Wave particle-interactions

- **Particle trapping in waves**
- **Exact nonlinear waves**
- **Quasilinear theory**
- **Diffusion equation**
- **Plateau formation**
- **Pitch-angle diffusion**
- **Anomalous resistivity**
- **Particle acceleration and heating**

Nonlinear phenomena

In the previous sections we have mainly dealt with ***linear approximations*** of space plasma physics problems. Here a well-developed apparatus of algebraic techniques is available.

- ***However, plasma physics is fundamentally nonlinear.***
- ***Yet, no general mathematical algorithms exist in nonlinear theory.***

Consequence:
--> **Perturbation theory** (quasilinear theory) of wave-particle and wave-wave interactions, analytical approach
--> **Direct numerical simulations** (not treated here)

In this section we discuss more qualitatively some selected nonlinear phenomena and physical effects.

Particle trapping in waves

One of the simplest nonlinear effects is ***trapping of particles*** in large-amplitude waves, in which the wave potential exceeds the particle kinetic energy. Trapping is largest for resonant particles, which are moving at the wave phase speed and see a nearly stationary electrostatic potential:

$$\phi(x,t) = \phi_0 \cos(kx - \omega t) = \phi_0 \cos(kx')$$

Here the coordinates were transformed into the wave frame by

$$x' = x - (\omega/k)t$$

The particle speed is also transformed by

$$v' = v - \omega/k$$

The particle's total energy in the wave frame is

$$W_e = \frac{1}{2}mv'^2 - e\phi_0 \cos(kx')$$

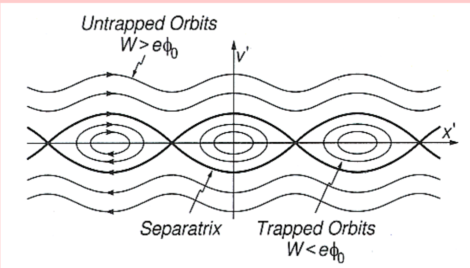
Trajectories of trapped particles

When considering electron motion in ***two-dimenensional phase space*** (x', v'), particles move along ***lines of constant energy*** W_e, as is shown below.

There appear to be two types of trajectories:

• ***closed*** **trajectories** with $W_e < 0$, ***trapped*** particles

• ***open*** **trajectories** with $W_e > 0$, ***untrapped*** particles



The trapped particles bounce back and force between the potential walls and oscillate periodically (expand the cosine potential) with the trapping frequency:

The larger the amplitude, ϕ_0, the faster is the oscillation.

$$\omega_b = |e\phi_0 k^2/m|^{1/2}$$

Exact nonlinear waves I

The ***kinetics of particles*** in large-amplitude electrostatic waves shows that one should distinguish between ***trapped*** and ***transmitted*** particles, determined by the Vlasov and Poisson equations. This complex system can usually not be solved exactly. Only in one dimension simple examples exist, such as the Bernstein-Green-Kruskal (BGK) waves. One uses comoving coordinates, $x \to x - vt$ and $v \to v - v_0$, such that a stationary system results, $\partial/\partial t \to 0$.

$$
\begin{aligned}
\upsilon\frac{\partial f_{s0}(x,\upsilon)}{\partial x} &= -\frac{q_s}{m_s}E(x)\frac{\partial f_{s0}(x,\upsilon)}{\partial \upsilon} \\
\frac{\partial E(x)}{\partial x} &= \sum_s \frac{q_s}{\epsilon_0}\int d\upsilon f_{s0}(x,\upsilon)
\end{aligned}
$$

Introducing the total energy, as a variable,

$$W_s = \frac{1}{2}m_s\upsilon_s^2 + q_s\phi(x)$$

and the potential ϕ by $E = -\partial\phi/\partial x$, one can rewrite the stationary Poisson equation as:

$$\frac{\partial^2\phi}{\partial x^2} = -\sum_s \int_{q_s\phi}^{\infty} \frac{f_{s<}(W_s) + f_{s>}(W_s)}{\epsilon_0\left[2m_s(W_s - q_s\phi)\right]^{1/2}} dW_s$$

Exact nonlinear waves II

Formally this can be regarded as the equation of motion for a ***pseudo 'particle' at 'position' ϕ in 'time' x***. Via multiplication with $\partial\phi/\partial x$ and integration with respect to x, we finally obtain:

$$\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi) = \text{const}$$

The ***pseudo-potential*** $V(\phi)$ is obtained by integration from the right-hand side, denoted for short by $G(\phi)$, as

$$V(\phi) = -\int_{\phi_0}^{\phi} d\phi G(\phi)$$

Depending on the geometrical shape of this pseudo-potential $V(\phi)$, one obtains ***periodic*** or ***aperiodic*** solutions for $\phi(x)$ by simple ***quadrature***:

$$x - x_0 = \frac{\pm 1}{\sqrt{2}}\int_{\phi_0}^{\phi} \frac{d\phi}{|V(\phi) - V(\phi_0)|^{1/2}}$$

Perturbation theory

Nonlinear interaction can lead to stationary states consisting of large-amplitude waves and related particle distributions of trapped and free populations. It is difficult to find these states, and often ***perturbation*** expansions are used, leading to what is called ***weak plasma turbulence*** theory. Starting point is the coupled system of **Maxwell's** (which we do not quote here) and **Vlasov's** equations, e.g. for the *s*-component of the plasma.

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0$$

We split the fields and velocity distributions into average slowly varying parts, f_{s0}, $\mathbf{E}_0$, and $\mathbf{B}_0$, and oscillating parts, δf_s, $\delta\mathbf{E}$ and $\delta\mathbf{B}$, and assume that the long-time and large-volume averages over fluctuations vanish, i.e.

$$\langle \delta f_s \rangle = \langle \delta \mathbf{E} \rangle = \langle \delta \mathbf{B} \rangle = 0$$

Averaging the resulting Vlasov equation gives the evolution for f_{s0}. No assumptions were yet made about the size of the fluctuations, but usually they are assumed to be much smaller than the background.

Quasilinear theory

The **Vlasov** equation for the slowly-varying ensemble averaged VDF of species *s* reads:

$$\begin{aligned} & \frac{\partial f_{s0}}{\partial t} + \mathbf{v} \cdot \nabla f_{s0} + \frac{q_s}{m_s}(\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_{s0}}{\partial \mathbf{v}} \\ = & -\frac{q_s}{m_s}\left\langle (\delta\mathbf{E} + \mathbf{v} \times \delta\mathbf{B}) \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \right\rangle \end{aligned}$$

The second-order interactions between the fluctuations, δf_s, $\delta\mathbf{E}$ and $\delta\mathbf{B}$, appear on the right-hand side. If they are small we speak of ***quasilinear theory***. This fundamental equation describes the nonlinear dynamics of the plasma. ***It results from the scattering of particle motion in the short-wavelength, high-frequency field fluctuations.***

Formally the perturbation series can, with the smallness parameter λ, be written as:

$$\begin{aligned} f_s &= f_{s0} + \lambda \delta f_{s1} + \lambda^2 \delta f_{s2} + \ldots \\ \delta \mathbf{E} &= \lambda \delta \mathbf{E}_1 + \lambda^2 \delta \mathbf{E}_2 + \ldots \end{aligned}$$

Weak gentle-beam turbulence I

The perturbation series is expected to converge rapidly, if λ is small. Assume

$$\lambda = \frac{\langle \epsilon_0 |\delta \mathbf{E}(\mathbf{x},t)|^2\rangle}{2\langle n\rangle k_B \langle T\rangle} \ll 1$$

Assume a gentle source of ***free energy*** in the form of a ***weak beam*** of electrons crossing the plasma and consider the associated excitation of Langmuir waves. Remember that the ***linear complex frequency*** is:

$$\omega(k) = \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_D^2\right)$$

$$\gamma(k,t) = \omega(k)\frac{\pi\omega_{pe}^2}{2k^2}\frac{\partial f_{0b}(v,t)}{\partial v}|_{v=\omega/k}$$

Consequently, the electric field evolves in time according to:

$$\delta E(k,t) = \delta E(k,0)\,\mathrm{exp}\left\{-\int_0^t [i\omega(k) - \gamma(k,\tau)]\,d\tau\right\}$$

Weak gentle-beam turbulence II

Consequently, the average particle VDF will also evolve in time according to:

$$\frac{\partial f_{0b}(v,t)}{\partial t} = \frac{e}{m_e}\left\langle \delta E \frac{\partial \delta f}{\partial v}\right\rangle$$

This quadratic correlation term can be calculated by help of Fourier transformation of the Vlasov equation for the fluctuations, yielding:

$$\delta f(k) = i\frac{e}{m_e}\frac{\delta E(k)}{\omega - kv}\frac{\partial f_{0b}(v,t)}{\partial v}$$

Inserting the first in the second gives a second-order term, δ*E*(**k**, *ω*) δ*E*(-**k**, -*ω*), which gives the ***wave power spectrum***. Hence we arrive at the diffusion equation for the beam distribution, with the general diffusion coefficient:

$$D(v,t) = Re\left\{\frac{ie^2}{m_e^2}\sum_k \frac{|\delta E(k)|^2}{kv - \omega(k) + i\gamma(k,t)} \mathrm{exp}\left[2\int_0^t \gamma(k,\tau)d\tau\right]\right\}$$

Diffusion equation

Through diffusion of particles in the wave field, the average VDF will slowly evolve in time according to:

$$\frac{\partial f_{0b}(v,t)}{\partial t} = \frac{\partial}{\partial v}\left[D(v,t)\frac{\partial f_{0b}(v,t)}{\partial v}\right]$$

This is a special case of a ***Fokker-Planck equation***, typically arising in quasilinear theory. The beam distribution will spread with time in velocity space under the action of the unstable Langmuir fluctuations. The resonant denominator may be replaced by a delta function giving:

$$D(v,t) = \frac{\pi e^2}{m_e^2}\int W_w(k,t)\delta(\omega - kv)dk$$

By differentiation of the wave electric field, $\delta E(\mathbf{x}, t)$, we obtain the evolution equation of the associated spectral density as follows:

$$\frac{\partial W_w(k,t)}{\partial t} = 2\gamma(k,t)W_w(k,t)$$

This completes the ***quasilinear equations*** of beam-excited Langmuir waves.

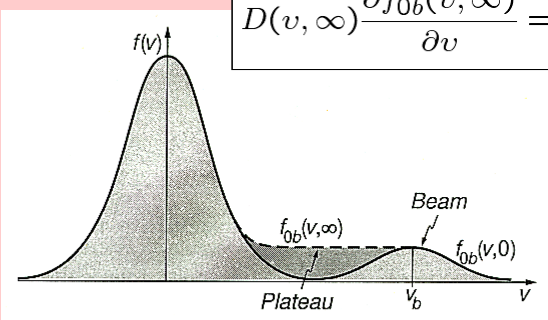
Plateau formation

The validity of the quasilinear equations requires that

$$\gamma/\omega \gg W_w/n_0 k_B T_e$$

Fortunately, the equations allow some clear insights into the underlying physics. Inspection of the diffusion equation shows, that a steady final VDF occurs if

$$D(v,\infty)\frac{\partial f_{0b}(v,\infty)}{\partial v} = 0$$



This implies that the gradient flattens and disappears, since the spectrum is positive definite, and the diffusion coefficient is only zero if the entire spectrum vanishes.

As a final result, a plateau will form by diffusion in velocity space.

Ion cyclotron wave turbulence

Electromagnetic waves below the ion gyrofrequency propagating parallel to the field can be excited by ion ***beams*** and a ***core temperature ansisotropy***.

A diffusion theory can be formulated, in which the ions form by pitch-angle scattering in the wave field a plateau, having the shape of concentric circles centered at the wave phase speed. Ions obey the resonance condition:

$$\boxed{v_{res} = \left[\omega(k) - \omega_{ge}\right]/k}$$

In the asymptotic limit the VDF attains gradients corresponding to zero growth on contours defined by:

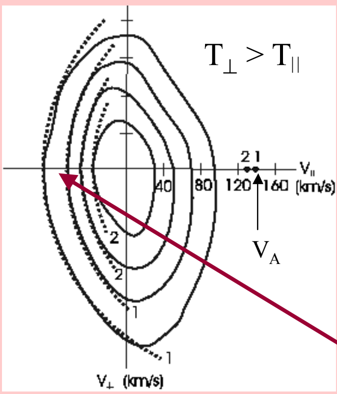
$$\boxed{(v_{\|} - \omega/k_{\|})^2 + v_{\perp}^2 = \mathrm{const}}$$

Hence, the final resonant part of the VDF depends only on the particle energy that is constant in the wave frame:

$$f_{0e}(v_{\perp}, v_{\|}, \infty) = f_{0e}\left[\frac{v_{\|}^2 + v_{\perp}^2}{2} - \int_0^{v_{res}} dv_{\|}\frac{\omega(v_{\|})}{k}\right]$$

Resonant pitch-angle diffusion and plateau formation of solar wind protons

Measured protons in the fast solar wind (Helios)



Marsch & Tu, JGR, **106**, 8357, 2001

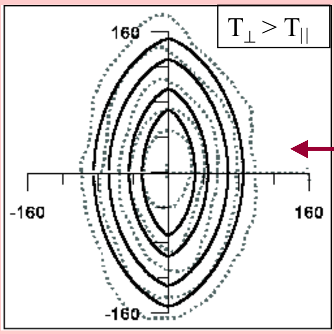
$$\frac{\partial}{\partial t} f_j(V_{\perp}, V_{\|}, t) = \sum_M \sum_{s=-\infty}^{+\infty} \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3k\, \mathcal{B}_M(\mathbf{k}) \times \frac{1}{V_{\perp}} \frac{\partial}{\partial \alpha}\left(V_{\perp}\, \nu_j(\mathbf{k}, s; V_{\|}, V_{\perp}) \frac{\partial}{\partial \alpha} f_j(V_{\perp}, V_{\|}, t)\right)$$

$$\frac{\partial}{\partial \alpha} = V_{\perp}\frac{\partial}{\partial V_{\|}} - \left(V_{\|} - \frac{\omega_M(\mathbf{k})}{k_{\|}}\right)\frac{\partial}{\partial V_{\perp}}$$

- **Resonant pitch-angle diffusion leads to quasilinear plateau (on the dotted circles)**

- **Particles conserve energy in wave frame**

Regulation of the solar wind proton core temperature anisotropy by wave-particle interactions



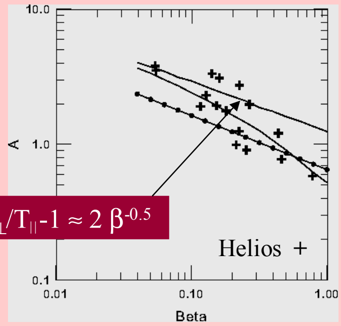
Anisotropic core VDF of fast protons (Helios at 0.3 AU).

Tu & Marsch, JGR, **107**, 1249, 2002

Quasilinear diffusion in the field of

- inward waves (local decay, instability)
- outward (coronal origin) LHP waves

shell distribution



$A = T_\perp/T_{||} - 1 \approx 2\,\beta^{-0.5}$

Anomalous resistivity I

The determination of ***transport coefficients*** is one of the most important aspects of microscopic plasma theory. The moments like density, flow speed, temperature are of macroscopic nature, and their gradients induce corresponding flows in the plasma related with diffusion, viscosity, or heat conduction, which will cause ***irreversibility*** in the system.

In a ***collisionless*** plasma irreversibility is the consequence of nonlinear interactions between field fluctuations (waves, turbulence) and particles. The associated transport coefficients are called ***anomalous***.

Consider an electron moving under friction in an external electric field:

$$\boxed{m_e dv_e/dt = -eE - m_e \nu v_e}$$

Under stationary conditions the current is, $j = -en_0 v_e = \sigma E$. The resistivity, $\eta = 1/\sigma$, is given by

$$\boxed{\eta = \nu/\omega_{pe}^2 \epsilon_0}$$

Anomalous resistivity II

For a collisional plasma with ***ion-electron Coulomb collisions***, we found that the Spitzer collision frequency can be expressed as:

$$\nu_{ei} \propto \frac{\omega_{pe}}{n_0 \lambda_D^3} = \frac{\omega_{pe} W_{tf}}{n_0 k_B T_e}$$

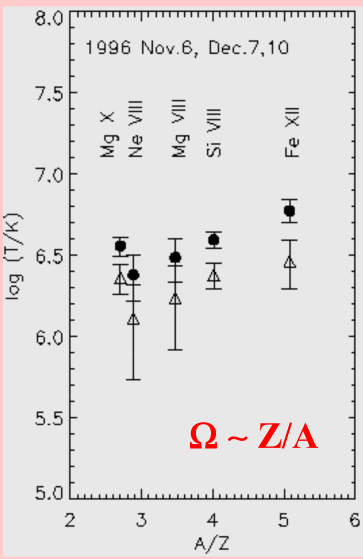
The frequency is proportional to the ratio of the thermal wave fluctuation energy density, W_{tf}, to the thermal energy per electron. The colliding electrons are assumed to be scattered by the oscillating electric field of thermally excited Langmuir waves.

The idea of anomalous collisions is to replace W_{tf} by the enhanced energy, W_w, of waves driven by a ***microinstability***. For large drifts, e.g. the ion acoustic instability occurs, and the corresponding ion-acoustic saturated energy density, determining the collision frequency in the **Sagdeev** formula:

$$\nu_{ia,an} \approx \omega_{pe} W_w / n_0 k_B T_e$$

Similar collision-frequency formulae can be derived for other instabilities, involving essentially the natural frequency of the waves considered.

Heating of coronal ions by cyclotron-wave resonance



Marsch & Tu, JGR, **106**, 227, 2001

$$\begin{pmatrix} \frac{\partial}{\partial t} U_{j\parallel} \\ \frac{\partial}{\partial t} V_{j\parallel}^2 \\ \frac{\partial}{\partial t} V_{j\perp}^2 \end{pmatrix} \begin{matrix} = \text{wave acceleration} \\ = \text{parallel wave heating} \\ = \text{perpendicular heating} \end{matrix}$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3k \sum_M \mathcal{B}_M(\mathbf{k}) \left(\frac{\Omega_j}{k}\right)^2 \frac{1}{1 - |\hat{\mathbf{k}} \cdot e_M(\mathbf{k})|^2}$$

$$\times \sum_{s=-\infty}^{+\infty} \mathcal{R}_j(\mathbf{k}, s) \begin{pmatrix} k_\parallel \\ 2k_\parallel w_j(\mathbf{k}, s) \\ s\Omega_j \end{pmatrix}$$

- **Wave opacity or absorption coefficient depends on the ion VDF**
- **Heating depends on the wave spectrum**

Tu, Marsch, & Wilhelm, Space Sci. Rev., **87**, 331, 1999

Particle acceleration and heating in wave fields

Particle ***acceleration*** and ***heating*** by various kinds of plasma waves is perhaps the most important mechanism. Wave turbulence is most easily generated in nonuniform plasmas by various types of free energy. This can often be described by a general diffusion equation of the form:

$$\frac{\partial f(\mathbf{v})}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{D}(\mathbf{v}) \cdot \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} \right] - \frac{f(\mathbf{v})}{\tau(\mathbf{v})}$$

The diffusion coefficient is a functional of the ***wave spectrum*** and ***particle VDF***. We have added a loss term as well. The spectrum evolves according to the wave kinetic equation, including wave damping, $\gamma_l(\mathbf{k})$, and losses or sources, $S_w(\mathbf{k})$. This equation may involve a diffusion term, describing the spreading of wave energy across k-space, or terms invoking a ***turbulent cascade***. It reads:

$$\frac{\partial W_w(\mathbf{k},t)}{\partial t} = \frac{\partial}{\partial \mathbf{k}} \cdot \left[\mathbf{D}_w(\mathbf{k}) \frac{\partial W_w(\mathbf{k},t)}{\partial \mathbf{k}} \right] \\ -\gamma_l(\mathbf{k}) W_w(\mathbf{k},t) + S_w(\mathbf{k})$$