Plasma waves in the fluid picture II

- Parallel electromagnetic waves
- Perpendicular electromagnetic waves
- Whistler mode wavs
- Cut-off frequencies
- Resonance (gyro) frequencies
- Ordinary and extra-ordinary waves
- Ion-cyclotron waves, Alfvén waves
- Lower-hybrid and upper-hybrid resonance

Parallel electromagnetic waves I

We use the wave electromagnetic field components:

$$\sqrt{2}\delta E_{R,L} = (\delta E_x \mp i\delta E_y)$$

They describe ***right-hand*** (R) and ***left-hand*** (L) ***polarized*** waves, as can be seen when considering the ratio

$$(\delta E_y/\delta E_x)_{R,L} = \pm i$$

This shows that the electric vector of the R-wave rotates in the positive while that of the L-wave in the negative *y* direction. The component transformation from $\delta E_{x,y}$ to $\delta E_{R,L}$ does not change the perpendicular electric field vector. Using the unitary matrix, **U**, makes the dielectric tensor diagonal:

$$\mathbf{U} = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ -i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{U} \cdot \epsilon \cdot \mathbf{U}^\dagger = \begin{bmatrix} \epsilon_R & 0 & 0 \\ 0 & \epsilon_L & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}$$

Parallel electromagnetic waves II

The components read:

$$\epsilon_{R,L} = 1 - \frac{\omega_{pe}^2}{\omega(\omega \mp \omega_{ge})}$$

The dispersion relation for the transverse R and L wave reads:

$$N^2 = \frac{k^2c^2}{\omega^2} = \epsilon_{R,L}$$

The right-hand circularly polarised wave has the ***refractive index***:

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega(\omega - \omega_{ge})}$$

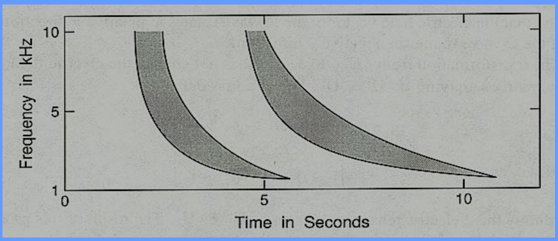
This refractive index diverges for ω -> 0 as well as for ω -> ω_{ge}, where k diverges.

Here $\omega_{R,res} = \omega_{ge}$ is the electron-cyclotron resonance frequency for the right-hand-polarised (RHP) parallel electromagnetic wave.

Parallel electromagnetic waves III

Resonances indicate a complex interaction of waves with plasma particles. Here $k \to \infty$ means that the wavelength becomes at constant frequency very short, and the wave momentum large. This leads to violent effects on a particle‘s orbit, while resolving the microscopic scales. During this resonant interaction the waves may give or take energy from the particles leading to ***resonant absorption or amplification*** (growth) of wave energy.

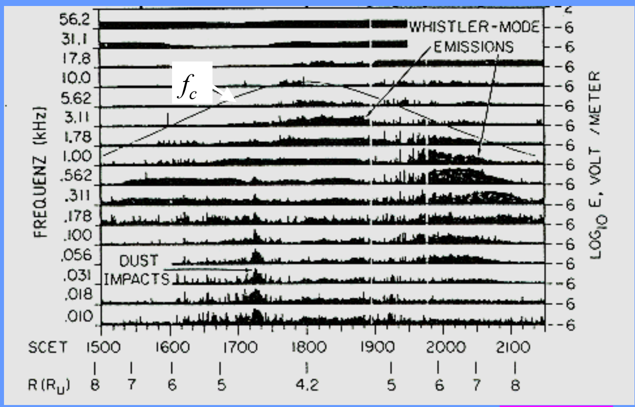
$$\omega/kc \sim \omega^{1/2}$$



At low frequencies, $\omega \ll \omega_{ge}$, the above dispersion simplifies to the electron ***Whistler mode,*** yielding the typical falling tone in a sonogram as shown above.

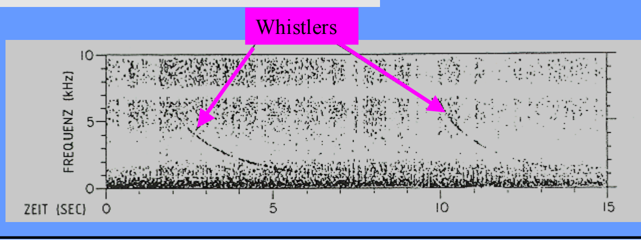
$$\omega = \frac{\omega_{ge}}{1 + \omega_{pe}^2/k^2c^2}$$

Whistlers in the magnetosphere of Uranus and Jupiter

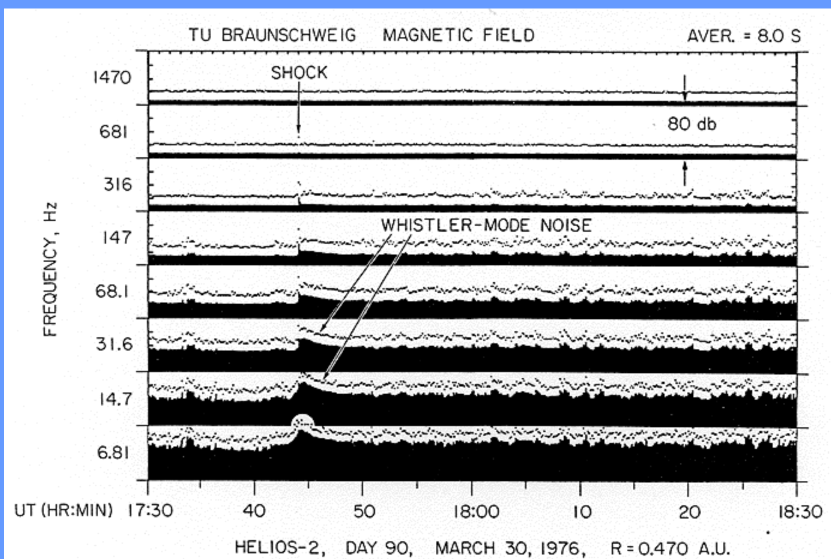


Wideband electric field spectra obtained by **Voyager** at Uranus on January 24, 1986.

Wave measurements made by Voyager I near the **moon Io** at a distance of 5.8 R_J from Jupiter.



Whistler mode waves at an interplanetary shock



Gurnett et al., JGR **84**, 541, 1979

$$\omega_w = \omega_{ge}(kc/\omega_{pe})^2$$

Cut-off frequencies

Setting the refractive index N for R-waves equal to zero, which means $k = 0$ at a finite ω, leads to a second-order equation with the roots:

$$\omega_{R,co} = \frac{1}{2}\left[\omega_{ge} + (\omega_{ge}^2 + 4\omega_{pe}^2)^{1/2}\right]$$

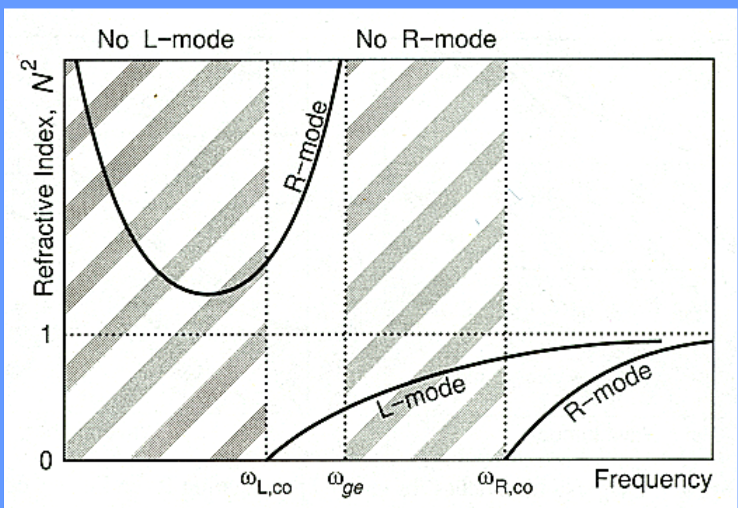
The left-hand circularly polarised wave has a refractive index given by:

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega(\omega + \omega_{ge})}$$

This refractive index does not diverge for $\omega \rightarrow \omega_{ge}$ and shows no cyclotron resonance. Moreover, since $N^2 < 1$ one has $\omega/k > c$.
The LHP waves have a low-frequency cut-off at

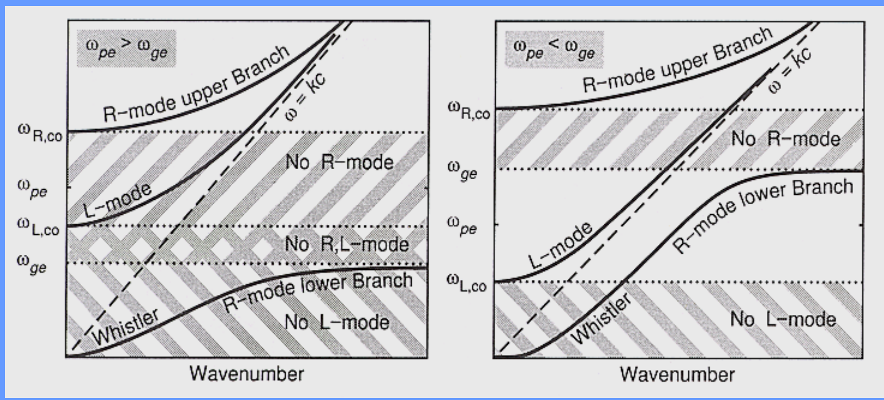
$$\omega_{L,co} = \frac{1}{2}\left[(\omega_{ge}^2 + 4\omega_{pe})^{1/2} - \omega_{ge}\right]$$

Refractive index for parallel R- and L-waves



There is no wave propagation for $N^2 < 0$, regions which are called ***stop bands*** or domains where the waves are ***evanescent***.

Dispersion branches for parallel R- and L-waves



The dispersion branches are for a ***dense*** (left) and ***dilute*** (right) plasma. Note the tangents to all curves, indicating that the group velocity is always smaller than *c*. Note also that the R- and L-waves can not penetrate below their cut-off frequencies. The R-mode branches are separated by ***stop bands.***

Perpendicular electromagnetic waves I

The other limiting case is purely perpendicular propagation, which means, $\mathbf{k} = \mathbf{k}_\perp$. In a uniform plasma we may chose **k** to be in the *x*-direction. The cold plasma dispersion relation reduces to:

$$Det \begin{bmatrix} -\epsilon_1 & i\epsilon_2 & 0 \\ -i\epsilon_2 & N_\perp^2 - \epsilon_1 & 0 \\ 0 & 0 & N_\perp^2 - \epsilon_3 \end{bmatrix} = 0$$

Apparently, $\delta E_\parallel$ decouples from to $\delta E_\perp$, and the third tensor element yields the dispersion of the ***ordinary mode***, which is denoted as ***O-mode***. It is transverse, is cut off at the local plasma frequency and obeys:

$$\omega_{om}^2 = \omega_{pe}^2 + k_\perp^2 c^2$$

The remaining dispersion relation is obtained by solving the two-dimensional determinant, which gives:

$$\epsilon_2^2 + \epsilon_1 (N_\perp^2 - \epsilon_1) = 0$$

Perpendicular electromagnetic waves II

When inserting the tensor elements one obtains after some algrabra (exercise!) the wave vector as a function of frequency in convenient form:

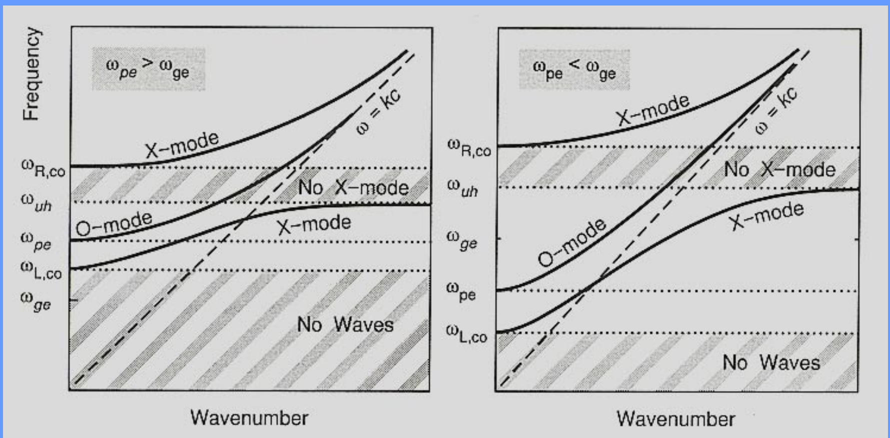
$$k_\perp^2 c^2 = \frac{(\omega^2 - \omega_{R,co}^2)(\omega^2 - \omega_{L,co}^2)}{\omega^2 - \omega_{ge}^2 - \omega_{pe}^2}$$

Apparently, δE_x is now coupled with δE_y, and this mode thus mixes longitudinal and transverse components. Therefore it is called the ***extraordinary mode***, which is denoted as ***X-mode***. It is resonant at the ***upper-hybrid frequency***:

$$\omega_{uh}^2 = \omega_{ge}^2 + \omega_{pe}^2$$

The lower-frequency branch of the ***X-mode*** goes in resonance at this upper-hybrid frequency, and from there on has a stop-band up to $\omega_{R,co}$.

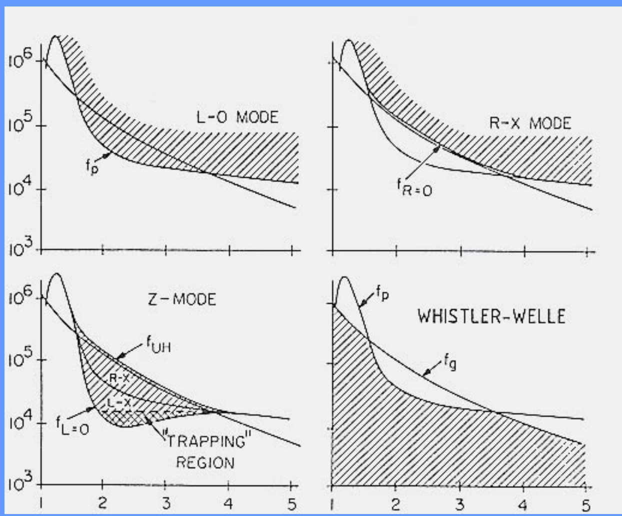
Dispersion for perpendicular O- and X-waves



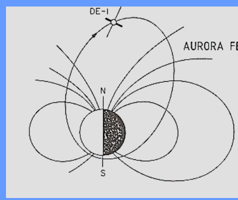
The dispersion branches are for a ***dense*** (left) and ***dilute*** (right) plasma. Note the tangents to all curves, indicating that the group velocity is always smaller than *c*. Note that the O- and X-waves can not penetrate below the cut-off frequencies. The X-mode branches are separated by ***stop bands.***

Frequency ranges of Z-, L-O- and R-X-mode waves

Frequency /Hz



Radial distance /R_E

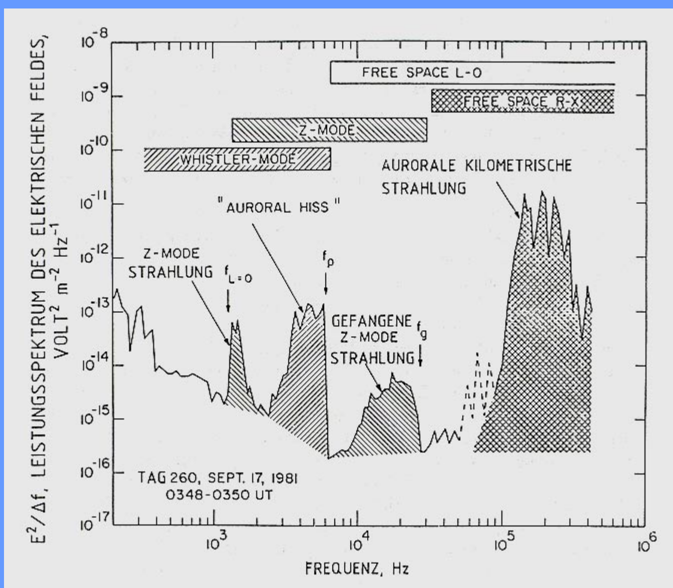


Top: Dynamics Explorer DE-1 satellite orbit

Left: Frequency ranges in the auroral region of the Earth magnetosphere

Whistler and Z-mode waves are trapped.

Electric field fluctuation spectra in the auroral zone



Measurements by the Dynamics Explorer DE-1 satellite in the Earth's high-latitude auroral zone.

Maximal field strength at a few mV/m. Wave excitation by fast electrons at relativistic cyclotron resonance.

$f_p = 9\ (n_e/cm^{-3})^{1/2}$ [kHz]

$f_g = 28\ B/nT$ [Hz]

Two-fluid plasma waves

At ***low frequencies*** below and comparable to ω_{gi}, the ***ion dynamics*** become important. Note that the ion contribution can be simply added to the electron one in the current and charge densities. The cold dielectric tensor is getting more involved. The elements read now:

$$\epsilon_1 = 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ge}^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{gi}^2}$$

$$\epsilon_2 = -\frac{\omega_{ge}}{\omega}\frac{\omega_{pe}^2}{\omega^2 - \omega_{ge}^2} + \frac{\omega_{gi}}{\omega}\frac{\omega_{pi}^2}{\omega^2 - \omega_{gi}^2}$$

$$\epsilon_3 = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2}$$

For ***parallel*** propagation, $k_\perp = 0$, the dispersion relation is:

$$N_{\parallel R,L}^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega \mp \omega_{ge})} - \frac{\omega_{pi}^2}{\omega(\omega \pm \omega_{gi})}$$

For ***perpendicular*** propagation, $k_\parallel = 0$, the dispersion relation can be written as:

$$N_\perp^2 = \epsilon_1 - \epsilon_2^2/\epsilon_1 = 0$$

Ion-related dispersion effects

The L-mode has a resonance at ω_{gi}, the ion gyrofrequency. The refractive index $N_\parallel$ -> 0 at the slightly modified cut-off frequency, which includes ionic corrections. The low-frequency ***R-mode dispersion*** becomes now:

$$\omega = \frac{\omega_{ge}}{2}\left(1 + \frac{\omega_{pe}^2}{k^2c^2}\right)^{-1}\left[\left(1 + \frac{4\omega_{pi}^2}{k^2c^2}\right)^{1/2} + 1\right]$$

The ***L-mode dispersion*** relation can be written analogously:

$$\omega = \frac{\omega_{ge}}{2}\left(1 + \frac{\omega_{pe}^2}{k^2c^2}\right)^{-1}\left[\left(1 + \frac{4\omega_{pi}^2}{k^2c^2}\right)^{1/2} - 1\right]$$

Here $\omega_{L,res} = \omega_{gi}$ is the ion-cyclotron resonance frequency for the left-hand-polarised (LHP) parallel electromagnetic wave.

For very long (in the MHD regime) wavelength, $k^2 < \omega_{pi}^2/c^2$, both dispersion relations reduce to the simple Alfvén waves.

Lower-hybrid resonance

For ***perpendicular propagation*** the dispersion relation can be written as:

$$N_\perp^2 = \epsilon_1 - \epsilon_2^2/\epsilon_1 = 0$$

At extremely low frequencies, we have the limits:

$$\lim_{\omega\to 0} \epsilon_1 = \lim_{\omega\to 0} \epsilon_2 = 1 + \frac{c^2}{v_A^2}$$

These are the dielectric constants for the ***X-mode*** waves. In that limit the refractive index is $N_\perp = \sqrt{\varepsilon_1}$, and the Alfvén wave dispersion results:

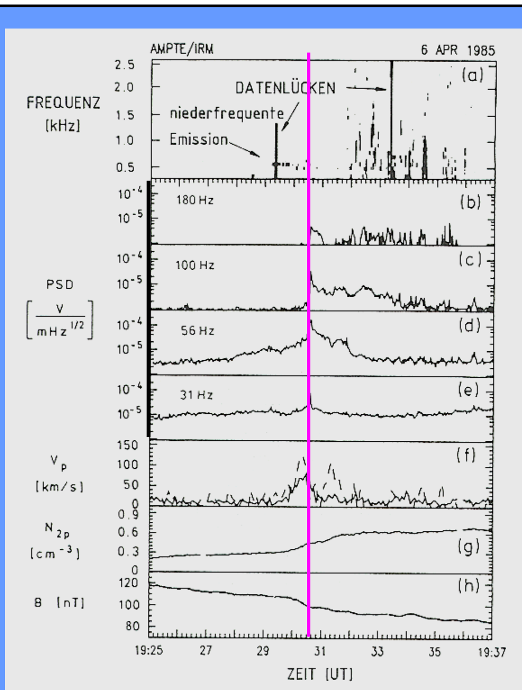
$$\omega = \pm k v_A \left(1 + \frac{v_A^2}{c^2}\right)^{-1/2}$$

For $\varepsilon_1 \to 0$, the ***lower-hybrid resonance*** occurs at:

$$\omega_{lh}^2 = \frac{\omega_{pi}^2 + \omega_{gi}^2}{1 + \omega_{pe}^2/\omega_{ge}^2}$$

It varies between the ion plasma and gyro frequency, and in dense plasma it is given by the ***geometric mean***:

$$\omega_{lh} = (\omega_{ge}\omega_{gi})^{1/2}$$



Waves at the lower-hybrid frequency

Measurements of the AMPTE satellite in the plasmasphere of the Earth near 5 R_E. Wave excitation by ***ion currents*** (modified two-stream instability).

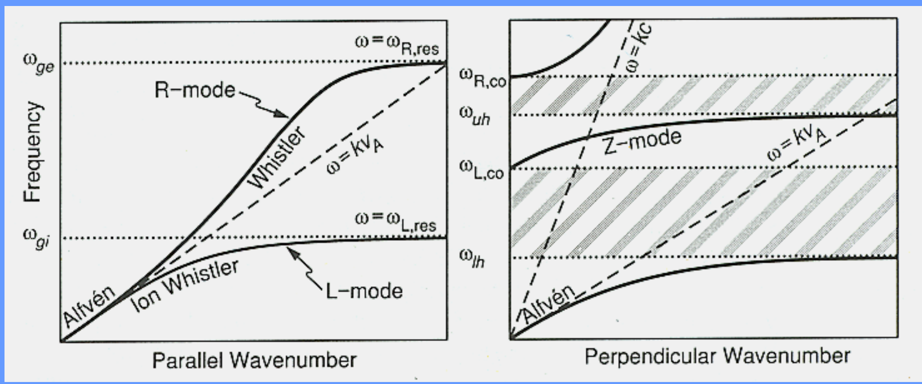
$N_e \approx 40$ cm^{-3}

$T_e \approx$ several eV

$\omega_{lh}/2\pi \approx 56$ Hz

$E_{max} \approx 0.6$ mV/m

Low-frequency dispersion branches



The dispersion branches are for a ***parallel*** (left) and ***perpendicular*** (right) propagation. Note the tangents to all curves, indicating that the group velocity is always smaller than c, and giving the Alfvén speed, v_A, for small $\boldsymbol{k}$. Note that the ***Z-mode waves*** can not penetrate below the cut-off frequency $\omega_{L,co}$ and is trapped below ω_{uh}. The X-mode branches are separated by ***stop bands.***

General oblique propagation

The previous theory can be generalized to ***oblique propagation*** and to multi-ion plasmas. Following Appleton and Hartree, the cold plasma dispersion relation (with no spatial dispersion) in the ***magnetoionic theory*** can be written as a biquadratic in the refractive index, $N^2 = (kc)^2/\omega^2$.

$$AN^4 - BN^2 + C = 0$$

The coefficients are given by the previous dielectric functions, and there is now an explicit dependence on the wave ***propagation angle***, θ, with respect to **B**.

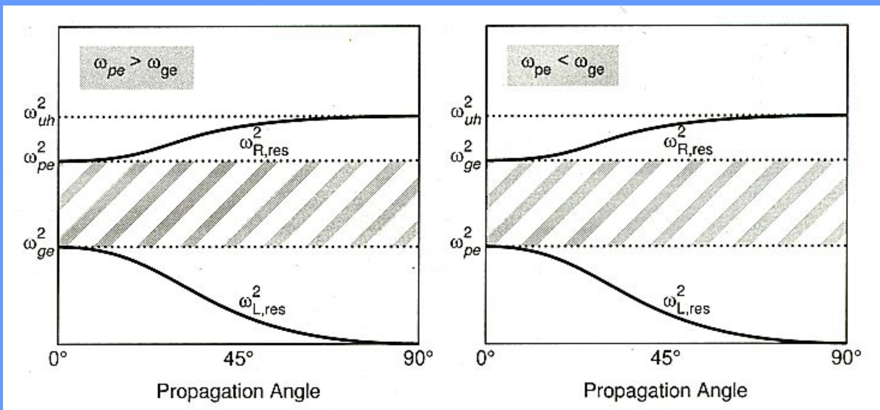
$$\begin{aligned} A &= \epsilon_1 \sin^2\theta + \epsilon_3 \cos^2\theta \\ B &= \epsilon_R \epsilon_L \sin^2\theta + \epsilon_1 \epsilon_3 (1 + \cos^2\theta) \\ C &= \epsilon_3 \epsilon_R \epsilon_L \end{aligned}$$

The coefficient A must vanish at the ***resonance***, $N \to \infty$, which yields the angular dependence of the resonance frequency on the angle θ_{res} as:

$$\tan^2\theta_{res} = -\frac{\epsilon_3}{\epsilon_1}$$

The coefficient C must vanish at the ***cut off***, $N \to 0$, which means the cut-offs do not depend on θ.

Angular variation of the resonance frequencies



The two resonance frequencies for a ***dense*** (left) and ***dilute*** (right) pure electron plasma, following from the biquadratic equation:

$$\omega_{res}^4 - \omega_{uh}^2\omega_{res}^2 + \omega_{ge}^2\omega_{pe}^2 \cos^2\theta = 0$$

Drift waves

The previous theory can be generalized to wave propagation in ***non-uniform media***, in which the wavelengths are comparable to the natural scales in the plasma, such as the gradient scales:

$L_n = |\nabla \ln n|^{-1}$; $L_B = |\nabla \ln B|^{-1}$; $L_T = |\nabla \ln T|^{-1}$ etc.

New waves occur, ***drift modes***, owing their existence to inhomogeneity.

With the drift speed, $\delta\upsilon_{ix} = \delta E_y/B_0 = -ik_y\delta\phi/B_0$, the linearized 1-D ion continuity equation is:

$$\frac{\partial \delta n_i}{\partial t} + \delta\upsilon_{ix}\frac{\partial n_0}{\partial x} = 0$$

Assuming quasineutrality and using Boltzmann‘s law:

$$\delta n_e = -\frac{e n_0}{k_B T_e}\delta\phi$$

Using a plane wave ansatz, $\exp(-i\omega t + ik_y y)$, we obtain the small electron ***drift-wave frequency***:

$$\omega_{de}/\omega_{ge} = k_y r_{ge}^2/L_n$$