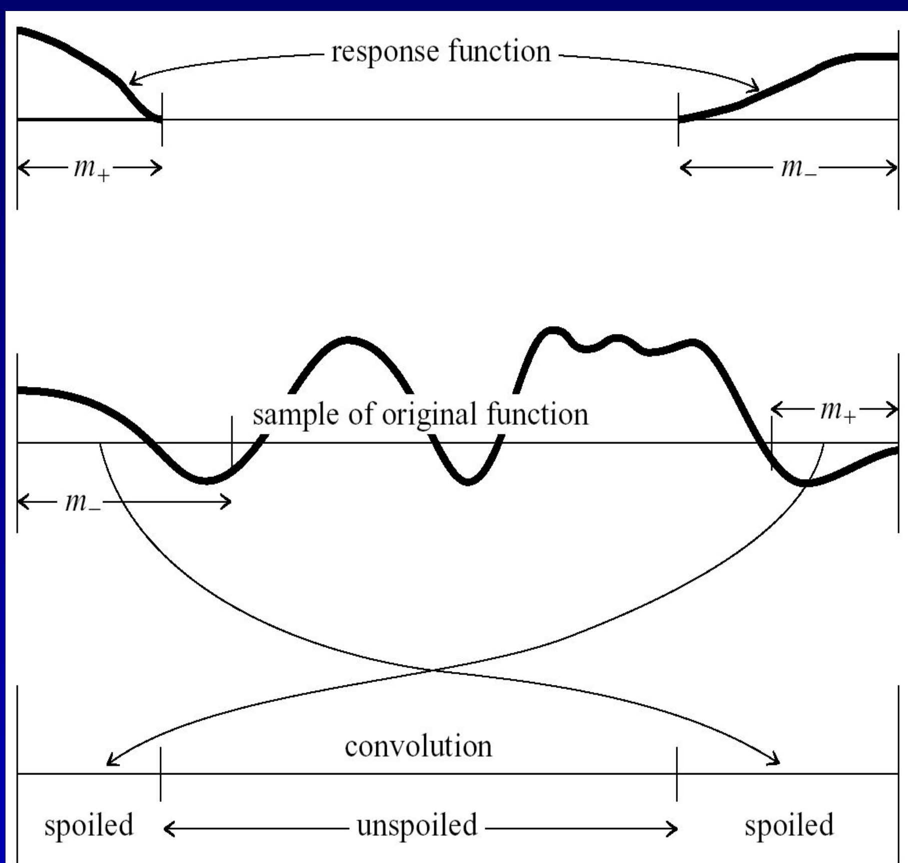
constraints:

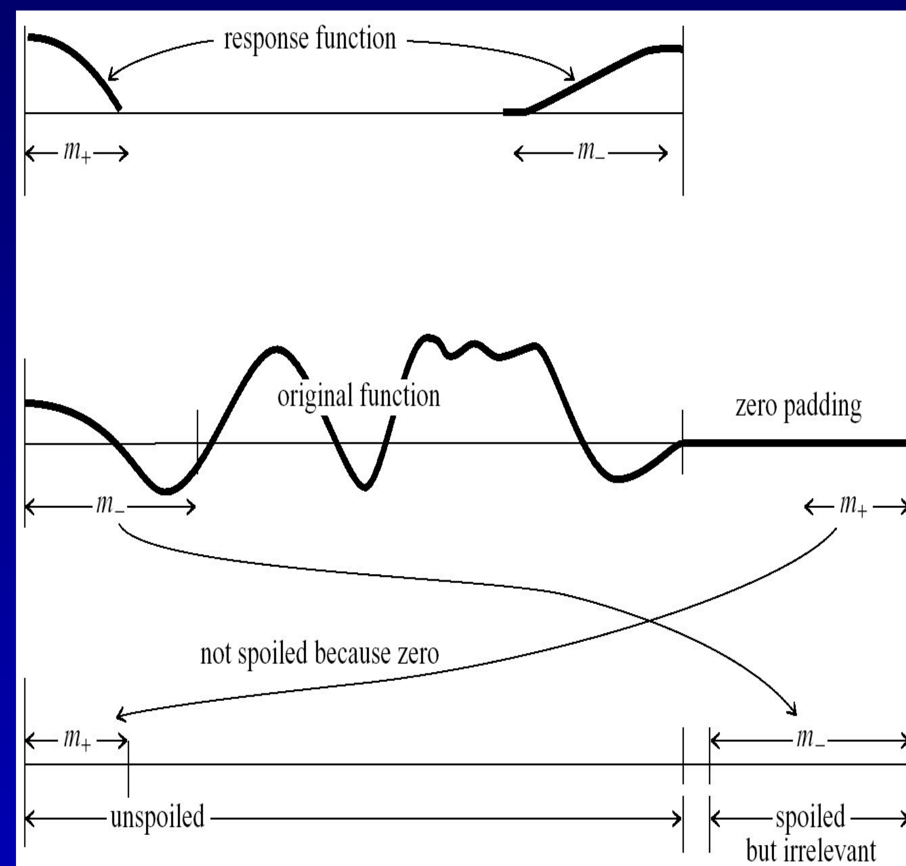
- duration of r and s are not the same
- signal is not periodic

Treatment of end effects by zero padding

constraint 1: simply expand response function to length N by padding it with zeros

constraint 2: extend data at one end with a number of zeros equal to the max. positive / negative duration of r (whichever is larger)





FFT for Convolution

1. zero-pad data
2. zero-pad response function
   (-> data and response function have N elements)
3. calculate FFT of data and response function
4. multiply FFT of data with FFT of response function
5. calculate inverse FFT for this product

Deconvolution

-> undo smearing caused by a response function

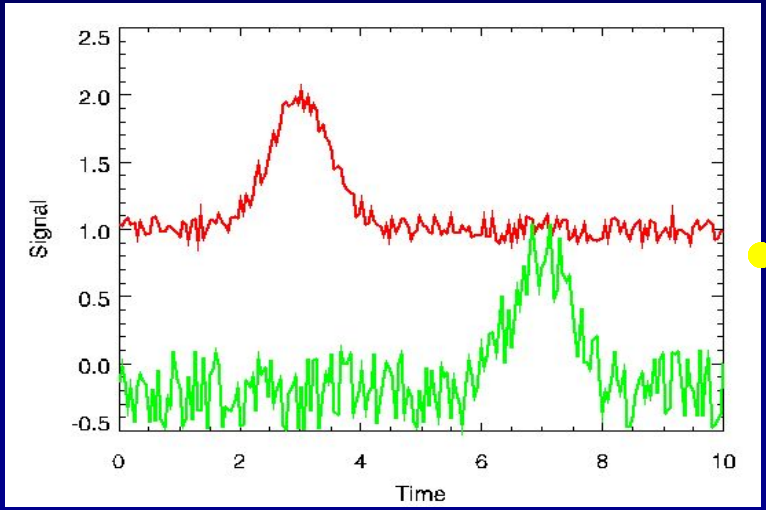
use steps (1-3), and then:

4. divide FFT of convolved data with FFT of response function
5. calculate inverse FFT for this product

Correlation / Autocorrelation with FFT

definition of correlation / autocorrelation see first lecture

$$Corr(g,h) = g*h = \int_{-\infty}^{\infty} g(t+\tau)h(\tau)\,d\tau$$



Correlation Theorem:

$$Corr(g,h) \overset{FT}{\Leftrightarrow} G(f)H^{*}(f)$$

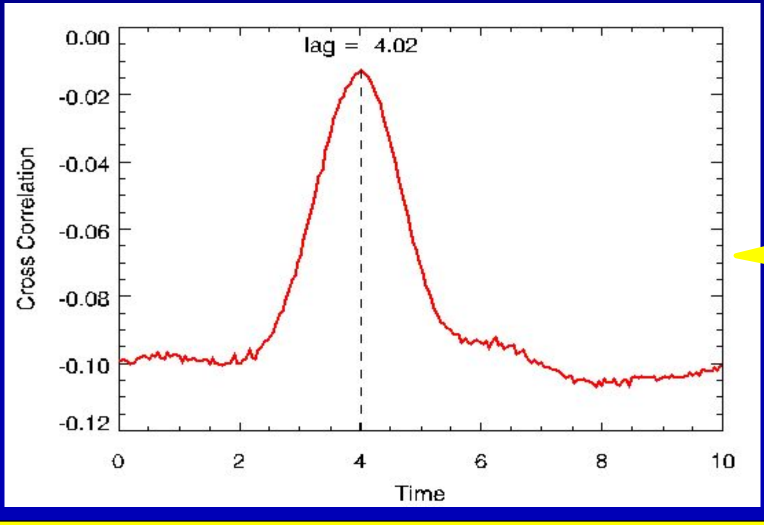
Auto-Correlation:

$$Corr(g,g) \overset{FT}{\Leftrightarrow} |G(f)|^{2}$$

discrete correlation theorem:

$$Corr(g,h)_j \equiv \sum_{k=0}^{N-1} g_{j+k}h_k$$

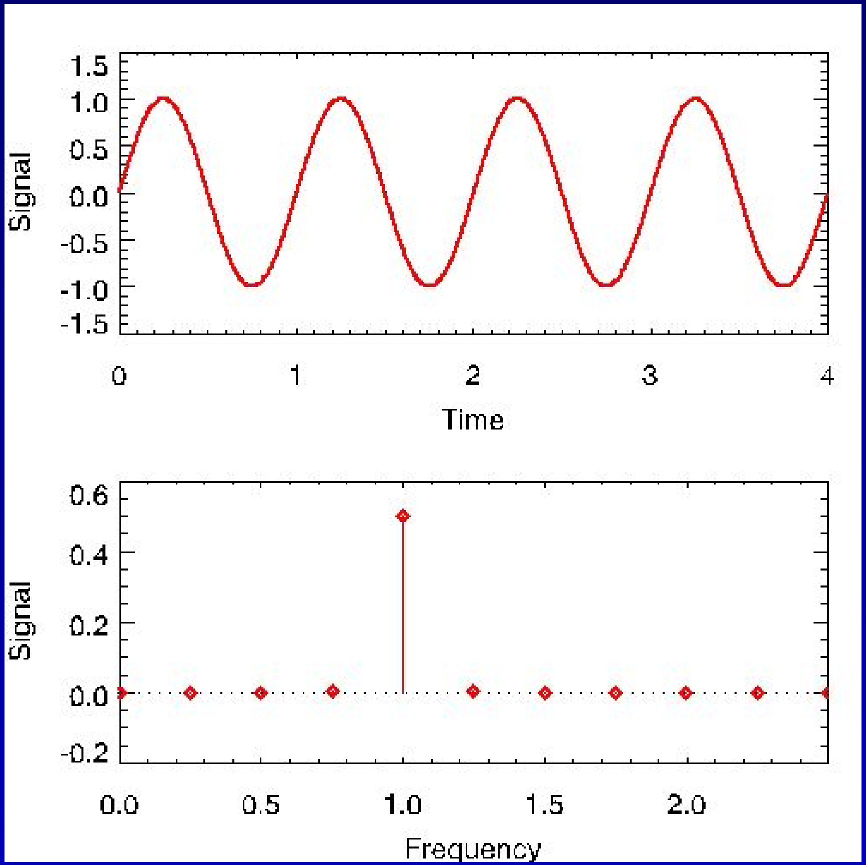
$$\overset{FT}{\Leftrightarrow} G_k H_k^{*}$$



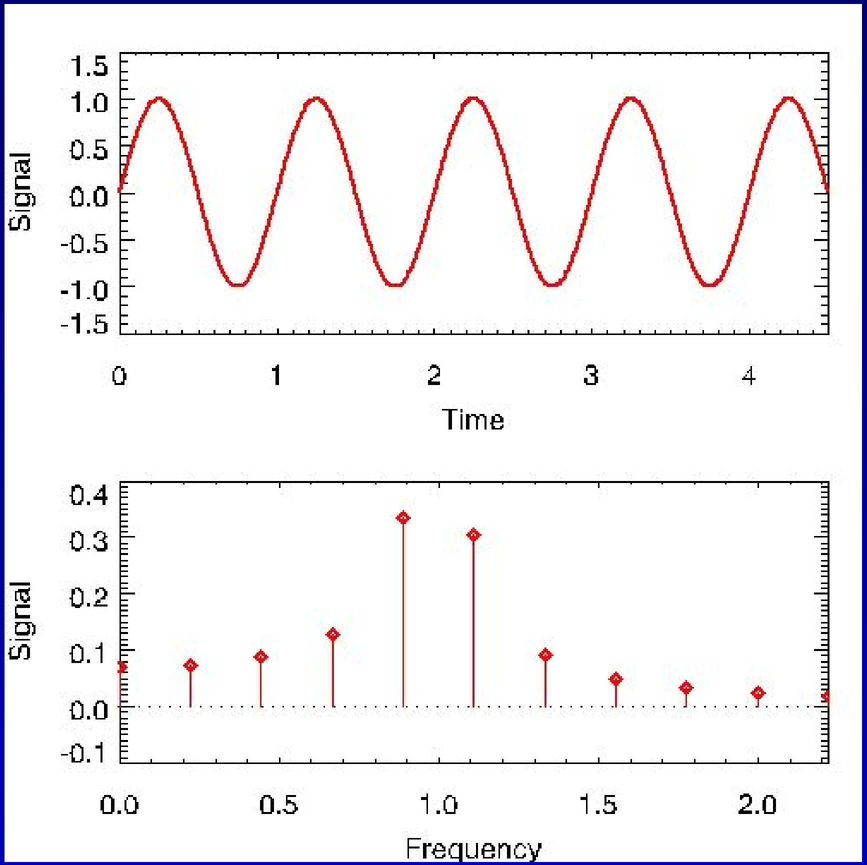
Fourier Transform - problems

spectral leakage

$T = nT_p$



$T \neq nT_p$



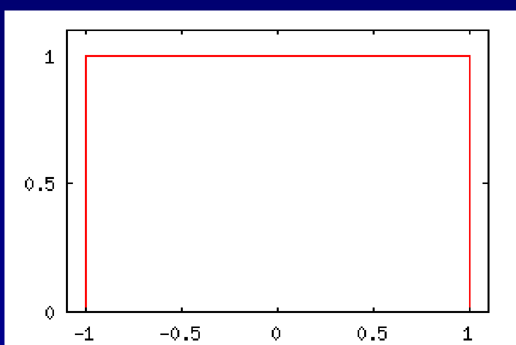
reducing leakage by windowing (1)

Applying windowing (apodizing) function to data record:

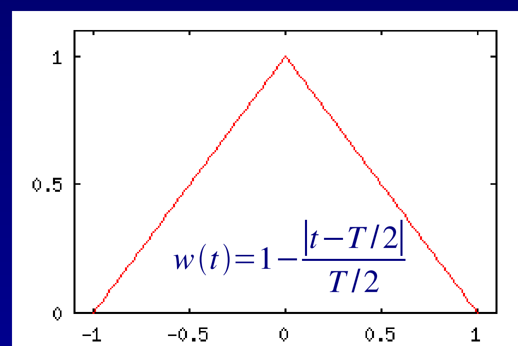
$$\bar{x}(t) = x(t)\,w(t) \qquad \text{(original data record x windowing function)}$$

$$\bar{x}_n = x_n w_n$$

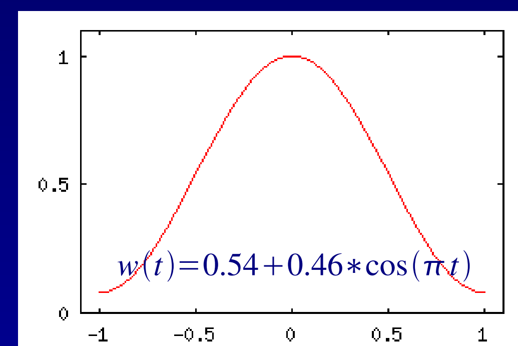
No Window:



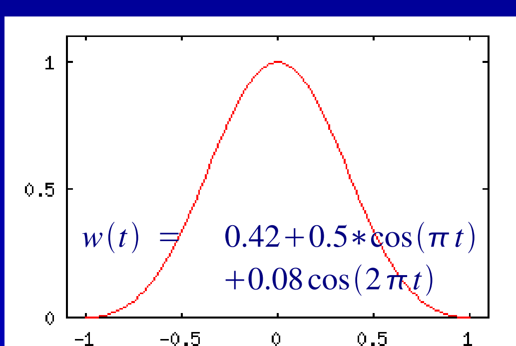
Bartlett Window:



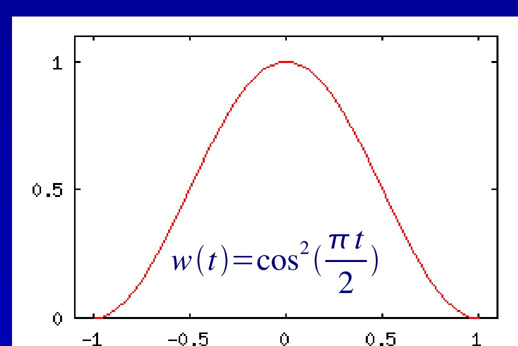
Hamming Window:



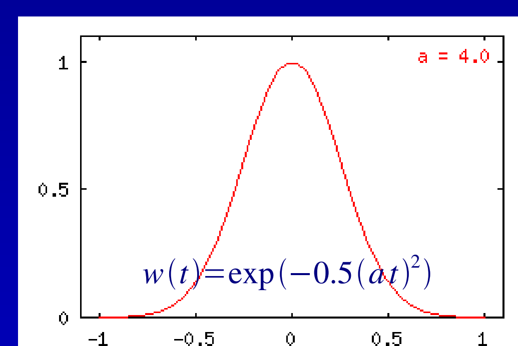
Blackman Window:



Hann Window:

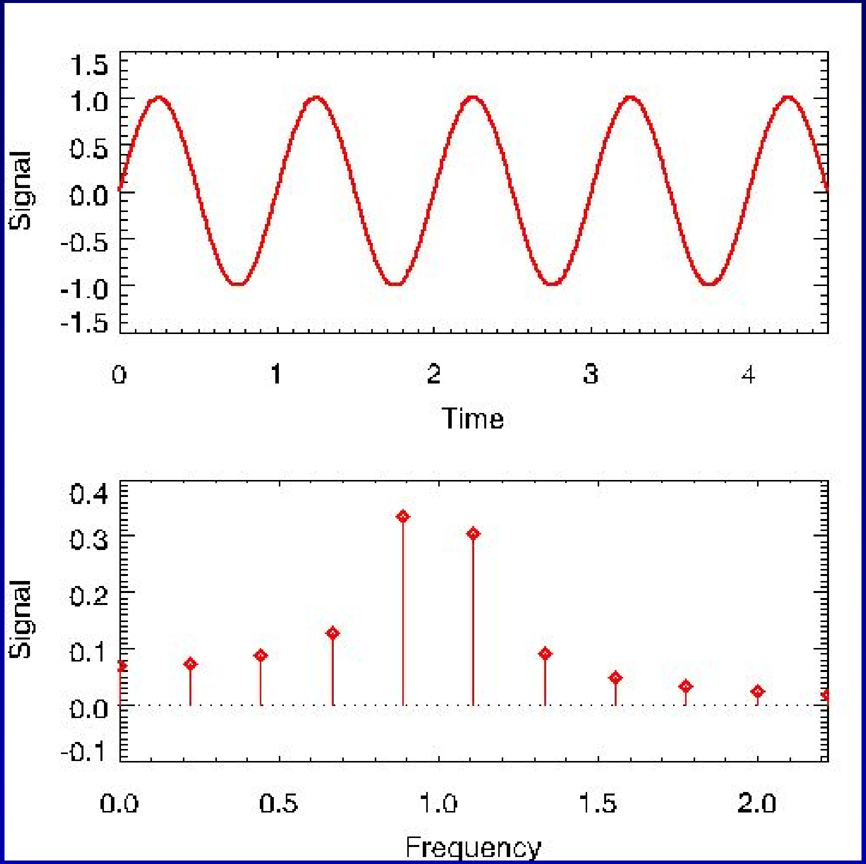


Gaussian Window:

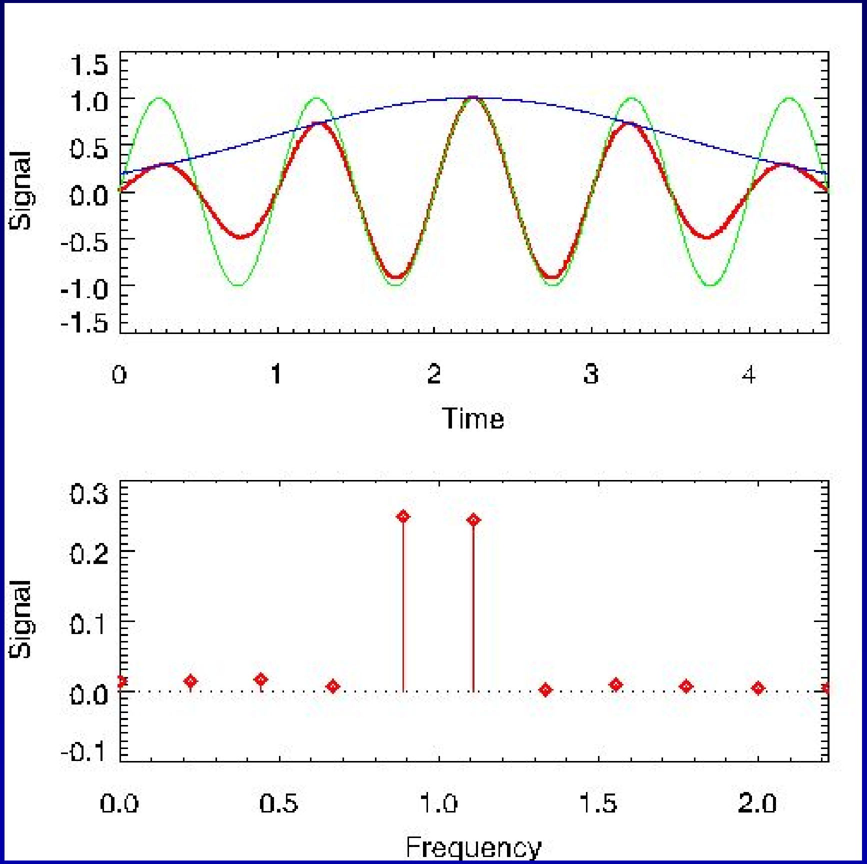


reducing leakage by windowing (2)

without windowing



with Gaussian windowing



No constant sampling frequency

Fourier transformation requires constant sampling (data points at equal distances)
-> not the case for most physical data

Solution: Interpolation

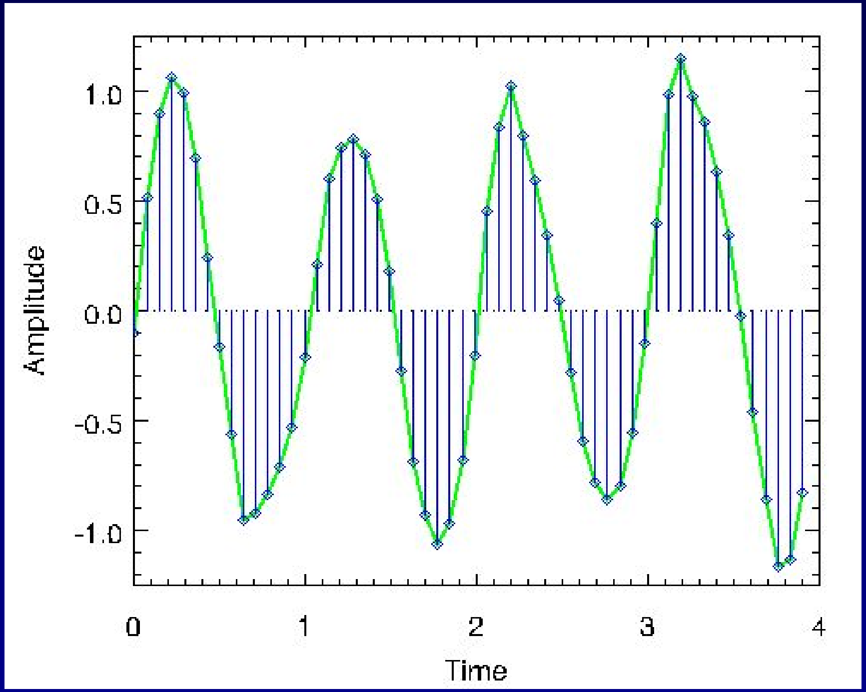
- linear:
  inear interpolation between y_k and y_{k+1}
  IDL> idata=interpol(data,t,t_reg)
- quadratic:
  quadratic interpolation using y_{k-1}, y_k and y_{k+1}
  IDL> idata=interpol(data,t,t_reg,/quadratic)
- least-square quadratic
  least-square quadratic fit using y_{k-1}, y_k, y_{k+1} and y_{k+2}
  IDL> idata=interpol(data,t,t_reg,/lsq)
- spline
  IDL> idata=interpol(data,t,t_reg,/spline)
  IDL> idata=spline(t,data,t_reg[,tension])

important:

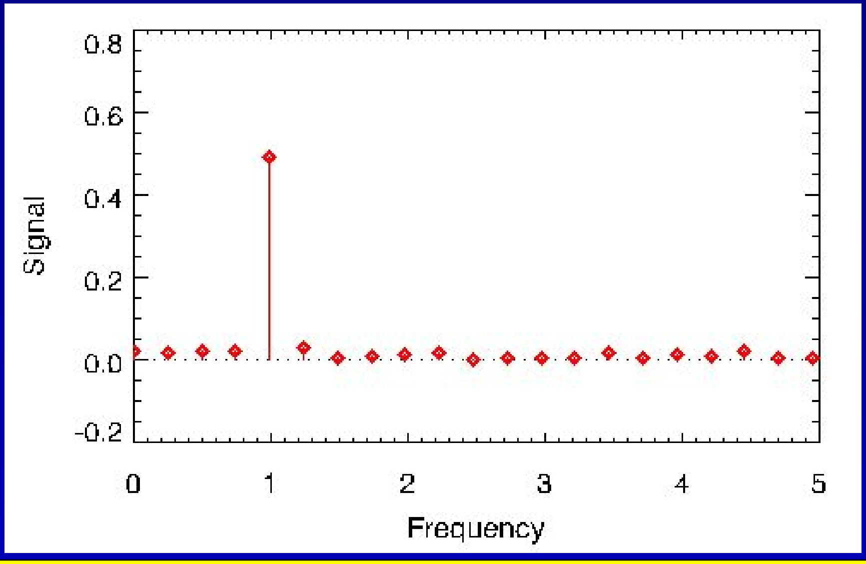
interpolation changes sampling rate!

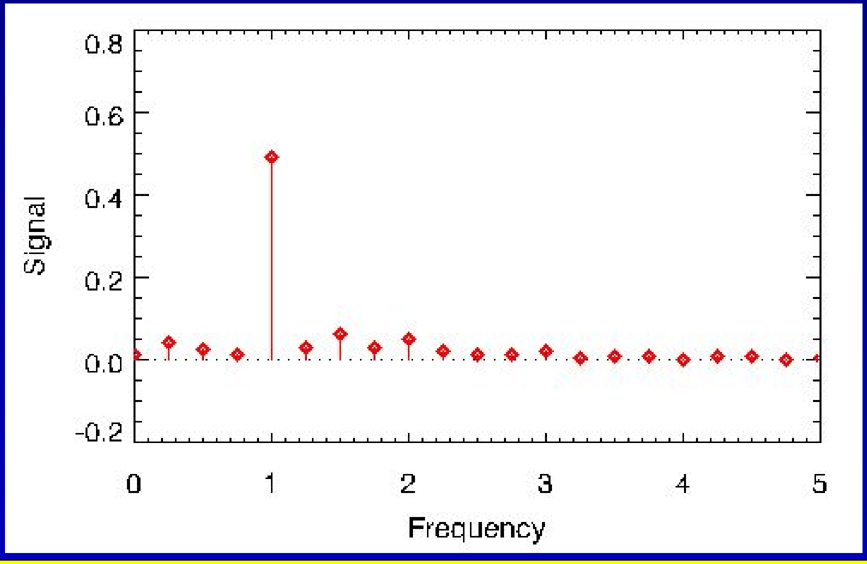
-> careful choice of new (regular) time grid necessary!

Fourier Transorm on irregular gridded data - Interpolation



- original data: sine wave + noise
- FT of original data
- irregular sampling of data (measurement)
- interpolation: linear, lsq, spline, quadratic
- 're-sampling'
- FT of interpolated data

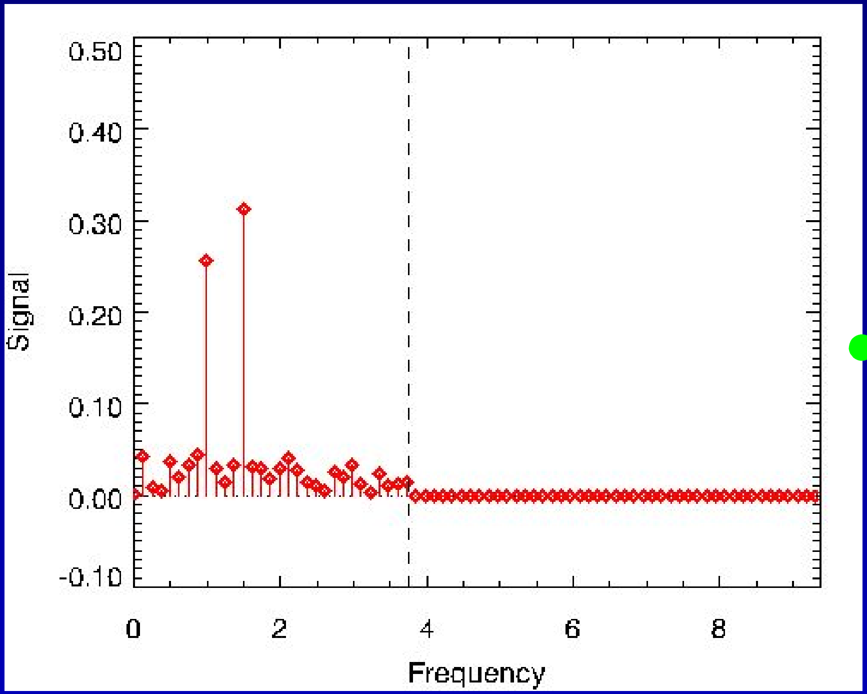


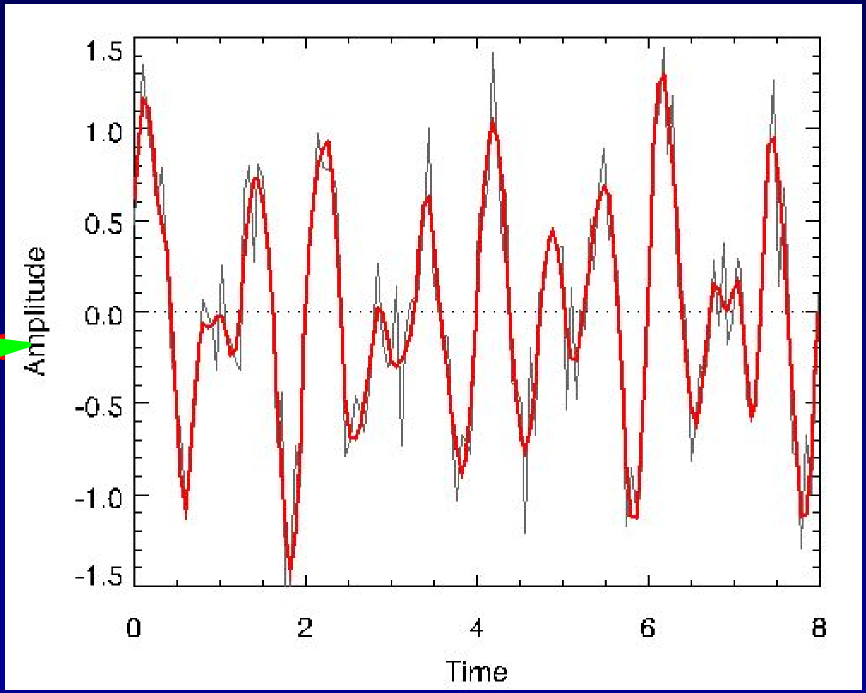


Noise removal

Frequency threshold (lowpass)

- make FT of data
- set high frequencies to 0
- transorm back to time domain

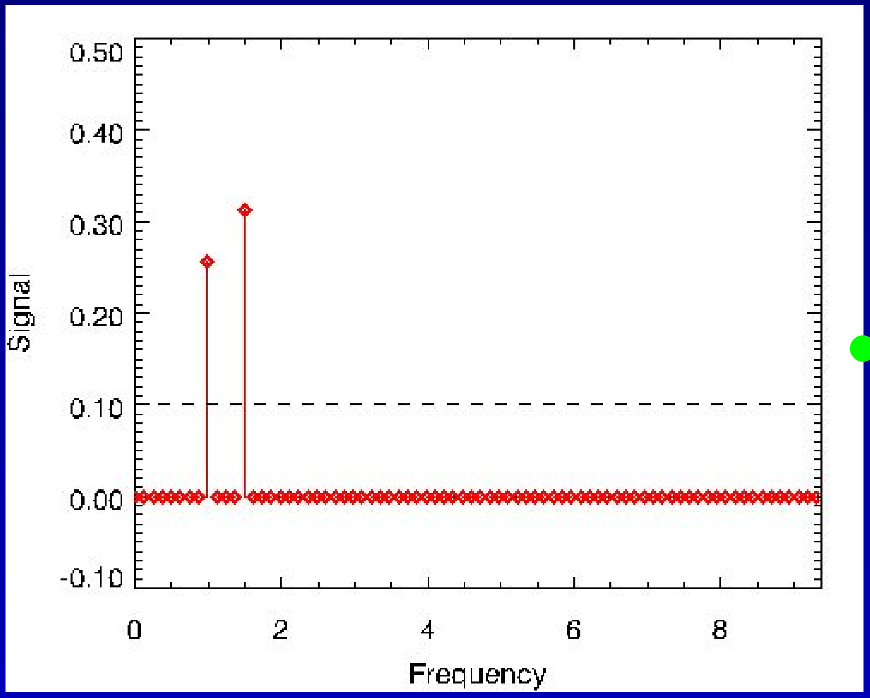


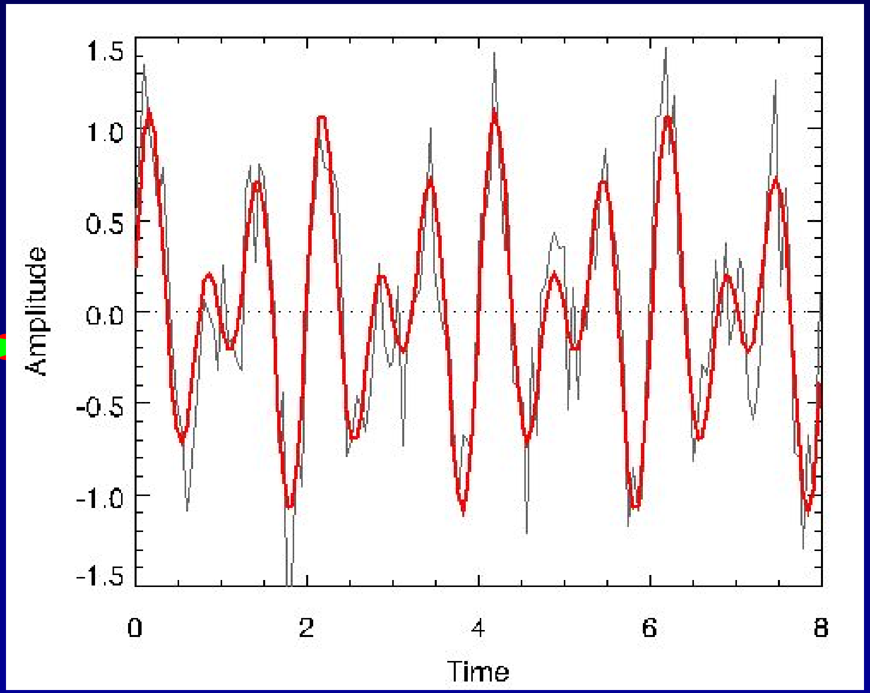


Noise removal

signal threshold for weak frequencies (dB-threshold)

- make FT of data
- set frequencies with amplitudes below a given threshold to 0
- transorm back to time domain





Optimal Filtering with FFT

normal situation with measured data:

underlying, uncorrupted signal u(t)
+ response function of measurement r(t)
= smeared signal s(t)

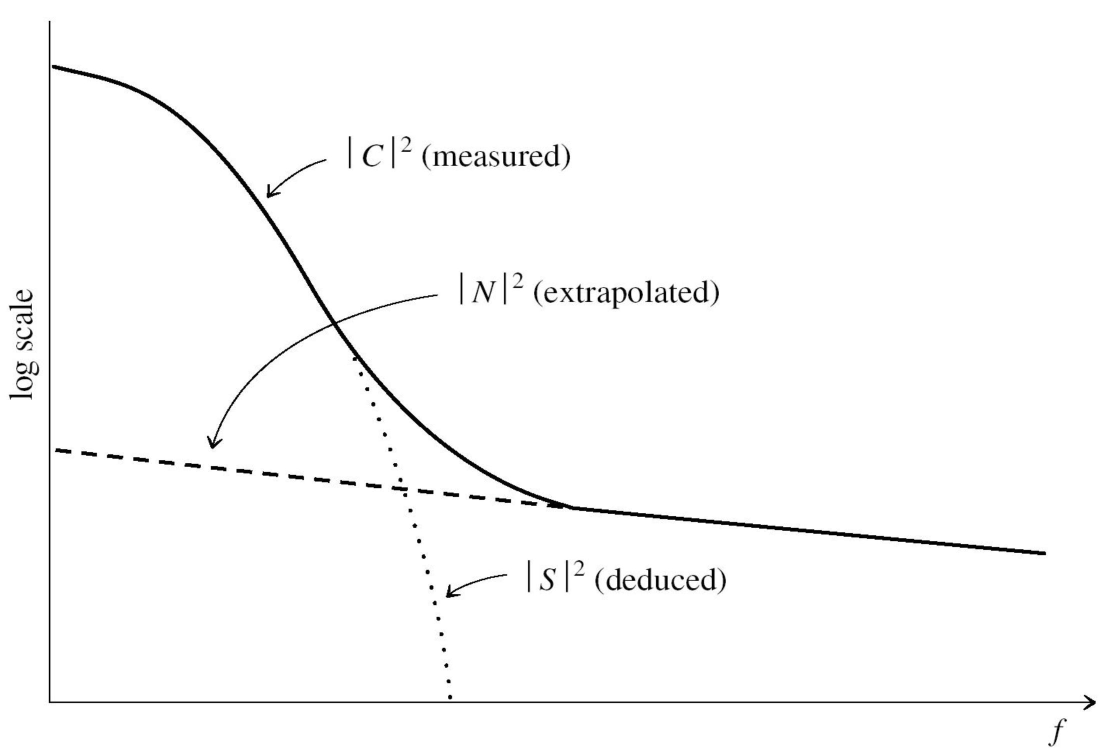
+ noise n(t)
= smeared, noisy signal c(t)

$$s(t) = \int_{-\infty}^{\infty} r(t-\tau)u(\tau)d\tau$$
$$c(t) = s(t)+n(t)$$

estimate true signal u(t) with:

$$\tilde{U}(f) = \frac{C(f)\Phi(f)}{R(f)}$$

$\Phi(f), \varphi(t)$ = optimal filter (Wiener filter)



Calculation of optimal filter

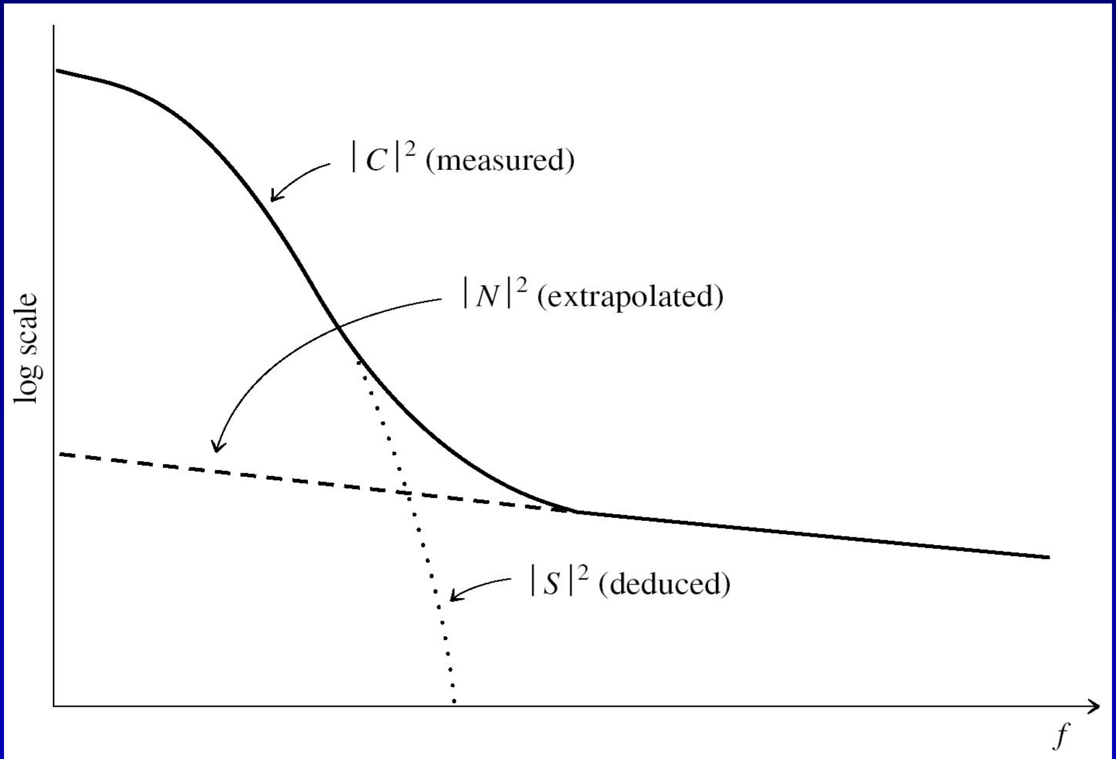
reconstructed signal and uncorrupted signal should be close in least-square sense:
-> minimize

$$\int_{-\infty}^{\infty} |\tilde{u}(t)-u(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{U}(f)-U(f)|^2 df$$

$$\Rightarrow \frac{\partial}{\partial \Phi(f)} \left| \frac{[S(f)+N(f)]\Phi(f)}{R(f)} - \frac{S(f)}{R(f)} \right|^2 = 0$$

$$\Rightarrow \Phi(f) = \frac{|S(f)|^2}{|S(f)|^2+|N(f)|^2}$$

additional information:
power spectral density can often be used to disentangle noise function N(f) from smeared signal S(f)



Using FFT for Power Spectrum Estimation

discrete Fourier transform of c(t)
-> Fourier coefficients:

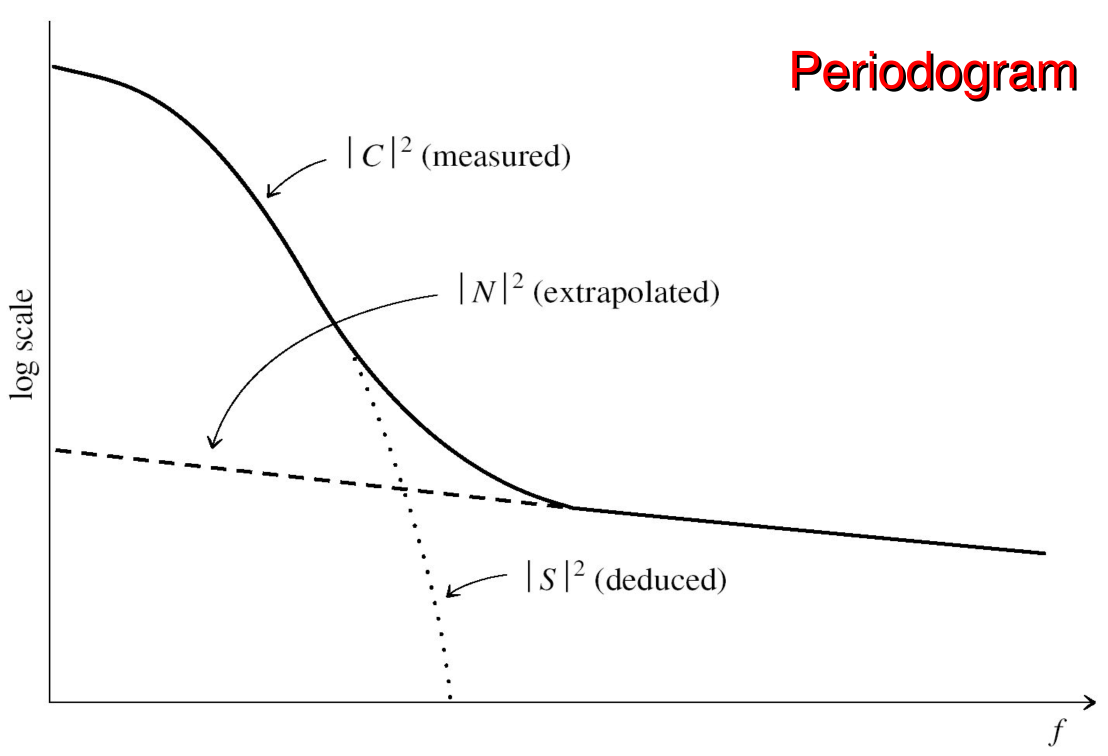
$$C_k = \sum_{j=0}^{N-1} c_j e^{2\pi i jk/N} \quad k = 0, \ldots, N-1$$

-> periodogram estimate of power spectrum:

$$P(0) = P(f_0) = \frac{1}{N^2} |C_0|^2$$

$$P(f_k) = \frac{1}{N^2} \left[|C_k|^2 + |C_{N-k}|^2 \right]$$

$$P(f_c) = P(f_{N/2}) = \frac{1}{N^2} |C_{N/2}|^2$$



end of FT

Spectral Analysis and Time Series

Andreas Lagg

Part I: fundamentals on time series

- classification
- prob. density func.
- auto-correlation
- power spectral density
- cross-correlation
- applications
- pre-processing
- sampling
- trend removal

Part II: Fourier series

- definition
- method
- properties
- convolution
- correlations
- leakage / windowing
- irregular grid
- noise removal

Part III: Wavelets

- why wavelet transforms?
- fundamentals: FT, STFT and resolution problems
- multiresolution analysis: CWT
- DWT

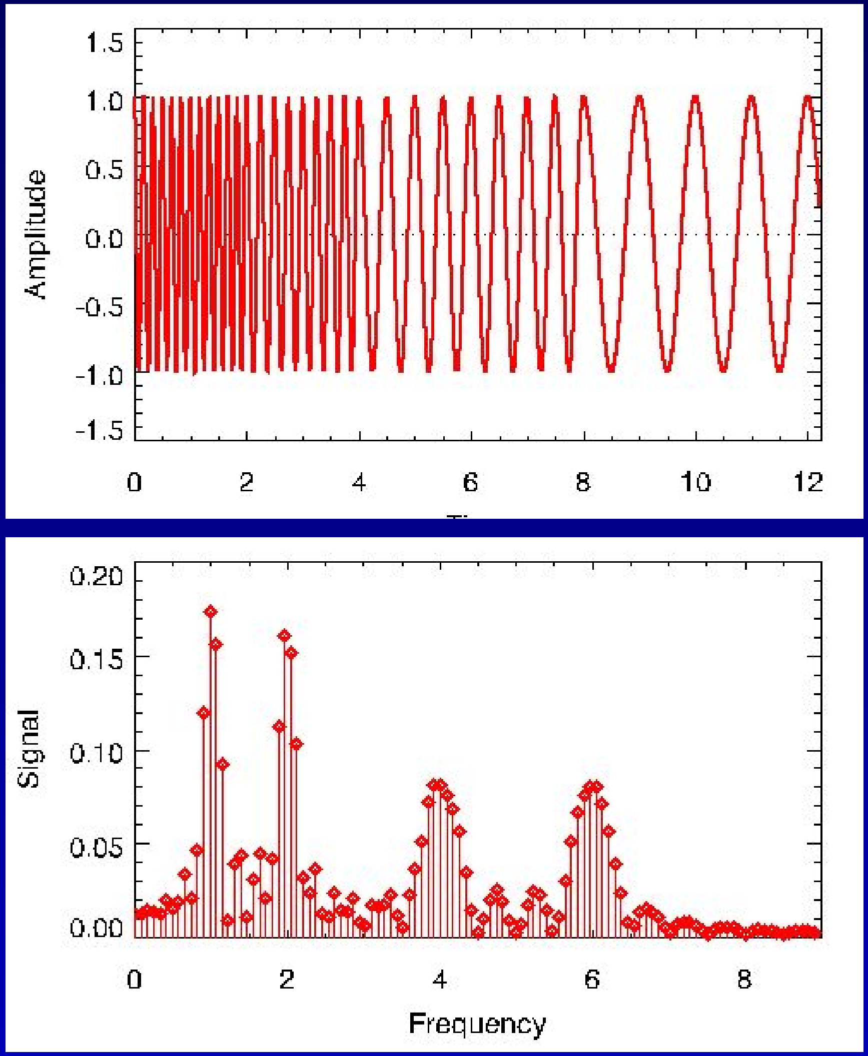
Exercises

Introduction to Wavelets

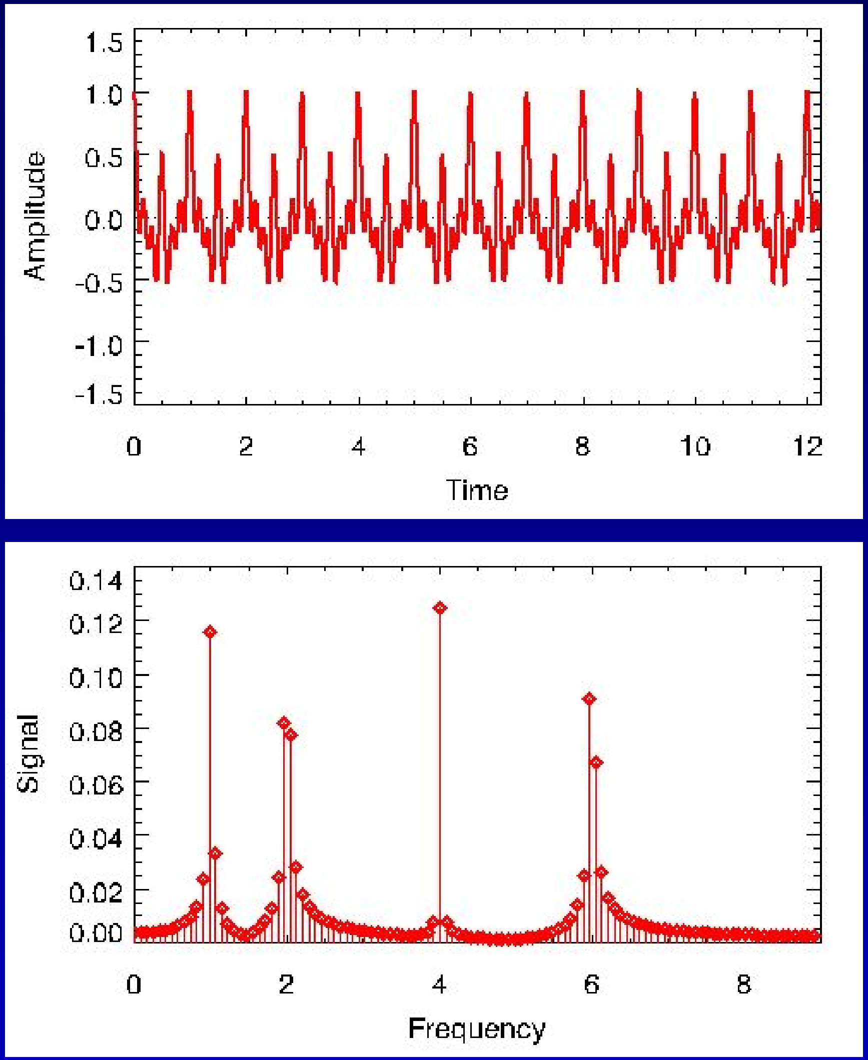
- why wavelet transforms?
- fundamentals: FT, short term FT and resolution problems
- multiresolution analysis:
  continous wavelet transform
- multiresolution analysis:
  discrete wavelet transform

Fourier: lost time information

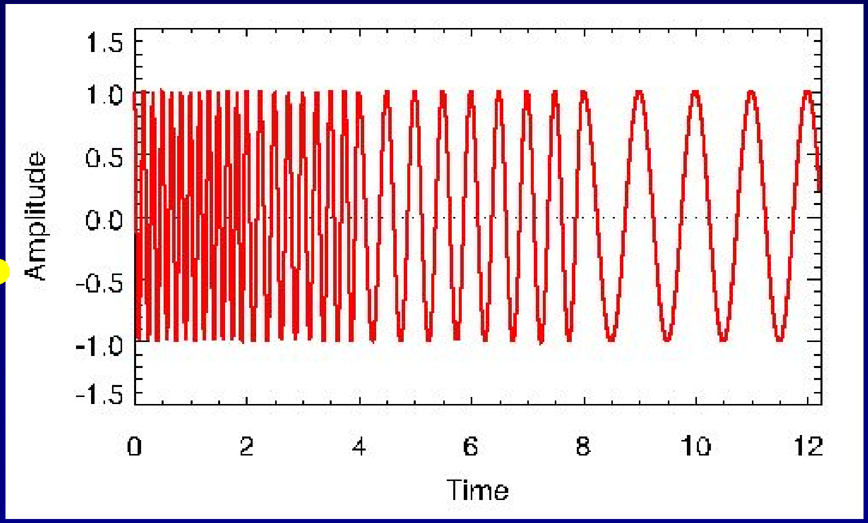
6 Hz, 4 Hz, 2 Hz, 1 Hz



6 Hz + 4 Hz + 2 Hz + 1 Hz



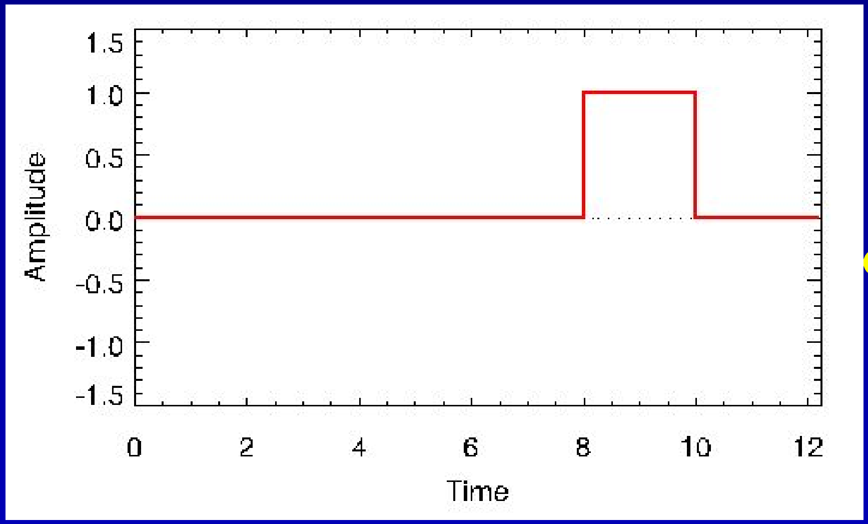
Solution: Short Time Fourier Transform
(STFT)

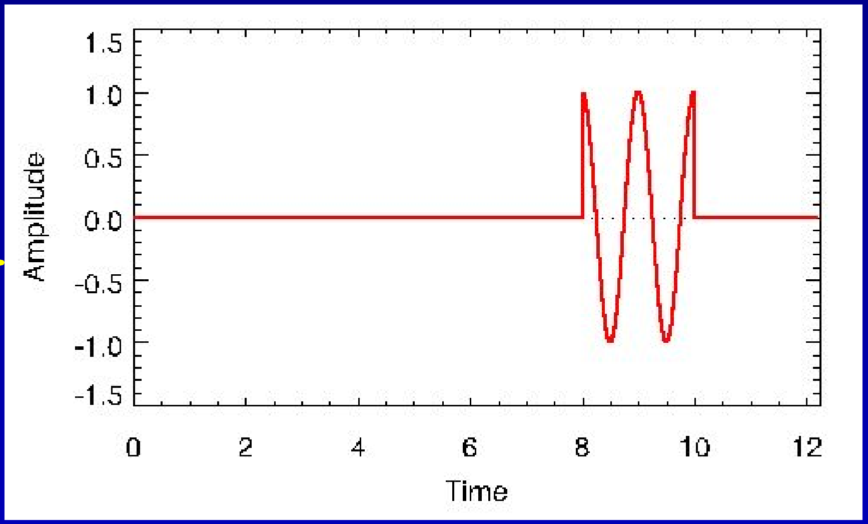


perform FT on 'windowed' function:

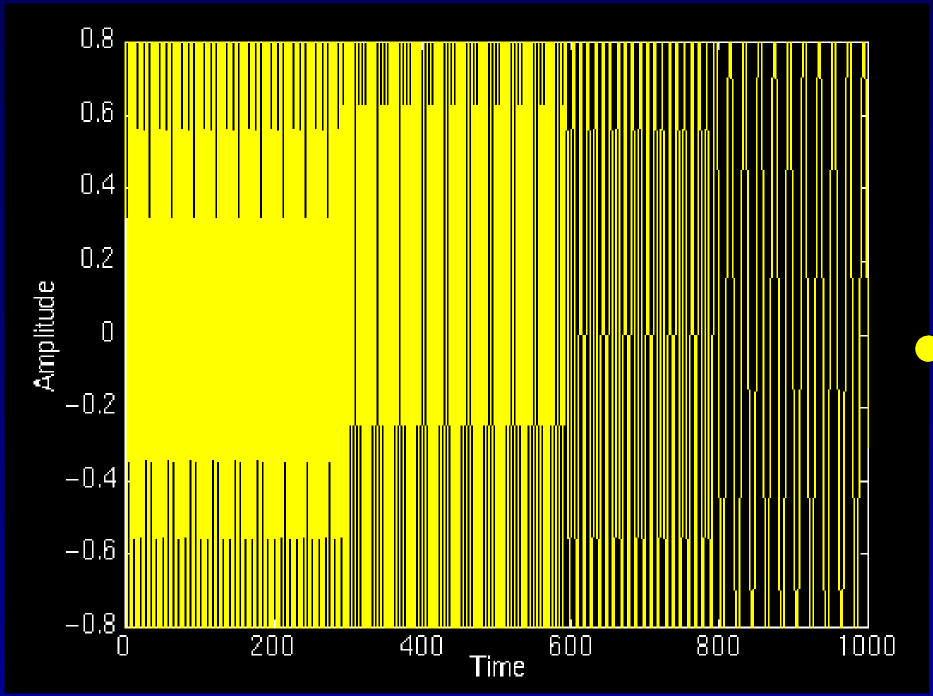
- example: rectangular window
- move window in small steps over data
- perform FT for every time step

$$STFT(f,t') = \int_t [x(t)\omega(t-t')] e^{-i2\pi f t} dt$$

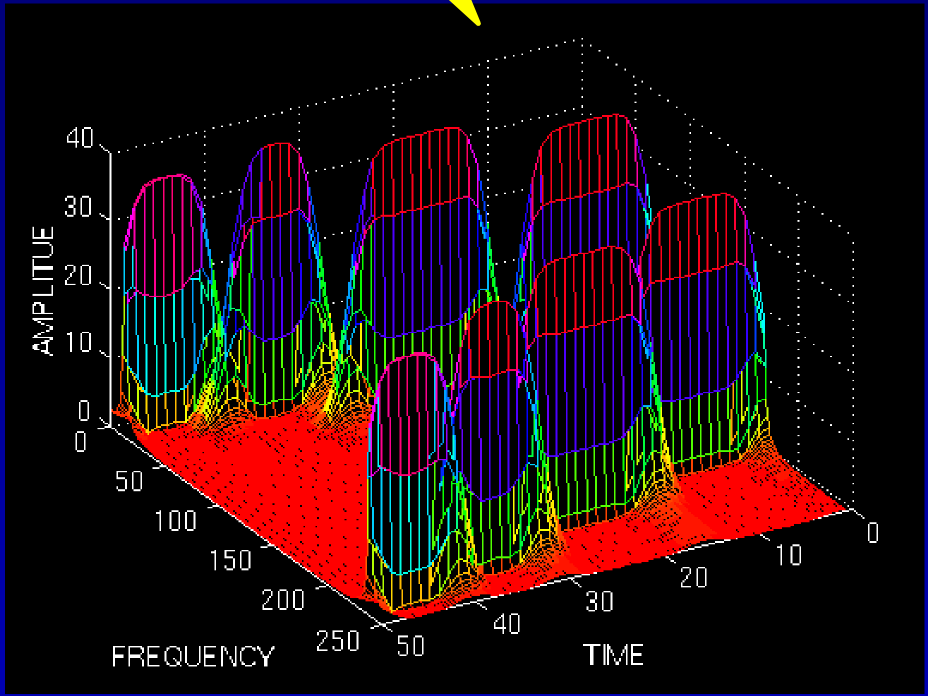




Short Time Fourier Transform



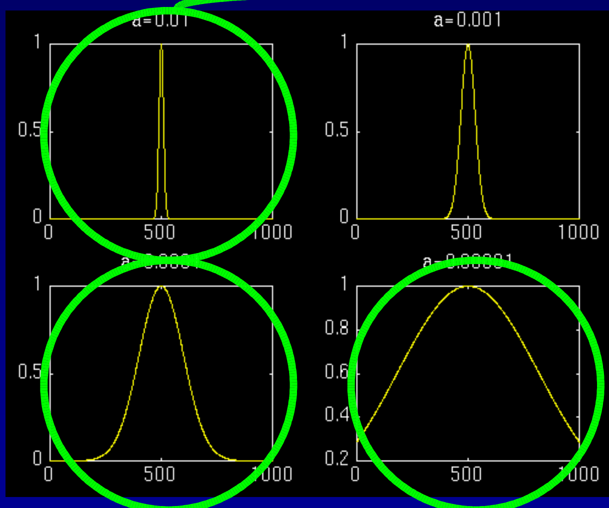
STFT



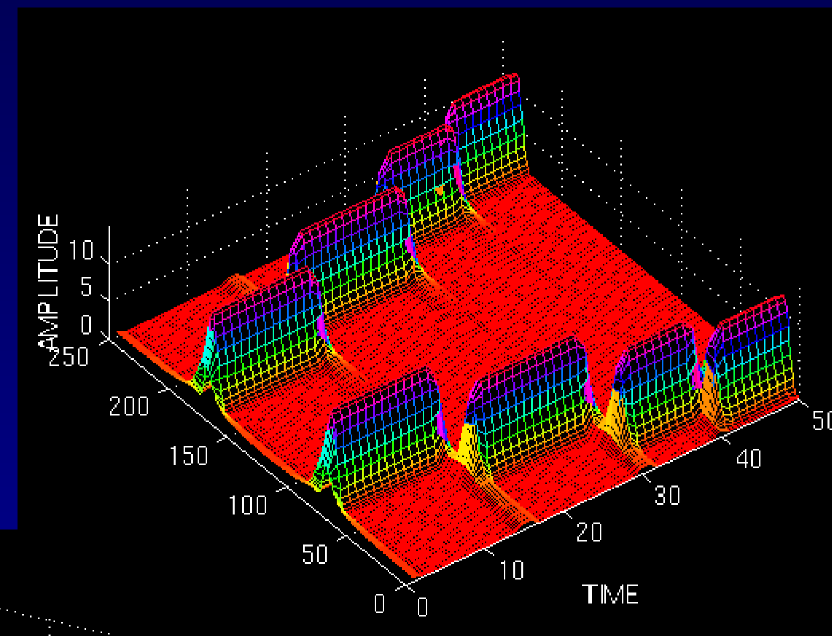
STFT-spectrogram shows both time and frequency information!

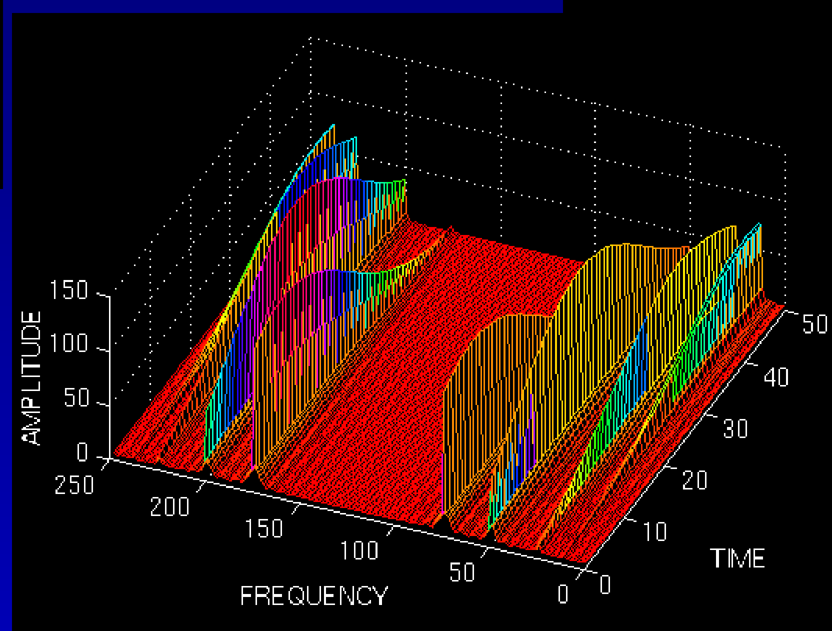
Short Time Fourier Transform: Problem

narrow window function -> good time resolution
wide window function -> good frequency resolution



Gauss-functions as windows





Solution: Wavelet Transformation

time vs. frequency resolution is intrinsic problem (Heisenberg Uncertainty Principle)
approach: analyze the signal at different frequencies with different resolutions

-> multiresolution analysis (MRA)

Continuous Wavelet Transform

similar to STFT:

- signal is multiplied with a function (the *wavelet*)
- transform is calculated separately for different segments of the time domain

but:

- the FT of the windowed signals are not taken
  (no negative frequencies)
- The width of the window is changed as the transform is computed for every single spectral component

Continuous Wavelet Transform

$$CWT_x^{\psi} = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int_t x(t)\, \psi^*\left(\frac{t-\tau}{s}\right) dt$$

τ ... translation parameter, s ... scale parameter

$\psi(t)$ ... mother wavelet $($= small wave$)$

mother wavelet:

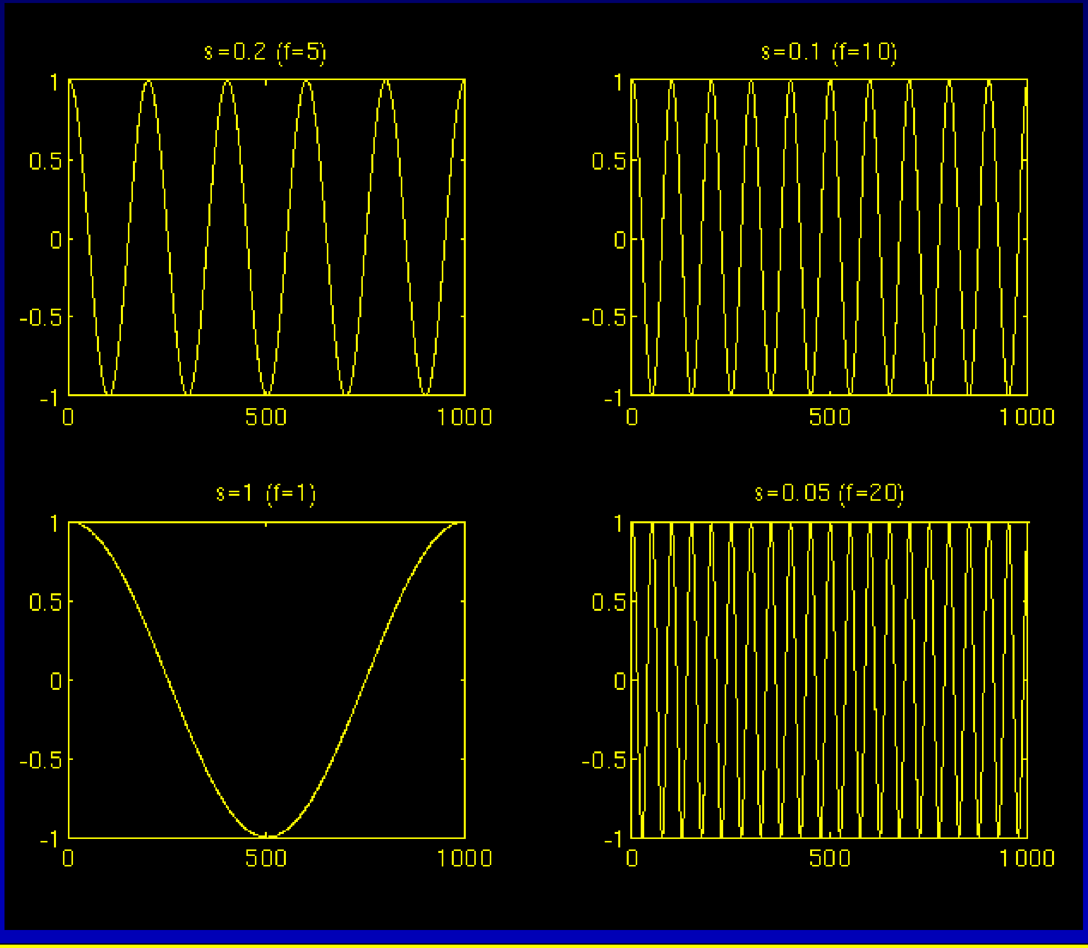
- finite length (compactly supported) ->'let'
- oscillatory ->'wave'
- functions for different regions are derived from this function -> 'mother'

scale parameter s replaces frequency in STFT

The Scale

similar to scales used in maps:

- high scale = non detailed global view (of the signal)
- low scale = detailed view



in practical applications:

- low scales (= high frequencies) appear usually as short bursts or spikes
- high scales (= low frequencies) last for entire signal

scaling dilates (stretches out) or compresses a signal:

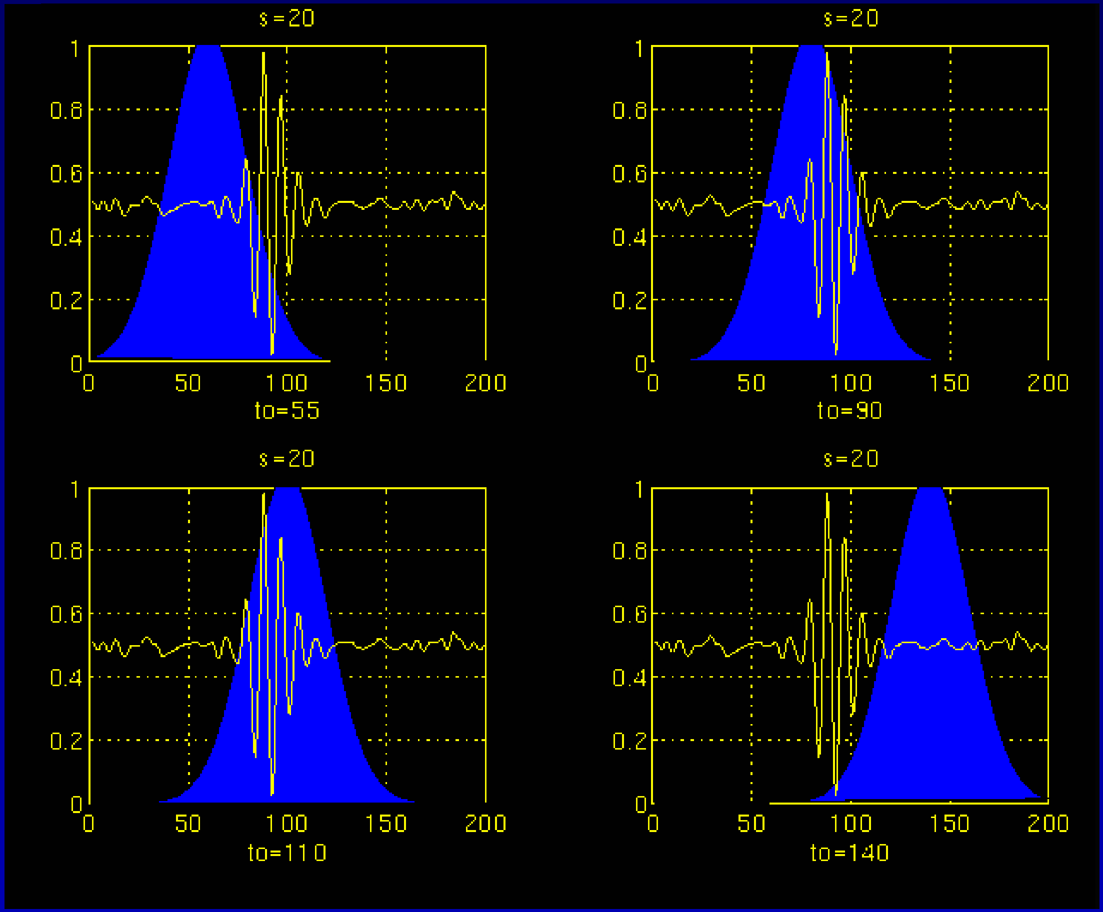
s > 1 -> dilation

s < 1 -> compression

Computation of the CWT

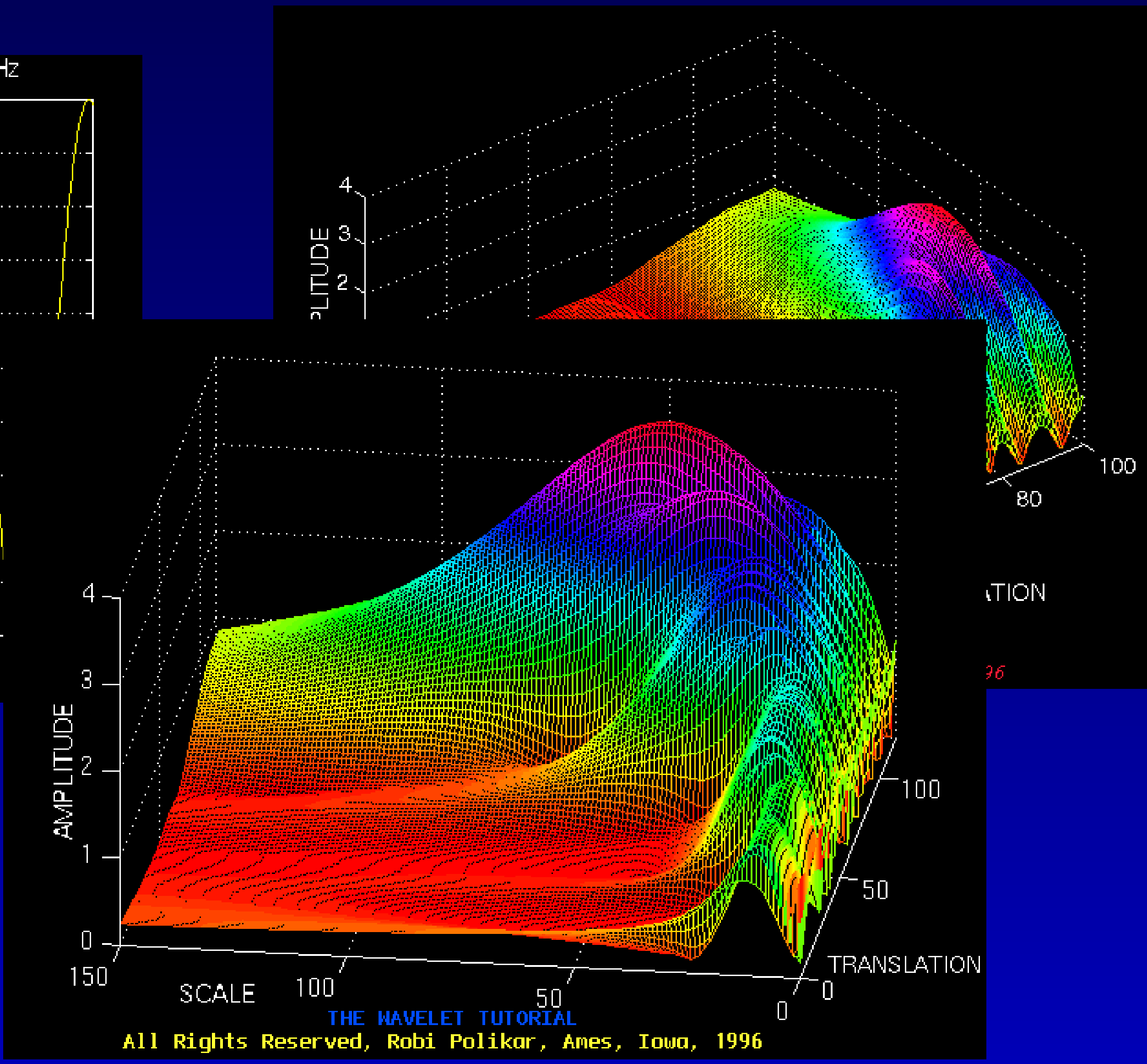
signal to be analyzed: x(t), mother wavelet: Morlet or Mexican Hat

- start with scale s=1 (lowest scale, highest frequency) -> most compressed wavelet
- shift wavelet in time from t_0 to t_1
- increase s by small value
- shift dilated wavelet from t_0 to t_1
- repeat steps for all scales



CWT - Example

signal x(t)



axes of CWT: translation and scale (not time and frequency)

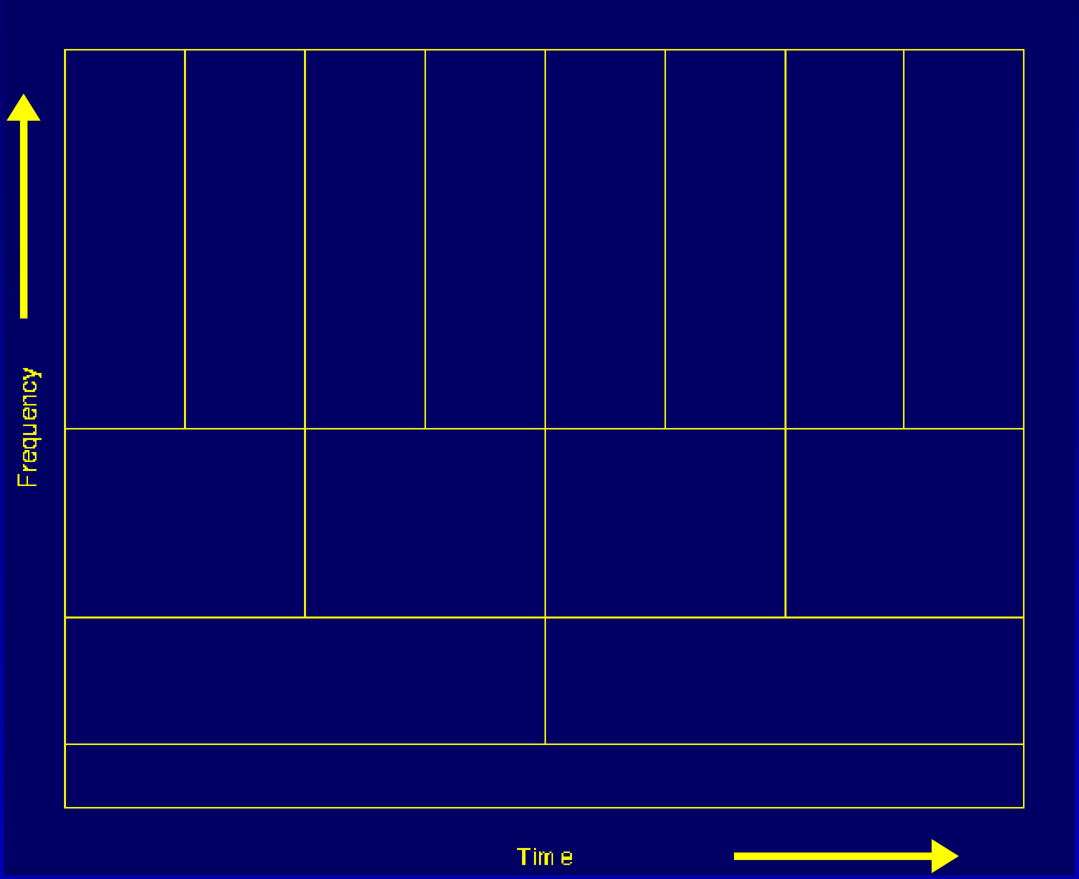
translation -> time
scale -> 1/frequency

Time and Frequency Resolution

every box corresponds to a value of the wavelet transform in the time frequency plane

- all boxes have constant area
  $\Delta f \, \Delta t = \text{const.}$
- low frequencies: high resolution in f, low time resolution
- high frequencies: good time resolution

STFT: time and frequency resolution is constant (all boxes are the same)



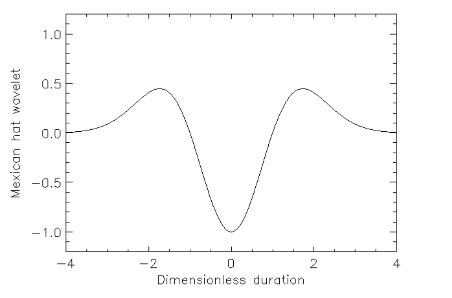
Wavelets: Mathematical Approach

WL-transform:

$$CWT_x^{\psi} = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int_t x(t) \psi^* \left(\frac{t-\tau}{s} \right) dt$$

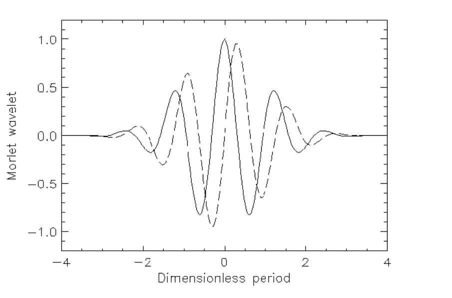
Mexican Hat wavelet:

$$\psi(t) = \frac{1}{\sqrt{s \pi \sigma^3}} e^{\frac{-t^2}{2\sigma^2}} \left(\frac{t^2}{\sigma^2} - 1 \right)$$



Morlet wavelet:

$$\psi(t) = e^{iat} e^{-\frac{t^2}{2\sigma}}$$



inverse WL-transform:

$$x(t) = \frac{1}{c_\psi^2} \int_s \int_\tau \Psi_x^{\psi}(\tau, s) \frac{1}{s^2} \psi \left(\frac{t-\tau}{s} \right) d\tau \, ds$$

admissibility condition:

$$c_\psi = \left\{ 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi \right\}^{1/2} < \infty \quad \text{with} \quad \hat{\psi}(\xi) \overset{FT}{\Leftrightarrow} \psi(t)$$

Discretization of CWT: Wavelet Series

-> sampling the time – frequency (or scale) plane

advantage:

- sampling high for high frequencies (low scales)
  scale s_1 and rate N_1
- sampling rate can be decreased for low frequencies (high scales)
  scale s_2 and rate N_2

$$N_2 = \frac{s_1}{s_2} N_1$$

$$N_2 = \frac{f_2}{f_1} N_1$$

continuous wavelet → discrete wavelet

$$\psi_{\tau,s} = \frac{1}{\sqrt{s}} \psi\left(\frac{t-\tau}{s}\right) \rightarrow \psi_{j,k}(t) = s_0^{-j/2} \psi(s_0^{-j} t - k\tau_0), \quad \psi_{j,k} \text{ orthonormal}$$

$$\Psi_x^{\psi\, j,k} = \int_t x(t) \psi_{j,k}^*(t)\, dt \qquad \text{WL-transformation}$$

$$x(t) = c_\psi \sum_j \sum_k \Psi_x^{\psi\, j,k} \psi_{j,k}(t) \qquad \text{reconstruction of signal}$$

Discrete Wavelet Transform

(DWT)

discretized continuous wavelet transform is only a sampled version of the CWT

The discrete wavelet transform (DWT) has significant advantages for implementation in computers.

excellent tutorial:
http://users.rowan.edu/~polikar/WAVELETS/WTtutorial.html

IDL-Wavelet Tools:
IDL> wv_applet

Wavelet expert at MPS:
Rajat Thomas

end of Wavelets

Exercises

Part I: Fourier Analysis
(Andreas Lagg)

Instructions:

http://www.linmpi.mpg.de/~lagg

Part II: Wavelets
(Rajat Thomas)

Seminar room
Time: 15:00

Exercise: Galileo magnetic field

data set from Galileo magnetometer (synthesized)

file: gll_data.sav, contains:

- total magnetic field
- radial distance
- time in seconds

Tips:

- restore,'gll_data.sav'
- use IDL-FFT
- remember basic plasma physics formula for the ion cyclotron wave:

$$\omega_{gyro} = \frac{q\,B}{m}, \quad f_{gyro} = \frac{\omega_{gyro}}{2\pi}$$

your tasks:

- Which ions are present?
- Is the time resolution of the magnetometer sufficient to detect electrons or protons?

Background:

If the density of ions is high enough they will excite ion cyclotron waves during gyration around the magnetic field lines. This gyration frequency only depends on mass per charge and on the magnitude of the magnetic field.

In a low-beta plasma the magnetic field dominates over plasma effects. The magnetic field shows only very little influence from the plasma and can be considered as a magnetic dipole.

http://www.sciencemag.org/cgi/content/full/274/5286/396

Literature

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Bendat and Piersol, Wiley Interscience, 1971

The Analysis of Time Series: An Introduction
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