Plasma waves in the fluid picture I

- Langmuir oscillations and waves
- Ion-acoustic waves
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- Ordinary electromagnetic waves
- General wave equation
- General dispersion equation
- Dielectric response function
- Dispersion in a cold electron plasma

Langmuir oscillations I

Consider high-frequency electron oscillations (ions remain at rest) with displacement $\delta x$, causing an electric field, $\delta E$, and corresponding force, $-e\delta E$, on each electron, and thus leading to a density perturbation, $\delta n$, as shown below:
**Langmuir oscillations II**

The electron fluid equations read:

- **Continuity:**
  \[
  \frac{\partial \delta n}{\partial t} = -n_e \frac{\partial \delta \nu_{e,x}}{\partial x}
  \]

- **Momentum:**
  \[
  \frac{\partial \delta \nu_{e,x}}{\partial t} = -\frac{e}{m_e} \delta E
  \]

- **Poisson:**
  \[
  \frac{\partial \delta E}{\partial x} = \frac{\epsilon}{\epsilon_0} \delta n
  \]

Taking the time derivative yields the linear wave equation:

\[
\frac{\partial^2 \delta n}{\partial t^2} + \frac{n_e e^2}{m_e \epsilon_0} \delta n = 0
\]

It can easily be solved by a simple temporal oscillation with the angular frequency:

\[
\omega_{pe}^2 = \frac{n_e e^2}{m_e \epsilon_0}
\]

**Electron plasma frequency**

**Langmuir waves**

The plasma oscillations are somewhat artificial since the thermal electrons can move and change their (adiabatic) pressure such that the force balance is:

\[
\frac{\partial \delta \nu_{e,x}}{\partial t} = -\frac{e}{m_e} \delta E - \frac{\gamma_e k_B T_e}{m_e n_e} \frac{\partial \delta n}{\partial x}
\]

The wave equation contains now spatial dispersion as well and reads:

\[
\frac{\partial^2 \delta n}{\partial t^2} - \frac{\gamma_e k_B T_e}{m_e} \frac{\partial^2 \delta n}{\partial x^2} + \omega_{pe}^2 \delta n = 0
\]

A plane wave ansatz yields the dispersion relation with a lower cutoff at the electron plasma frequency. The electron thermal speed is \(v_{te} = (k_B T_e/m_e)^{1/2}\).

\[
\omega_i^2 = \omega_{pe}^2 + k^2 \gamma_e v_{te}^2
\]

-> **Langmuir oscillations become travelling electrostatic waves for \(k \neq 0\).**
**Ion acoustic waves I**

At frequencies below the electron oscillations the ions come into play. They contribute their own plasma frequency:

\[ \omega_{pi} = \left( \frac{n_i Z^2 e^2}{m_i \epsilon_0} \right)^{1/2} \]

which is for protons \((Z=1)\) by a large factor, \((m_i/m_e)^{1/2} = 43\), smaller than \(\omega_{pe}\).

When we assume quasineutrality, \(n_e = n_i\), the electron dynamics reduces to

\[ e \delta E = -\gamma_e k_B T_e \frac{\partial \ln n_e}{\partial x} \]

Upon linearization, \(n_e = n_0 + \delta n_e\), and with the electric field, \(\delta E = -\partial \Phi / \partial x\), the density fluctuation becomes proportional to the potential fluctuation:

\[ n_e = n_0 \exp \left( \frac{e \delta \phi}{\gamma_e k_B T_e} \right) \quad \frac{\partial \delta n_i}{\partial t} = -n_i \frac{\partial \delta v_{i,x}}{\partial x} \]

The linearized ion equations of motion:

**Ion acoustic waves II**

We neglect the ion pressure, assuming \(T_i \ll T_e\) and exploit charge-neutrality of the fluctuations, \(\delta n_e = \delta n_i = \delta n\), and thus arrive at a single wave equation:

\[ \frac{\partial^2 \delta n}{\partial t^2} - \frac{\gamma_e k_B T_e}{m_i} \frac{\partial^2 \delta n}{\partial x^2} = 0 \]

Its plane wave solution gives the dispersion of ion acoustic waves.

Upon division by \(k^2\) one finds the phase velocity, the ion acoustic speed, in which ions provide inertia and electrons the restoring force.

\[ c_{ia} = \left( \frac{\gamma_e k_B T_e}{m_i} \right)^{1/2} \]

*No electrostatic wave can propagate between \(\omega_{pe}\) and \(\omega_{pi}\) in an unmagnetized plasma.*
Ordinary electromagnetic waves I

The occurrence of electrostatic waves is a particular property of a plasma with free charges, which can also contribute to current oscillations. These become the source of electromagnetic waves, of which the magnetized plasma can carry a large variety. An electromagnetic wave of frequency \( \omega \) will set an electron in motion, creating a current density:

\[ \delta j_{em} = -en_0 \delta v_e \]

The velocity disturbance follows from the equation of motion in the electromagnetic plane wave field, \( \delta E \), yielding:

\[ \delta v_e = -\frac{ie}{\omega m_e} \delta E \]

From this we read of the current density, \( \delta j_{em} = \sigma_{em} \delta E \), resulting in the conductivity:

\[ \sigma_{em} = \frac{i\epsilon_0 \omega_e}{\omega} \]

The dispersion relation in terms of the refractive index, \( N \), is given by:

In vacuo, \( N^2 = 1 \). In plasma unity is replaced by the dielectric constant:

\[ \frac{k^2 c^2}{\omega^2} = \epsilon(\omega, k) \]

\[ N^2 = \frac{k^2 c^2}{\omega^2} \]
Ordinary electromagnetic waves II

There is a unique relation between conductivity and dielectric constant, which in our case can be written as:

$$\varepsilon(\omega) = 1 + \frac{i \sigma_{em}(\omega)}{\varepsilon_0 \omega} = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

With the help of this expression, we find the dispersion relation (of the ordinary mode):

$$\omega_{om}^2 = \omega_{pe}^2 + c^2 k^2$$

The wave number vanishes at the plasma frequency, which is a cut-off for the ordinary mode. Here $N^2$ becomes formally negative, the wave is reflected. See on the right side the ionospheric reflection of radio waves.

Ulysses wave data - day 73 in 1995

McDowell and Kellog, 2001

Power in grey scale in dB above noise
General wave equation

Maxwell’s equations including external sources, \( j_{\text{ex}} \) and \( \rho_{\text{ex}} \), read:

\[
\begin{align*}
\nabla \times \mathbf{B} &= \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \left( j + j_{\text{ex}} \right) \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \cdot \mathbf{E} &= \frac{1}{\varepsilon_0} (\rho + \rho_{\text{ex}})
\end{align*}
\]

Taking the time derivative of the first and replacing the magnetic field with the second, yields a general wave equation for the electric field:

\[
\nabla^2 \mathbf{E} - \nabla (\nabla \cdot \mathbf{E}) - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \left( \frac{\partial j}{\partial t} + \frac{\partial j_{\text{ex}}}{\partial t} \right)
\]

The conductivity tensor

We have included \( j_{\text{ex}} \) and \( \rho_{\text{ex}} \) explicitly, which may be imposed from outside on the plasma, but since the equations are linear in charge density and current can simply be added to the internal induced ones. They may be assumed to be given in linear response to the total field by Ohm’s law:

\[
j = \int d^3 x' \int_{-\infty}^{t} dt' \sigma \left( x, x', t, t' \right) \cdot \mathbf{E}(t', x')
\]

The integration from \(-\infty\) to \( t \) reflects causality (no effect before a cause).

The relation is constitutive for the material properties of a plasma and involves all microscopic particle motions. If the medium is stationary and uniform only the coordinate differences enter, and thus Ohm’s law reduces then to:

\[
j(t, x) = \int d^3 x' \int_{-\infty}^{t} dt' \sigma \left( x - x', t - t' \right) \cdot \mathbf{E}
\]
**General wave dispersion equation I**

Interpreting the electric field as a superposition of plane waves (Fourier analysis) with amplitude

$$\delta E(\omega, k) = \delta E_0(\omega, k) \exp(ik \cdot x - i\omega t)$$

Making use of the folding integral theorem transforms, after substantial formal algebra (left as an exercise), the wave differential into an algebraic (dyadic tensor) equation for the electric field amplitude:

$$\left[ \left( k^2 - \frac{\omega^2}{c^2} \right) I - kk - i\omega\mu_0\sigma(\omega, k) \right] \cdot \delta E_0(\omega, k) = 0$$

The requirement that the electric field be real leads to the symmetry conditions:

$$\delta E^*(\omega, k) = \delta E(-\omega, -k)$$
$$\sigma^*(\omega, k) = \sigma(-\omega, -k)$$

*Nontrivial solutions require: the determinant of the dynamic matrix vanishes.*

**General wave dispersion equation II**

In dielectric media it is convenient to use the electric induction and dielectric tensor, $\varepsilon(k, \omega)$, via the relation

$$\delta D = \varepsilon \cdot \delta E$$

With its help the current density is

$$\delta j(\omega, k) = -i\omega\varepsilon_0 \left[ \varepsilon(\omega, k) - I \right] \cdot \delta E(\omega, k)$$

Using Ohm’s law gives the general dielectric tensor and the dispersion relation defined as

$$\varepsilon(\omega, k) = I + \frac{i}{\omega\varepsilon_0} \sigma(\omega, k)$$

$$\text{Det} \left[ \frac{k^2 c^2}{\omega^2} \left( \frac{kk}{k^2} - I \right) + \varepsilon(\omega, k) \right] = 0$$

*Its solutions describe linear eigenmodes at frequency $\omega_0(k)$ and wavevector $k$.***
### Dispersion in an isotropic plasma

In this case only \( k \) defines a symmetry direction, and thus the unit tensor can be decomposed as follows:

\[
\begin{align*}
I_L &= \frac{kk}{k^2} \\
I_T &= I - \frac{kk}{k^2}
\end{align*}
\]

Corresponding to **longitudinal** and **transverse** components such that

\[
\epsilon(\omega, k) = \epsilon_L(\omega, k) I_L + \epsilon_T(\omega, k) I_T
\]

<table>
<thead>
<tr>
<th>( \epsilon_L(\omega, k) )</th>
<th>( \epsilon_T(\omega, k) )</th>
</tr>
</thead>
</table>
| 0 | \(
\frac{k \cdot \epsilon(\omega, k) \cdot k}{k^2}
\)
| \(
\frac{\epsilon(\omega, k)}{2}
\) |

*The dispersion relation in isotropic media splits into two separate parts, electrostatic and purely electromagnetic waves.*

### Dispersion in MHD-fluid theory

One-fluid magnethydrodynamics is only valid at low frequencies, \( \omega \ll (\omega_i, \omega_p) \), for long wavelengths, and for small phase speeds, such that \( \omega k \ll c \).

Near \( \omega_i, \omega_p, \omega_{ei}, \) and \( \omega_{ie} \) the ion and electron inertia becomes important. At high frequencies new waves appear which require single- or multi-fluid equations for their adequate description, to account for the natural **kinetic scales** (which MHD does not have) in a multi-component plasma.

*To derive the dispersion equation the induced current density must be calculated. The simplest model is the cold electron fluid in a strong field. Each new species introduces new dispersion branches.*
Dispersion in a cold electron plasma I

For cold (zero pressure) electrons the magnetic field is not affected by the electron motion and can be included in the gyrofrequency vector \( \omega_{ge} = eB_0/m_e \). The equations of motions for the fluctuations read:

\[
\begin{align*}
\frac{d\delta v_\parallel}{dt} &= \frac{e}{m_e} \delta E_\parallel \\
\frac{d\delta v_\perp}{dt} &= \frac{e}{m_e} \delta E_\perp + \omega_{ge} \times \delta v_\perp
\end{align*}
\]

Time differentiation yields the driven oscillator equation:

\[
\frac{\partial^2 \delta v_\perp}{\partial t^2} + \omega_{ge}^2 \delta v_\perp = \frac{e}{m_e} \left( \frac{\partial \delta E}{\partial t} + \omega_{ge} \times \delta E_\perp \right)
\]

Assuming a cold plasma means all electrons have the same speed, and thus the current density is simply:

\( \delta j = -en_0 \delta v = \sigma \cdot \delta E \)

Dispersion in a cold electron plasma II

For vanishing perpendicular electric field, the electrons perform a pure gyromotion. For the inhomogeneous solution part we make the ansatz of a periodic oscillation, \( \delta v = \exp(-i\omega t) \). After some vector algebra (left as an exercise) one obtains the frequency-dependent conductivity tensor (parallel component is independent of the field):

\[
\sigma(\omega) = \epsilon_0 \omega_{pe}^2 \begin{bmatrix}
\frac{i\omega}{\omega^2 - \omega_{pe}^2} & \frac{\omega_{ge}}{\omega^2 - \omega_{ge}^2} & 0 \\
\frac{\omega_{ge}}{\omega^2 - \omega_{ge}^2} & \frac{\omega_{pe}}{\omega^2 - \omega_{pe}^2} & 0 \\
0 & 0 & \frac{i}{\omega}
\end{bmatrix}
\]

From this equation the dielectric tensor follows by definition.

\[
\epsilon_{cold}(\omega) = \begin{bmatrix}
1 + \frac{\omega_{pe}^2}{\omega^2 - \omega_{ge}^2} & -\frac{i\omega_{pe}}{\omega} & \frac{\omega_{pe}}{\omega} \\
\frac{i\omega_{ge}}{\omega} & 1 + \frac{\omega_{pe}^2}{\omega^2 - \omega_{ge}^2} & 0 \\
0 & 0 & 1 - \frac{\omega_{pe}^2}{\omega^2}
\end{bmatrix}
\]
Dispersion in a cold electron plasma III

The cold electron plasma dispersion relation thus reads:

\[
\text{Det} \left[ \frac{k^2 c^2}{\omega^2} \left( I - \frac{\mathbf{k} \mathbf{k}}{k^2} \right) - \epsilon_{\text{cold}} \right] = 0
\]

We can write as shorthand for the dielectric tensor elements:

\[
\begin{bmatrix}
\epsilon_1 & -i \epsilon_2 & 0 \\
-i \epsilon_2 & \epsilon_1 & 0 \\
0 & 0 & \epsilon_3
\end{bmatrix}
\]

\[
\begin{align*}
\epsilon_1 &= 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ge}^2} \\
\epsilon_2 &= -\frac{\omega_{ge}}{\omega} \frac{\omega_{pe}^2}{\omega^2 - \omega_{ge}^2} \\
\epsilon_3 &= 1 - \frac{\omega_{pe}^2}{\omega^2}
\end{align*}
\]

Dispersion in a cold electron plasma IV

By using the vectorial refractive index, \( N = \mathbf{k} \cdot \mathbf{\hat{a}} \) with \( N^2 = N_{\perp}^2 + N_{\parallel}^2 \), and without loss of generality, \( k_y = 0 \), and \( \mathbf{k} \) in the \((x, z)\) plane, we obtain:

\[
\text{Det} \begin{bmatrix}
N_{\parallel}^2 - \epsilon_1 & i \epsilon_2 & -N_{\parallel} N_{\perp} \\
-i \epsilon_2 & N^2 - \epsilon_1 & 0 \\
-N_{\parallel} N_{\perp} & 0 & N_{\perp}^2 - \epsilon_3
\end{bmatrix} = 0
\]

Basic dispersion relation for a zero-temperature charge-compensated electron plasma, which is valid only for: \( k << 1/\rho_{ge} \) and \( v_{th} << \omega/k \).

We distinguish between parallel, \( N_{\parallel} = 0 \), and perpendicular, \( N_{\parallel} = 0 \), propagation, in which cases the dispersion relation factorises. The electric field has the components: \( \mathbf{E}_\parallel = E_y \mathbf{\hat{z}} + E_x \mathbf{\hat{x}} \), which suggests to use instead right-hand (R) and left-hand (L) circularly polarised components:

\[
\sqrt{2} \delta E_{R,L} = (\delta E_x \mp i \delta E_y)
\]
Plasma waves and frequencies in the solar wind

Non-uniformity leads to strong radial variations of the plasma parameters!

Wave phase velocities in the supersonic solar wind

Doppler shift: \( \omega' = \omega + k \cdot V_{sw} \)