Fundamentals of wave kinetic theory

• Introduction to the subject
• Perturbation theory of electrostatic fluctuations
• Landau damping - mathematics
• Physics of Landau damping
• Unmagnetized plasma waves
• The plasma dispersion function
• The dielectric tensor of a magnetized plasma

Introduction to the subject

The most general theory of plasma waves uses kinetic theory.

• Velocity distributions based on the Vlasov equation
• Wave equation based on the kinetic form of the induced current density (Maxwell’s equations unchanged)
• The dielectric tensor includes particle dynamics
• Self-consistent charge separation fields and currents become important
• Wave-particle interactions are accounted for
• Thermal effects lead to spatial dispersion and dissipation
Perturbation theory of electrostatic fluctuations

Consider a one-dimensional unmagnetized plasma. Vlasov equation:

\[
\frac{\partial f_{e,i}(v, x, t)}{\partial t} + v \frac{\partial f_{e,i}(v, x, t)}{\partial x} + \frac{e}{m_{e,i}} E(x, t) \frac{\partial f_{e,i}(v, x, t)}{\partial v} = 0
\]

Purely electrostatic field satisfies the Poisson equation:

\[
\frac{\partial E(x, t)}{\partial x} = \frac{e}{\varepsilon_0} \int_{-\infty}^{\infty} dv \left[ f_i(v, x, t) - f_{e}(v, x, t) \right]
\]

Consider fluctuations (waves) on a quiet background, such that the decomposition holds:

\[
f_{e,i} = f_{e,i0} + \delta f_{e,i}
\]

- Assume that the perturbations are linear, \(|\delta f| \ll f_0\)
- Assume stationary background VDF, \(f_0 = f_0(v)\)

Langmuir waves

Consider high-frequency fluctuations and electrons with immobile ions. The Vlasov-Poisson system reduces to the two equations:

\[
\frac{\partial \delta f(v, x, t)}{\partial t} + v \frac{\partial \delta f(v, x, t)}{\partial x} = \frac{e}{m_e} \delta E(x, t) \frac{\partial f_0(v, x, t)}{\partial v}
\]

\[
\frac{\partial \delta E(x, t)}{\partial x} = -\frac{e}{\varepsilon_0} \int_{-\infty}^{\infty} dv \delta f(v, x, t)
\]

Because the system is linear we may solve it by Fourier transformation in space. Note that \(\partial / \partial x \) transforms into \(ik\), such that we get the coupled system:

\[
\delta E(k, t) = \frac{ie}{\varepsilon_0 k} \int_{-\infty}^{\infty} dv \delta f(k, v, t)
\]

\[
\frac{\partial \delta f(k, v, t)}{\partial t} + ikv \delta f(k, v, t) - \frac{e}{m_e} \delta E(k, t) \frac{\partial f_0(k, v, t)}{\partial v} = 0
\]

We can solve this system by Laplace transformation.
Laplace transformation

The Laplace transform (variable $p = \gamma - i\omega$) and its inversion are

$$[\delta f(k, \nu, p), \delta E(k, p)] = \int_0^\infty dt \, [\delta f(k, \nu, t), \delta E(k, t)] e^{-pt}$$

$$\delta E(k, t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} dp e^{pt} \delta E(k, p)$$

Here $a$ is a real, large enough constant, and the integration contour is a line parallel to the imaginary axis in the complex $p$ plane, so that all singularities of the integrand are to the right in order to warrant convergence of the integral.

Laplace transform of the electric field I

Exercise: Calculate the Fourier-Laplace transform of the perturbations:

$$\delta f(k, \nu, p) = (p + ik\nu)^{-1} \left[ \frac{e}{m_e} \delta E(k, p) \frac{\partial f_0(\nu)}{\partial \nu} + g(k, \nu) \right]$$

$$\delta E(k, p) = \frac{ie}{\epsilon_0 k \epsilon(k, p)} \int_{-\infty}^\infty d\nu \frac{g(k, \nu)}{p + ik\nu}$$

The inhomogeneity $g(k, \nu) = \delta f(k, \nu, t=0)$ is the initial perturbation of the VDF. The electric field has poles at $p = -ik\nu$. Here the new term $\epsilon(k, p)$ is the well known dielectric function, which only depends on the speed-gradient of the background distribution function and reads:

$$\epsilon(k, p) = 1 - \frac{i\omega^2}{n_0 k} \int_{-\infty}^\infty d\nu \frac{\partial f_0(k, \nu, p)}{p + ik\nu}$$

The Laplace integral will have poles where $\epsilon(k, p) = 0$. The related solutions may be called, $\rho(k) = \gamma - i\omega$, where $p$ is split into its real and imaginary part.
Laplace transform of the electric field II

Integrating along \( a = \text{const} \) and then deforming the contours, whereby we pull \( a \) into the negative direction to position \( a' \) far beyond all poles which become encircled. The integral will be the sum of all residua, \( r_i \), at the poles, \( p_i(k) \), and of the contribution from the piecewise continuous path parallel to the imaginary axis, where use has been made of the Cauchy's integral theorem (check in a functional analysis book).

\[
\delta E(k, t) = \sum_i r_i(p_i) \exp[p_i(k)t] + (2\pi i)^{-1} \int_{a' - i\infty}^{a' + i\infty} dp e^{pt} \delta E(k, p)
\]

The integral contribution taken at \( a' \) vanishes in the long-time limit, \( t \to \infty \), as:

\[
\lim_{t \to \infty} \exp(-|a'|t) \to 0
\]

Of all residua only the one with smallest real part survives and yields as time-asymptotic solution the weakly damped eigen oscillation:

\[
\delta E(k, t) \propto \exp[\gamma_i(k)t - i\omega_i(k)t]
\]

Landau damping I

Langmuir waves when treated kinetically:

- Large number of wave modes (spread in VDF)
- Harmonic waves only appear asymptotically in time
- Collisionless damping appears, if \( \gamma_l(k) < 0 \).
- Plasma instability arises, if \( \gamma_l(k) > 0 \).

Plasma in thermal equilibrium, 1-D Maxwell VDF:

\[
f_0(v) = n_0 \left( \frac{m_e}{2\pi k_BT_c} \right)^{3/2} \exp \left( -\frac{m_v^2}{2k_BT_c} \right)
\]

Then the dielectric function (after partial) integration reads:

\[
\epsilon(k, p) = 1 - \frac{\omega_{pe}^2}{n_0k^2} \int_{-\infty}^{\infty} dv \frac{\partial f_0(v)/\partial v}{v - ip/k} = 1 + \frac{\omega_{pe}^2}{n_0} \int_{-\infty}^{\infty} dv \frac{f_0(v)}{(p + ikv)^2}
\]
Landau damping II

The Laplace integral may have poles where $\varepsilon(k, p) = 0$. Note that this is a complex function. The solutions may be called $i\omega_i(k) = \omega_i + i\gamma_i$. The integration is carried out in the complex $v$-plane. Integration contours for three possible positions of the pole:

$$
\int_{-\infty}^{\infty} dv \frac{\partial f_0(v)}{\partial v} \left( \frac{1}{v - ip/k} \right)
$$

$$
= \begin{cases} 
\int_{-\infty}^{\infty} dv \frac{\partial f_0(v)}{\partial v} 
& \gamma > 0 \\
\int_{-\infty}^{\infty} dv \frac{\partial f_0(v)}{\partial v} + 2\pi i \left. \frac{\partial f_0(v)}{\partial v} \right|_{v = ip/k} 
& \gamma < 0 
\end{cases}
$$

Contribution from negative pole

General damping rate

Let us split the dielectric function $\varepsilon(k, \omega, \gamma)$ in its real and imaginary part and expand about the real axis, assuming $\omega \gg \gamma$. This gives:

$$
\varepsilon(k, \omega, \gamma) = \varepsilon_r(k, \omega, 0) + i\gamma \left. \frac{\partial \varepsilon_r(k, \omega, \gamma)}{\partial \omega} \right|_{\gamma = 0} + i\varepsilon_i(k, \omega, 0) = 0
$$

Setting the real and imaginary parts separately equal to zero leads to the general solution for electrostatic waves:

$$
\varepsilon_r(k, \omega, 0) = 0
$$

$$
\gamma(k, \omega) = \frac{\varepsilon_i(k, \omega, 0)}{\varepsilon_r(k, \omega, \gamma) / \partial \omega|_{\gamma = 0}}
$$

The first equation gives the real frequency of the eigenmode, the second the damping rate of any weakly damped mode.
Damped Langmuir waves

Expanding \((p+ik\nu)^2\) in the real part of the dielectric function \(\varepsilon(k, p)\) gives:

\[
\varepsilon(k, p) = 1 + \frac{\omega_{pe}^2}{n_0p^2} \int_{-\infty}^{\infty} dv_r f_0(v_r) \left( 1 - \frac{2iv_r\nu_r}{p} - \frac{3k^2\nu_r^2}{p^2} \right) - 2\pi i \frac{\omega_{pe}^2}{n_0k^2} \frac{\partial f_0(v)}{\partial v} \Big|_{v=ip/k}
\]

**Exercise:** Carry out the three integrations (first moments of the Maxwellian), a procedure which yields the dispersion of Langmuir waves:

\[
\omega_\nu = \pm \omega_{pe} \left( 1 + \frac{3}{2}k^2\lambda_D^2 \right) + i\gamma(k)
\]

\[
\gamma(k) = -\left( \frac{\pi}{8} \right)^{1/2} \omega_{pe} \frac{\omega_{pe}}{k^3\lambda_D^3} \exp \left( -\frac{1}{2k^2\lambda_D^2} - \frac{3}{2} \right)
\]

The first equation gives the frequency of the Langmuir mode, the second is the Landau damping term due to thermal decorrelation effects. Note that for \(T_e \to 0\), \(\lambda_D \to 0\), and thus \(\gamma \to 0\).

Physics of Landau damping I

The collisionless dissipation of plasma oscillations is due to the subtle effects of the few particles being in resonance with the waves, i.e. with speeds close to the phase speed: \(v = v_{ph} = \omega/k\).

Maxwellian (left) and schematic wave-electron interaction
(wave as a quantum of momentum and energy)
Physics of Landau damping II

Individual wave-particle interaction is considered as an elastic collision conserving energy and momentum. **Why then wave damping?**
The reason is the asymmetry of the Maxwellian VDF at $v_{ph} = \omega / k$; there are more slow than fast particles.

- Wave looses more momentum/energy to slow particles
- Wave gains less momentum/energy from fast particles

The retarded and accelerated particles, right and left of the resonance, are accumulated at $\omega / k$. The VDF deforms and flattens, so as to locally balance gain and loss, $\Rightarrow$ **plateau formation.**

---

Ion acoustic waves I

Landau damping effects all wave modes in a thermal plasma. In addition, there are new modes owing their existance to the finite temperature. Consider an ion-electron plasma. The dispersion equation (with $\nu_p = \omega + i\gamma$) reads:

$$
\epsilon(k, p) = 1 + \frac{\omega_{pe}^2}{n_{0e}k^2} \int_{-\infty}^{\infty} \frac{dv f_{0e}(v)}{(v - ip/k)^2} + \frac{\omega_{pi}^2}{n_{0i}k^2} \int_{-\infty}^{\infty} \frac{dv f_{0i}(v)}{(v - ip/k)^2} = 0
$$

**Exercise:** Expand the electron and ion integrals such that the inequalities are fulfilled:

$$
\frac{k_B T_i}{m_i} \ll \frac{\omega^2}{k^2} \ll \frac{k_B T_e}{m_e}
$$

Such an expansion of $(v - ip/k)^{-2}$ in the dielectric function $\epsilon(k, \omega)$ gives the approximate real part of the dispersion relation:

$$
\epsilon(k, \omega) = 1 + \frac{1}{k^2 \lambda_D^2} \frac{\omega_{pe}^2}{\omega^2} \left( 1 + \frac{3k^2}{2} \frac{k_B T_i}{m_i} \right)
$$
Ion acoustic waves II

Solving the previous equation interatively gives the modified ion acoustic dispersion containing finite ion temperature effects:

$$\omega_{ia}^2 = \frac{\omega_{pi}^2}{1 + \frac{1}{k^2 \lambda_D^2}} \left[ 1 + \frac{3 T_i}{T_e} \left( 1 + k^2 \lambda_D^2 \right) \right]$$

In the long-wavelength limit, \((k \lambda_D)^2 \ll 1\), this yields the dispersionless ion acoustic wave, \(\omega = \pm k c_{ia}'\), with a slightly modified ion acoustic speed.

In the long-wavelength limit and for cold ions \((T_i \ll T_e)\) the damping is only small.

$$\gamma_{ia} \approx \omega_{ia} (\pi / 8)^{1/2} (m_e / m_i)^{1/2}$$

In the long-wavelength limit, \((k \lambda_D)^2 \ll 1\), this yields the dispersionless ion acoustic wave, \(\omega = \pm k c_{ia}'\), with a slightly modified ion acoustic speed.

Ion acoustic waves at a shock

\[ \omega = \omega_s + k \mathbf{V} \]

\[ \omega_s = c_s k / (1 + k^2 \lambda_D^2)^{1/2} \]

Gurnett et al., JGR 84, 541, 1979
Electron acoustic waves

Consider an electron plasma with two component, a hot \( n_h \) and cold \( n_c \) one, with \( T_c << T_h \), such as core and halo in the solar wind electron VDF. Then an \textit{electron acoustic wave} may exist, with the dispersion:

\[
\omega_{\text{ea}}^2 = \frac{\omega_{pe}^2}{1 + \frac{1}{k^2 \lambda_{Dh}^2}} \left( 1 + 3k^2 \lambda_{Dc}^2 + 3 \frac{n_{0h}}{n_{0c}} \frac{T_c}{T_h} \right)
\]

Electromagnetic waves in unmagnetized plasma

In previous lectures we derived the general wave and dispersion equations. What needs to be calculated kinetically is the \textit{induced current density}, by means of the perturbed VDF. Since we are interested in the final oscillating state, we can simply use a \textit{plane wave ansatz} in space and time and \textit{Fourier transform} the perturbed Vlasov equation. This gives:

\[
j(k, \omega) = -\sum_{s=e,i} \frac{q_s}{m_s} \int d^3 v \frac{\partial f_{s0}(v)}{\partial v} \frac{1}{\omega - k \cdot v} \delta E(k, \omega)
\]

The resulting dispersion relation for a warm unmagnetized plasma reads:

\[
(\omega^2 - k^2 c^2) I = -\sum_{s=e,i} \frac{\omega_{ps}^2}{n_0} \int d^3 v \frac{\partial f_{s0}(v)}{\partial v} \frac{1}{\omega - k \cdot v}
\]

Result: Dispersion of a free ordinary wave mode for large phase velocities \( (\omega \gg k v) \). It is practically \textit{undamped} as long as relativistic particle effects do not matter.

\[
\omega_{om}^2 = k^2 c^2 + \omega_{pe}^2
\]
The plasma dispersion function

In the calculation of the warm plasma dispersion relations one continuously encounters singular integrals of the kind:

\[ Z(\zeta) = \int_{-\infty}^{\infty} \frac{dx}{x - \zeta} \]

where \( f_0(x) \) is some equilibrium function, which is usually an analytic function of its arguments, \( x \), that is interpreted as the real part of a complex variable, \( x = x + iy \). The integral is taken along the entire real axis. For a Maxwellian this function is called the plasma dispersion function, which is related to the complex error function, \( Z(\zeta) = i\sqrt{\pi} \text{erf}(\zeta) \).

\[ Z(\zeta) = -\frac{1}{2} \pi \int_{-\infty}^{\infty} \frac{dx}{x - \zeta} \exp(-x^2) \]

For ions (electrons) and electrostatic waves the argument is: \( \zeta_{le} = \omega/kv_{thl,e} \).

Dispersion relation for a magnetized plasma

What we have to calculate here kinetically is the induced current density, by means of the perturbed VDF. The linearized Vlasov equation reads:

\[ \left( \frac{\partial}{\partial t} + v \cdot \nabla + \frac{q}{m} v \times B \cdot \frac{\partial}{\partial v} \right) \delta f(v, x, t) = -q/m (\delta E + v \times \delta B) \cdot \frac{\partial f_0(v)}{\partial v} \]

One can integrate this Vlasov equation in time over the unperturbed helical particle orbits to obtain \( \delta f(v) \), and then sum over the current contributions of the various species (left as a tedious exercise....) with a gyrotropic VDF. After considerable algebra, the full dielectric tensor is:

\[ \varepsilon(\omega, k) = \left( I - \sum_s \frac{\omega_{ps}^2}{\omega^2} \right) - \sum_{l=\infty}^{l=\infty} \frac{2\pi \omega_{ps}^2}{n_{0s} \omega^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_{||} d\nu_{||} d\nu_{\perp} \left( \frac{\partial f_{0s}}{\partial v_{||}} + \frac{\omega_{gs}}{v_{\perp}} \frac{\partial f_{0s}}{\partial v_{\perp}} \right) S_{ls}(v_{||}, v_{\perp}) \right) k_{||} v_{||} + \omega_{gs} - \omega \]
Particle resonances

As in the discussion of the Landau method for electrostatic modes, the damping of the eigenmodes of a magnetized plasma is largely determined by the poles in the integrand of the dielectric tensor. They are at the resonance positions, where

\[ \omega - k_|| v_|| - l \omega_{gs} = 0 \]

This corresponds to cyclotron resonance or in case, \( l = 0 \), to the Landau resonance, where the particle speed matches the phase speed.

The Doppler-shifted frequency of a resonant particle is a multiple harmonic of their gyrofrequency \( \rightarrow \) constant electric field
(in a circularly polarized wave). \( \rightarrow \) acceleration or deceleration

The resonant particles are responsible for the kinetic effects (wave damping and growth) in a magnetized warm plasma.

Electrostatic plasma waves

- Magnetized Langmuir and ion-acoustic waves
- Electron and ion Bernstein waves
- Lower- and upper-hybrid waves

Electron-cyclotron or Bernstein wave dispersion for \( k_\perp = 0 \). Here \( \Lambda_l \) is the modified Bessel function, with \( \eta_e = 0.5 \left( k_\perp v_{\perp e} / \omega_{pe} \right)^2 \).

\[
\omega_{ec}^2 = l^2 \omega_{ge}^2 \left[ 1 + \frac{2 \omega_{pe}^2}{\omega_{ge}^2} \frac{\Lambda_l(\eta_e)}{\eta_e} \right]
\]
Electromagnetic plasma waves

\[ \text{Det} \left[ \frac{k^2 c^2}{\omega^2} \left( \frac{kk}{k^2} - I \right) + \epsilon(\omega, k) \right] = 0 \]

- Whistler mode waves
- Ion cyclotron waves
- Kinetic Alfvén waves

**Kinetic Alfvén** waves propagate across the magnetic field and obey \( k_\perp \ll k_\parallel \approx 1/r_{gi}. \) They contain thermal effects of the ions. For cold electrons the dispersion is:

\[ \omega_{\text{ska}}^2 = k_\parallel^2 u_\parallel^2 \frac{1 + k_\perp^2 r_{gi}^2}{1 + k_\perp^2 c^2/\omega_{pe}^2} \]