Plasma waves in the fluid picture II

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Parallel electromagnetic waves I

We use the wave electromagnetic field components:

\[ \sqrt{2} \delta E_{R,L} = (\delta E_x \mp i \delta E_y) \]

They describe right-hand (R) and left-hand (L) polarized waves, as can be seen when considering the ratio

\[ \frac{\delta E_y}{\delta E_x} \bigg|_{R,L} = \pm i \]

This shows that the electric vector of the R-wave rotates in the positive while that of the L-wave in the negative y direction. The component transformation from \( \delta E_{x,y} \) to \( \delta E_{R,L} \) does not change the perpendicular electric field vector. Using the unitary matrix, \( U \), makes the dielectric tensor diagonal:

\[
U = \begin{bmatrix}
1/\sqrt{2} & -i/\sqrt{2} & 0 \\
-i/\sqrt{2} & 1/\sqrt{2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
U \cdot \varepsilon \cdot U^\dagger = \begin{bmatrix}
\varepsilon_R & 0 & 0 \\
0 & \varepsilon_L & 0 \\
0 & 0 & \varepsilon_3
\end{bmatrix}
\]
Parallel electromagnetic waves II

The components read:

\[ \epsilon_{R,L} = 1 - \frac{\omega_{pe}^2}{\omega(\omega \mp \omega_{ge})} \]

The dispersion relation for the transverse R and L wave reads:

\[ N^2 = \frac{k^2 c^2}{\omega^2} = \epsilon_{R,L} \]

The right-hand circularly polarised wave has the **refractive index**:

\[ \frac{k^2 c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega(\omega - \omega_{ge})} \]

This refractive index diverges for \( \omega \to 0 \) as well as for \( \omega \to \omega_{ge} \), where \( k \) diverges.

*Here \( \omega_{pe,\text{res}} = \omega_{ge} \) is the electron-cyclotron resonance frequency for the right-hand-polarised (RHP) parallel electromagnetic wave.*

Parallel electromagnetic waves III

**Resonances** indicate a complex interaction of waves with plasma particles. Here \( k \to \infty \) means that the wavelength becomes at constant frequency very short, and the wave momentum large. This leads to violent effects on a particle’s orbit, while resolving the microscopic scales. During this resonant interaction the waves may give or take energy from the particles leading to **resonant absorption or amplification** (growth) of wave energy.

\[ \omega/kc \sim \omega^{1/2} \]

At low frequencies, \( \omega \ll \omega_{ge} \), the above dispersion simplifies to the electron **Whistler mode**, yielding the typical falling tone in a sonogram as shown above.

\[ \omega = \frac{\omega_{ge}}{1 + \omega_{pe}^2/k^2 c^2} \]
Whistlers in the magnetosphere of Uranus and Jupiter

Wideband electric field spectra obtained by Voyager at Uranus on January 24, 1986.

Wave measurements made by Voyager I near the moon Io at a distance of 5.8 R_J from Jupiter.

Whistler mode waves at an interplanetary shock

\[ \omega_w = \omega_{ge}(kc/\omega_{pe})^2 \]

Gurnett et al., JGR 84, 541, 1979
Cut-off frequencies

Setting the refractive index $N$ for R-waves equal to zero, which means $k = 0$ at a finite $\omega$, leads to a second-order equation with the roots:

$$\omega_{R,co} = \frac{1}{2} \left[ \omega_{ge} + (\omega_{ge}^2 + 4\omega_{pe}^2)^{1/2} \right]$$

The left-hand circularly polarised wave has a refractive index given by:

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega(\omega + \omega_{ge})}$$

This refractive index does not diverge for $\omega \to \omega_{ge}$ and shows no cyclotron resonance. Moreover, since $N^2 < 1$ one has $\omega/k > c$.

The LHP waves have a low-frequency cut-off at

$$\omega_{L,co} = \frac{1}{2} \left[ (\omega_{ge}^2 + 4\omega_{pe})^{1/2} - \omega_{ge} \right]$$

Refractive index for parallel R- and L-waves

There is no wave propagation for $N^2 < 0$, regions which are called stop bands or domains where the waves are evanescent.
Dispersion branches for parallel R- and L-waves

The dispersion branches are for a dense (left) and dilute (right) plasma. Note the tangents to all curves, indicating that the group velocity is always smaller than \( c \). Note also that the R- and L-waves can not penetrate below their cut-off frequencies. The R-mode branches are separated by stop bands.

Perpendicular electromagnetic waves I

The other limiting case is purely perpendicular propagation, which means, \( \mathbf{k} = k_\perp \). In a uniform plasma we may chose \( \mathbf{k} \) to be in the \( x \)-direction. The cold plasma dispersion relation reduces to:

\[
\begin{vmatrix}
-\epsilon_1 & \imath \epsilon_2 & 0 \\
-\imath \epsilon_2 & N_\perp^2 - \epsilon_1 & 0 \\
0 & 0 & N_\perp^2 - \epsilon_3
\end{vmatrix} = 0
\]

Apparently, \( \delta E_r \) decouples from \( \delta E_\perp \), and the third tensor element yields the dispersion of the ordinary mode, which is denoted as O-mode. It is transverse, is cut off at the local plasma frequency and obeys:

\[
\omega_{om}^2 = \omega_{pe}^2 + k_\perp^2 c^2 \\
\epsilon_2^2 + \epsilon_1 (N_\perp^2 - \epsilon_1) = 0
\]
Perpendicular electromagnetic waves II

When inserting the tensor elements one obtains after some algebra (exercise!) the wave vector as a function of frequency in convenient form:

\[ k_1^2 e^2 = \frac{(\omega^2 - \omega_{R,co}^2)(\omega^2 - \omega_{L,co}^2)}{\omega^2 - \omega_{ge}^2 - \omega_{pe}^2} \]

Apparently, \( \delta E_x \) is now coupled with \( \delta E_y \), and this mode thus mixes longitudinal and transverse components. Therefore it is called the extraordinary mode, which is denoted as X-mode. It is resonant at the upper-hybrid frequency:

\[ \omega_{uh}^2 = \omega_{ge}^2 + \omega_{pe}^2 \]

The lower-frequency branch of the X-mode goes in resonance at this upper-hybrid frequency, and from there on has a stop-band up to \( \omega_{R,co} \).

Dispersion for perpendicular O- and X-waves

The dispersion branches are for a dense (left) and dilute (right) plasma. Note the tangents to all curves, indicating that the group velocity is always smaller than \( c \). Note that the O- and X-waves can not penetrate below the cut-off frequencies. The X-mode branches are separated by stop bands.
Two-fluid plasma waves

At low frequencies below and comparable to $\omega_{gi}$, the ion dynamics become important. Note that the ion contribution can be simply added to the electron one in the current and charge densities. The cold dielectric tensor is getting more involved. The elements read now:

$$
\begin{align*}
\epsilon_1 &= 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ge}^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{gi}^2} \\
\epsilon_2 &= \frac{\omega_{ge}}{\omega} \frac{\omega_{pe}^2}{\omega^2 - \omega_{ge}^2} + \frac{\omega_{gi}}{\omega} \frac{\omega_{pi}^2}{\omega^2 - \omega_{gi}^2} \\
\epsilon_3 &= 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2}
\end{align*}
$$

For parallel propagation, $k_\perp = 0$, the dispersion relation is:

$$
N_{R,L}^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega \mp \omega_{ge})} - \frac{\omega_{pi}^2}{\omega(\omega \pm \omega_{gi})}
$$

For perpendicular propagation, $k_\parallel = 0$, the dispersion relation can be written as:

$$
N_\perp^2 = \epsilon_1 - \frac{\epsilon_2^2}{\epsilon_1} = 0
$$

Lower-hybrid resonance

For perpendicular propagation the dispersion relation can be written as:

$$
N_\perp^2 = \epsilon_1 - \frac{\epsilon_2^2}{\epsilon_1} = 0
$$

At extremely low frequencies, we have the limits:

$$
\lim_{\omega \to 0} \epsilon_1 = \lim_{\omega \to 0} \epsilon_2 = 1 + \frac{c^2}{v_A^2}
$$

These are the dielectric constants for the X-mode waves. In that limit the refractive index is $N_\perp = \sqrt{\epsilon_1}$, and the Alfvén wave dispersion results:

$$
\omega = \pm k v_A \left(1 + \frac{v_A^2}{c^2}\right)^{-1/2}
$$

For $\epsilon_1 \to 0$, the lower-hybrid resonance occurs at:

$$
\omega_{lh}^2 = \frac{\omega_{pi}^2 + \omega_{gi}^2}{1 + \omega_{pe}^2/\omega_{ge}^2}
$$

It varies between the ion plasma and gyro frequency, and in dense plasma it is given by the geometric mean:

$$
\omega_{lh} = (\omega_{ge}\omega_{gi})^{1/2}
$$
Waves at the lower-hybrid frequency

Measurements of the AMPTE satellite in the plasmasphere of the Earth near 5 Re. Wave excitation by ion currents (modified two-stream instability).

\[ N_e \approx 40 \text{ cm}^{-3} \]
\[ T_e \approx \text{several eV} \]

\[ \omega_L/2\pi \approx 56 \text{ Hz} \]
\[ E_{\text{max}} \approx 0.6 \text{ mV/m} \]

Low-frequency dispersion branches

The dispersion branches are for parallel (left) and perpendicular (right) propagation. Note the tangents to all curves, indicating that the group velocity is always smaller than \( c \), and giving the Alfvén speed, \( v_A \), for small \( k \). Note that the Z-mode waves cannot penetrate below the cut-off frequency \( \omega_{\text{L,co}} \) and is trapped below \( \omega_{\text{L}} \). The X-mode branches are separated by stop bands.
General oblique propagation

The previous theory can be generalized to oblique propagation and to multi-ion plasmas. Following Appleton and Hartree, the cold plasma dispersion relation (with no spatial dispersion) in the magnetonic theory can be written as a biquadratic in the refractive index, \( N^2 = \frac{kc}{\omega} \).

\[
AN^4 - BN^2 + C = 0
\]

The coefficients are given by the previous dielectric functions, and there is now an explicit dependence on the wave propagation angle, \( \theta \), with respect to \( B \).

The coefficient \( A \) must vanish at the resonance, \( N \to \infty \), which yields the angular dependence of the resonance frequency on the angle \( \theta_{\text{res}} \) as:

\[
\tan^2 \theta_{\text{res}} = -\frac{\epsilon_3}{\epsilon_1}
\]

The coefficient \( C \) must vanish at the cut off, \( N \to 0 \), which means the cut-offs do not depend on \( \theta \).

Angular variation of the resonance frequencies

The two resonance frequencies for a dense (left) and dilute (right) pure electron plasma, following from the biquadratic equation:

\[
\omega_{\text{res}}^4 - \omega_{\text{pe}}^2 \omega_{\text{res}}^2 + \omega_{\text{pe}}^2 \omega_{\text{res}}^2 \cos^2 \theta = 0
\]
Frequency ranges of Z-, L-O- and R-X-mode waves

Top: Dynamics Explorer DE-1 satellite orbit

Left: Frequency ranges in the auroral region of the Earth magnetosphere

Radial distance /R_E

Whistler and Z-mode waves are trapped.

Electric field fluctuation spectra in the auroral zone

Measurements by the Dynamics Explorer DE-1 satellite in the Earth’s high-latitude auroral zone.

Maximal field strength at a few mV/m. Wave excitation by fast electrons at relativistic cyclotron resonance.

f_p = 9 (n_e/cm^3)^1/2 [kHz]

f_g = 28 B/nT [Hz]