Magnetohydrodynamic waves

- **Ideal MHD equations**
- **Linear perturbation theory**
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- **Phase velocities**
- **Dispersion relations (polar plot)**
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Ideal MHD equations

Plasma equilibria can easily be perturbed and small-amplitude ***waves and fluctuations*** can be excited. Conveniently, while considering waves one starts from ideal plasmas. The damping of the waves requires consideration of some kind of disspation, which will not be done here. **MHD** (***with ideal Ohm's law and no space charges***) ***in the standard form*** (***and with* P(*n*) *given***) ***reads***:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

$$\frac{\partial (nm\mathbf{v})}{\partial t} + \nabla \cdot (nm\mathbf{v}\mathbf{v}) = -\nabla \cdot \left(\mathbf{P} + \frac{B^2}{2\mu_0}\mathbf{I}\right) + \frac{1}{\mu_0}\nabla \cdot (\mathbf{B}\mathbf{B})$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

$$\nabla \cdot \mathbf{B} = 0$$

MHD equilibrium and fluctuations

We assume ***stationary*** ideal ***homogeneous*** conditions as the intial state of the single-fluid plasma, with vanishing average electric and velocity fields, overal ***pressure equilibrium*** and no magnetic stresses. These assumptions yield:

$$\begin{aligned} \mathbf{v}_0 &= 0 \\ \mathbf{E}_0 &= 0 \\ \nabla\left(p_0 + B_0^2/2\mu_0\right) &= 0 \\ \left(\mathbf{B}_0 \cdot \nabla\right)\mathbf{B}_0 &= 0 \end{aligned}$$

These fields are decomposed as sums of their background initial values and space- and time-dependent ***fluctuations*** as follows:

$$\begin{aligned} n &= n_0 + \delta n \\ \mathbf{v} &= \delta\mathbf{v} \\ \mathbf{E} &= \delta\mathbf{E} \\ \mathbf{B} &= \mathbf{B}_0 + \delta\mathbf{B} \end{aligned}$$

Linear perturbation theory

Because the MHD equations are nonlinear (advection term and pressure/stress tensor), the fluctuations must be small.
-> Arrive at a uniform set of linear equations, giving the dispersion relation for the eigenmodes of the plasma.
-> Then all variables can be expressed by one, say the magnetic field.

Usually, in space plasma the background magnetic field is sufficiently strong (e.g., a planetary dipole field), so that one can assume the fluctuation obeys:

$$|\delta\mathbf{B}| \ll B_0$$

In the uniform plasma with straight field lines, the field provides the only ***symmetry axis*** which may be chosen as z-axis of the coordinate system such that: $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_{||}$.

Linearized MHD equations I

Linarization of the MHD equations leads to three equations for the three fluctuations, δn, $\delta\mathbf{v}$, and $\delta\mathbf{B}$:

$$\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \delta\mathbf{v} = 0$$

$$m_i n_0 \frac{\partial \delta\mathbf{v}}{\partial t} = -\nabla\left(\delta p + \frac{1}{\mu_0}\mathbf{B}_0 \cdot \delta\mathbf{B}\right) + \frac{1}{\mu_0}\left(\mathbf{B}_0 \cdot \nabla\right)\delta\mathbf{B}$$

$$\frac{\partial \delta\mathbf{B}}{\partial t} = \left(\mathbf{B}_0 \cdot \nabla\right)\delta\mathbf{B} - \mathbf{B}_0\left(\nabla \cdot \delta\mathbf{v}\right)$$

Using the adabatic pressure law, and the derived sound speed, $c_s^2 = p_0/m_i n_0$, leads to an equation for δp and gives:

$$\frac{\partial \delta p}{\partial t} = m_i c_s^2 \frac{\partial \delta n}{\partial t} = -m_i n_0 c_s^2 \nabla \cdot \delta\mathbf{v}$$

Linearized MHD equations II

Inserting the continuity and pressure equations, and using the Alfvén velocity, $\mathbf{v}_A = \mathbf{B}_0/(\mu_0 n m_i)^{1/2}$, two coupled vector equations result:

$$\frac{\partial \delta\mathbf{v}}{\partial t} = v_A^2 \nabla_{\|}\left(\frac{\delta\mathbf{B}_\perp}{B_0}\right) - \nabla\left(\frac{\delta p}{m_i n_0}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\delta\mathbf{B}}{B_0}\right) = \nabla_{\|}\delta\mathbf{v}_\perp - \hat{\mathbf{e}}_{\|}\left(\nabla_\perp \cdot \delta\mathbf{v}_\perp\right)$$

Time differentiation of the first and insertion of the second equation yields a second-order wave equation which can be solved by ***Fourier transformation***.

$$\begin{aligned}\frac{\partial^2 \delta\mathbf{v}}{\partial t^2} &= c_{ms}^2 \nabla\left(\nabla \cdot \delta\mathbf{v}\right) \\ &+ v_A^2\left(\nabla_{\|}^2 \delta\mathbf{v} - \nabla\nabla_{\|}\delta v_{\|} - \hat{\mathbf{e}}_{\|}\nabla_{\|}\nabla \cdot \delta\mathbf{v}\right)\end{aligned}$$

Dispersion relation

The ansatz of ***travelling plane waves***,

$$\delta \mathbf{v} = \delta \mathbf{v}_0 \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$$

with arbitrary constant amplitude, $\delta\mathbf{v}_0$, leads to the system,

$$\left[\left(\omega^2 - k_{\|}^2 v_A^2\right)\mathbf{I} - c_{ms}^2 \mathbf{k}\mathbf{k} + \left(\mathbf{k}\hat{\mathbf{e}}_{\|} + \hat{\mathbf{e}}_{\|}\mathbf{k}\right) k_{\|} v_A^2\right]\cdot \delta\mathbf{v}_0 = 0$$

To obtain a nontrivial solution the determinant must vanish, which means

$$\begin{bmatrix} \omega^2 - v_A^2 k_\|^2 - c_{ms}^2 k_\perp^2 & 0 & -c_s^2 k_\| k_\perp \\ 0 & \omega^2 - v_A^2 k_\|^2 & 0 \\ -c_s^2 k_\| k_\perp & 0 & \omega^2 - c_s^2 k_\|^2 \end{bmatrix} \begin{bmatrix} \delta v_{0x} \\ \delta v_{0y} \\ \delta v_{0\|} \end{bmatrix} = 0$$

Here the ***magnetosonic speed*** is given by $c_{ms}^2 = c_s^2 + v_A^2$. The wave vector component perpendicular to the field is oriented along the x-axis, $\mathrm{k} = k_{\|}\hat{\mathbf{e}}_z + k_{\perp}\hat{\mathbf{e}}_x$.

Alfvén waves

Inspection of the determinant shows that the fluctuation in the *y*-direction decouples from the other two components and has the linear dispersion

$$\omega_A = \pm k_{\|} v_A$$

This ***transverse wave*** travels parallel to the field. It is called ***shear Alfvén wave***. It has no density fluctuation and a constant group velocity, $\mathbf{v}_{gr,A} = \mathbf{v}_A$, which is always oriented along the background field, along which the wave energy is transported.

The transverse velocity and ***magnetic field components are (anti)-correlated*** *according to:* $\delta v_y/v_A = \pm\, \delta B_y/B_0$, *for* ***parallel (anti-parallel) wave propagation****. The wave electric field points in the x-direction:* $\delta E_x = \delta B_y/v_A$

Magnetosonic waves

The remaing four matrix elements couple the fluctuation components, $\delta v_{||}$ and $\delta v_{\perp}$. The corresponding determinant reads:

$$\omega^4 - \omega^2 c_{ms}^2 k^2 + c_s^2 v_A^2 k^2 k_{\|}^2 = 0$$

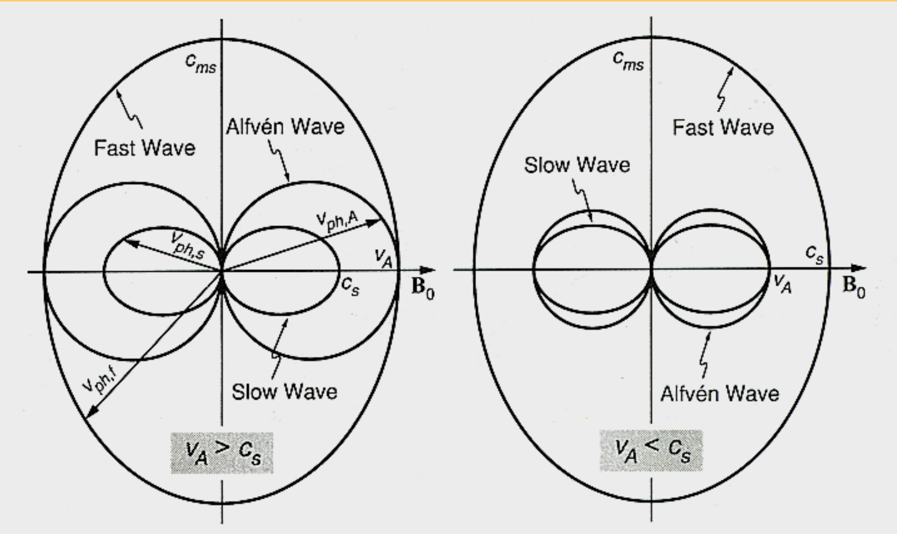
This bi-quadratic equation has the roots:

$$\omega_{ms}^2 = \frac{k^2}{2}\left\{c_{ms}^2 \pm \left[\left(v_A^2 - c_s^2\right)^2 + 4 v_A^2 c_s^2 \frac{k_\perp^2}{k^2}\right]^{1/2}\right\}$$

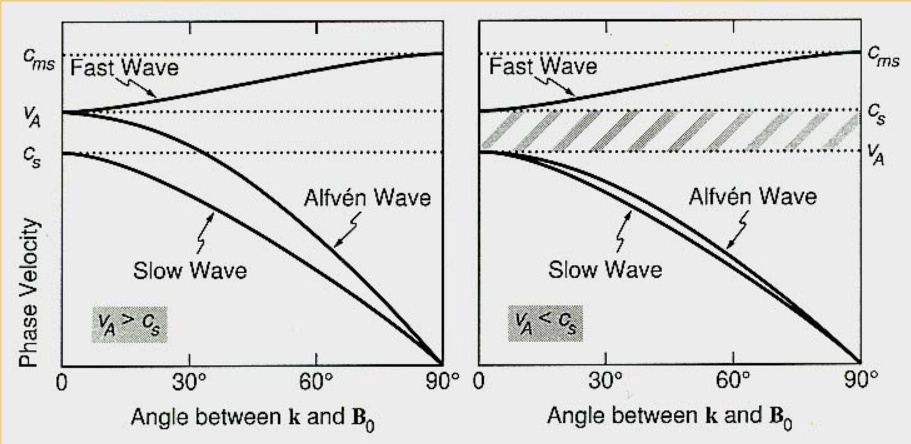
which are the phase velocities of the compressive ***fast and slow magnetosonic waves***. They depend on the propagation angle θ, with $k_\perp^2/k^2 = \sin^2\theta$. For $\theta = 90^0$ we have: $\omega = kc_{ms}$, and $\theta = 0^0$:

$$\omega^2 = \frac{1}{2}k^2\left[c_s^2 + v_A^2 \pm \left(c_s^2 - v_A^2\right)\right]$$

Phase-velocity polar diagram of MHD waves



Dependence of phase velocity on propagation angle



Magnetosonic wave dynamics

In order to understand what happens physically with the dynamic variables, δv_x, δB_x, $\delta B_{||}$, $\delta v_{||}$, δp, and δn, inspect again the equation of motion written in components:

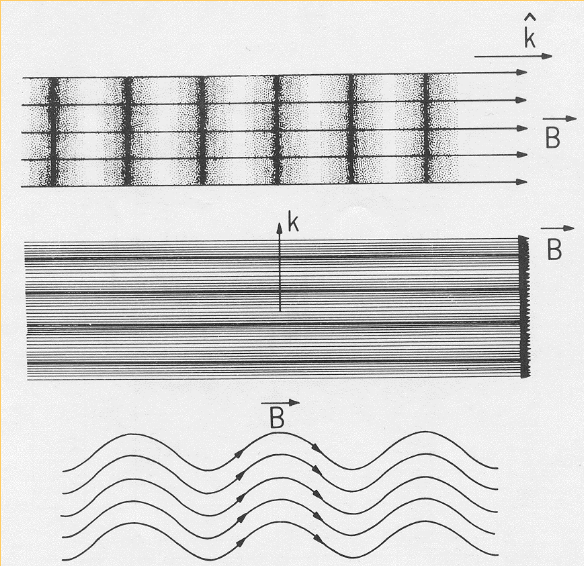
$$\omega \delta \mathbf{v} = \frac{\mathbf{k}}{m_i n_0}\left(\delta p + \frac{1}{\mu_0}\mathbf{B}_0 \cdot \delta \mathbf{B}\right) - \frac{\mathbf{k}\cdot\mathbf{B}_0}{\mu_0 m_i n_0}\delta \mathbf{B}$$

Parallel direction: $$\omega v_{||} = \frac{k_{||}\delta p}{m_i n_0}$$ *Parallel pressure variations cause parallel flow.*

Oblique direction: $$\omega\left(k_{||}v_{||} + k_{\perp}v_x\right) = \frac{k^2 \delta p_{tot}}{m_i n_0}$$

Total pressure variations ($p_{tot}=p+B^2/2\mu_0$) accelerate (or decelerate) flow, for in-phase (or out-of-phase) variations of δp and δB, *leading to the* ***fast and slow mode waves.***

Magnetohydrodynamic waves



- Magnetosonic waves

compressible

- parallel slow and fast
- perpendicular fast

$c_{ms} = (c_s^2 + v_A^2)^{-1/2}$

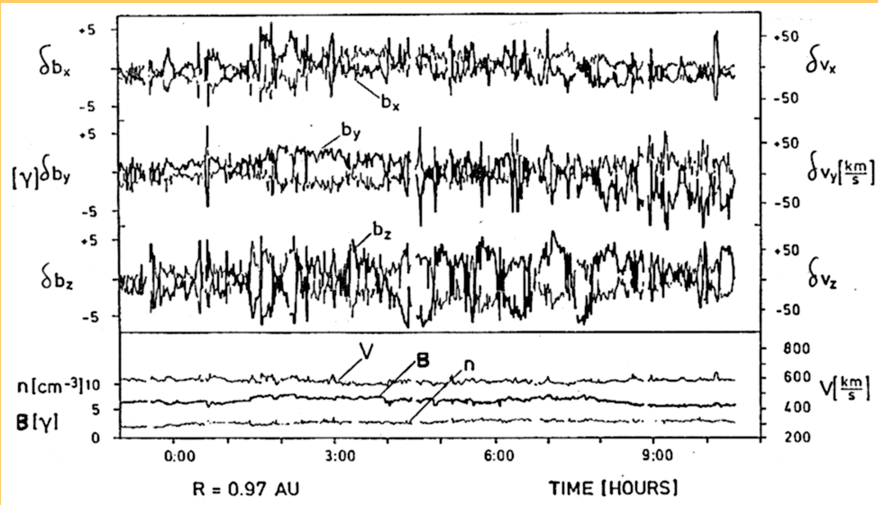
- Alfvén wave

incompressible

parallel and oblique

$v_A = B/(4\pi\rho)^{1/2}$

Alfvén waves in the solar wind



Neubauer et al., 1977

Helios

$\delta v = \pm\, \delta v_A$

Alfvénic fluctuations

Ulysses observed many such waves (4-5 per hour) in fast wind over the poles:

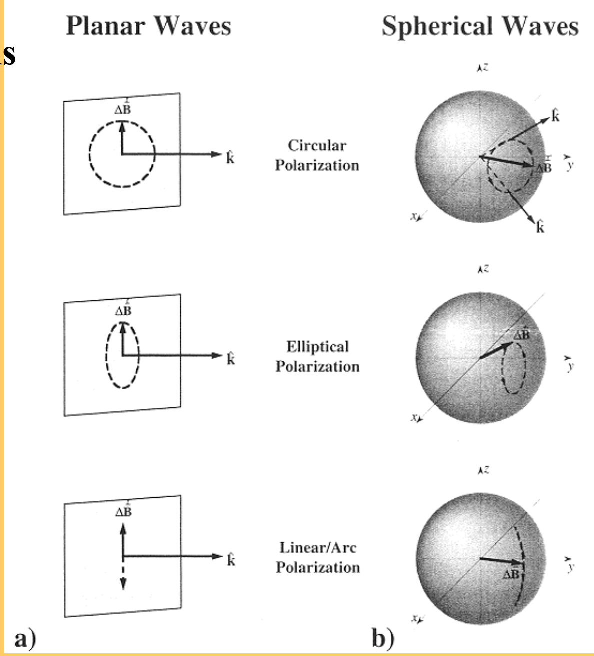
- Arc-polarized waves
- Phase-steepened

Rotational discontinuity:

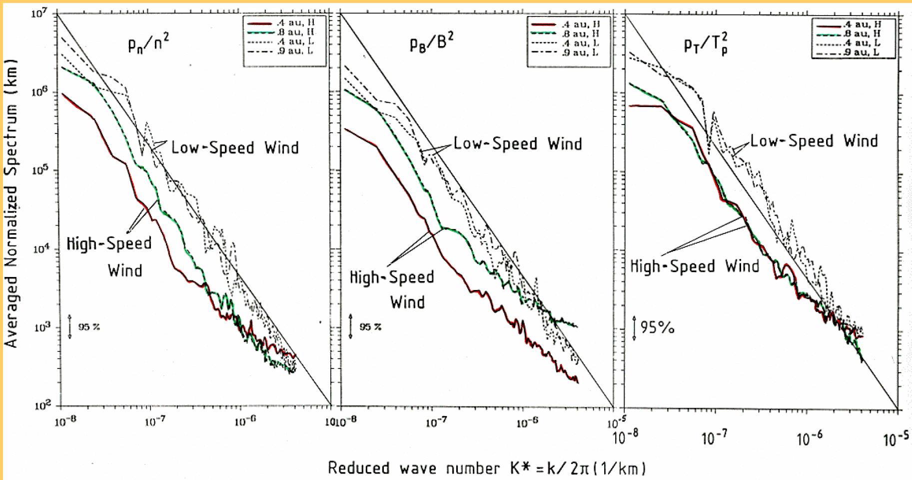
$\Delta \mathbf{V} = \pm \Delta \mathbf{V}_A$

Finite jumps in velocities over gyrokinetic scales

Tsurutani et al., 1997



Compressive fluctuations in the solar wind



Marsch and Tu, JGR, **95**, 8211, 1990

Kolmogorov-type turbulence

Magnetohydrodynamic turbulence in space plasmas

Integral invariants of ideal MHD

$E = 1/2 \int d^3x \, (V^2 + V_A^2)$ Energy

$H_c = \int d^3x \, (\mathbf{V} \cdot \mathbf{V}_A)$ Helicity

$H_m = \int d^3x \, (\mathbf{A} \cdot \mathbf{B})$ Magnetic helicity

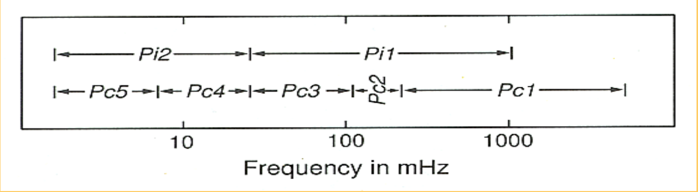
$\mathbf{B} = \nabla \times \mathbf{A}$

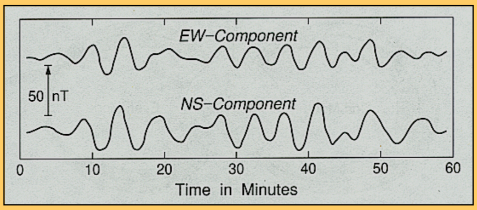
Elsässer variables: $\mathbf{Z}^{\pm} = \mathbf{V} \pm \mathbf{V}_A$

$E^{\pm} = 1/2 \int d^3x \, (\mathbf{Z}^{\pm})^2 = \int d^3x \, e^{\pm}(\mathbf{x})$

Geomagnetic pulsations

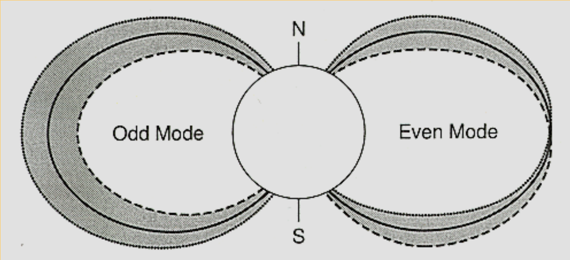
Fast magnetic fluctuations of the Earth's surface magnetic field (few mHz up to a few Hz), corresponding to oscillation periods from several hundreds to fractions of seconds. The pulsating quasi-sinusoidal (*continuous pulsations*) disturbances are observed globally and associated with Alfvén (and magneto-acoustic) waves. The *irregular pulsations* are short-lived and localized.





Field line resonances

The shown Pc5 pulsations are caused by oscillations of the Earth magnetic field and explained as standing single-fluid shear Alfvén waves, whose wavelengths must fit the geometry. The length, l, of the fieldline between two reflection point must be a multiple of half the wavelength, λ, implying: $\nu_h\lambda=2l$, with $\nu_h=1,2,3,\ldots$ From the dispersion relation, with the average Alfvén velocity, $\langle v_A \rangle$, along the fieldline one finds:

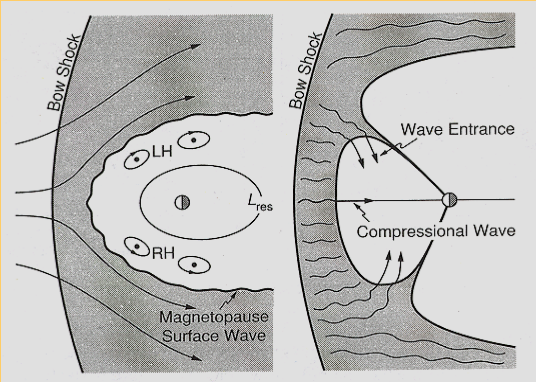


$$\omega_h = \frac{\nu_h \pi \langle v_A \rangle}{\ell}$$

Fundamental poloidal field-line resonances

Magnetosheath turbulence as source of pulsations

Excitation of global magnetospheric pulsations through driver at matching frequency: $\omega_{ex} = \omega_{res}$.



Excitation scenarios:

• Surface waves excited by ***Kelvin-Helmholtz instability*** driven by flow around magnetopause

• Compressional waves ***leaking from magnetosheath*** through polar cusp into magnetosphere