Fundamentals of wave kinetic theory

- **Introduction to the subject**
- **Perturbation theory of electrostatic fluctuations**
- **Landau damping - mathematics**
- **Physics of Landau damping**
- **Unmagnetized plasma waves**
- **The plasma dispersion function**
- **The dielectric tensor of a magnetized plasma**

Introduction to the subject

The most general theory of plasma waves uses kinetic theory.

- Velocity distributions based on the Vlasov equation
- Wave equation based on the kinetic form of the induced current density (Maxwell's equations unchanged)
- The dielectric tensor includes particle dynamics
- Self-consistent charge separation fields and currents become important
- Wave-particle interactions are accounted for
- Thermal effects lead to spatial dispersion and dissipation

Perturbation theory of electrostatic fluctuations

Consider a one-dimensional unmagnetized plasma. Vlasov equation:

$$\frac{\partial f_{e,i}(v,x,t)}{\partial t} + v\frac{\partial f_{e,i}(v,x,t)}{\partial x} \pm \frac{e}{m_{e,i}}E(x,t)\frac{\partial f_{e,i}(v,x,t)}{\partial v} = 0$$

Purely electrostatic field satisfies the Poisson equation:

$$\frac{\partial E(x,t)}{\partial x} = \frac{e}{\epsilon_0}\int_{-\infty}^{\infty} dv\,[f_i(v,x,t) - f_e(v,x,t)]$$

Consider fluctuations (waves) on a quiet background, such that the decomposition holds:

$$f_{e,i} = f_{e,i0} + \delta f_{e,i}$$

- Assume that the perturbations are linear, $|\delta f| << f_0$
- Assume stationary background VDF, $f_0 = f_0(v)$

Langmuir waves

Consider high-frequency fluctuations and electrons with immobile ions. The Vlasov-Poisson system reduces to the two equations:

$$\frac{\partial \delta f(v,x,t)}{\partial t} + v\frac{\partial \delta f(v,x,t)}{\partial x} = \frac{e}{m_e}\delta E(x,t)\frac{\partial f_0(v,x,t)}{\partial v}$$

$$\frac{\partial \delta E(x,t)}{\partial x} = -\frac{e}{\epsilon_0}\int_{-\infty}^{\infty} dv\,\delta f(v,x,t)$$

Because the system is linear we may solve it by ***Fourier transformation*** in space. Note that $\partial/\partial \mathbf{x}$ transforms into $i\mathbf{k}$, such that we get the coupled system:

$$\delta E(k,t) = \frac{ie}{\epsilon_0 k}\int_{-\infty}^{\infty} dv\,\delta f(k,v,t)$$

$$\frac{\partial \delta f(k,v,t)}{\partial t} + ikv\,\delta f(k,v,t) - \frac{e}{m_e}\delta E(k,t)\frac{\partial f_0(k,v,t)}{\partial v} = 0$$

We can solve this system by ***Laplace transformation.***

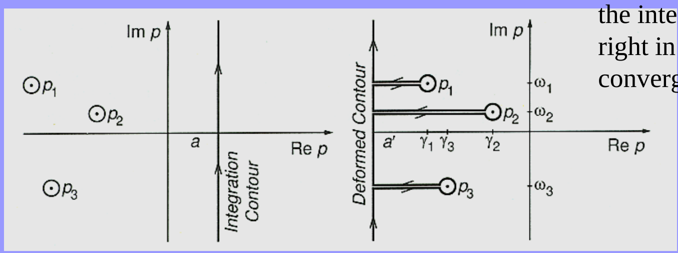
Laplace transformation

The Laplace transform (variable $p = \gamma - i\omega$) and its inversion are

$$[\delta f(k,v,p), \delta E(k,p)] = \int_0^\infty dt\, [\delta f(k,v,t), \delta E(k,t)]\, e^{-pt}$$

$$\delta E(k,t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} dp e^{pt} \delta E(k,p)$$

Here a is a real, large enough constant, and the integration contour is a line parallel to the imaginary axis in the complex p plane, so that all singularities of the integrand are to the right in order to warrant convergence of the integral.



Laplace transform of the electric field I

Exercise: Calculate the Fourier-Laplace transform of the perturbations:

$$\delta f(k,v,p) = (p + ikv)^{-1} \left[\frac{e}{m_e} \delta E(k,p) \frac{\partial f_0(v)}{\partial v} + g(k,v) \right]$$

$$\delta E(k,p) = \frac{ie}{\epsilon_0 k \epsilon(k,p)} \int_{-\infty}^{\infty} dv \frac{g(k,v)}{p + ikv}$$

The inhomogeneity $g(k,v) = \delta f(k,v,t{=}0)$ is the initial perturbation of the VDF. The electric field has poles at $p = -ikv$. Here the new term $\varepsilon(k,p)$ is the well known ***dielectric function***, which only depends on the speed-gradient of the background distribution function and reads:

$$\epsilon(k,p) = 1 - \frac{i\omega_{pe}^2}{n_0 k} \int_{-\infty}^{\infty} dv \frac{\partial f_0(k,v,p)/\partial v}{p + ikv}$$

The Laplace integral will have ***poles*** where $\varepsilon(k,p) = 0$. The related solutions may be called, $p_i(k) = \gamma_i - i\omega_i$, where p is split into its real and imaginary part.

Laplace transform of the electric field II

Integrating along a = const and then ***deforming the contours***, whereby we pull a into the negative direction to position a' far beyond all poles which become encircled. The integral will be the sum of all ***residua***, $r_i(k)$, at the poles, $p_i(k)$, and of the contribution from the picewise continuous path parallel to the imaginary axis, where use has been made of the ***Cauchy's intergral theorem*** (check in a functional analysis book).

$$\delta E(k,t) = \sum_i r_i(p_i) \exp\left[p_i(k)t\right] + (2\pi i)^{-1} \int_{a'-i\infty}^{a'+i\infty} dp e^{pt} \delta E(k,p)$$

The integral contribution taken at a' vanishes in the long-time limit, t -> ∞, as:

$$\lim_{t\to\infty} \exp(-|a'|t) \to 0$$

Of all residua only the one with smallest real part survives and yields as time-asymptotic solution the weakly damped ***eigen oscillation***

$$\delta E(k,t) \propto \exp\left[\gamma_l(k)t - i\omega_l(k)t\right]$$

Landau damping I

Langmuir waves when treated kinetically:

- **Large number of wave modes (spread in VDF)**
- **Harmonic waves only appear asymptotically in time**
- **Collisionless damping appears, if $\gamma_l(k) < 0$.**
- **Plasma instability arises, if $\gamma_l(k) > 0$.**

Plasma in thermal equilibrium, 1-D Maxwell VDF:

$$f_0(v) = n_0 \left(\frac{m_e}{2\pi k_B T_e}\right) \exp\left(-\frac{mv^2}{2k_B T_e}\right)$$

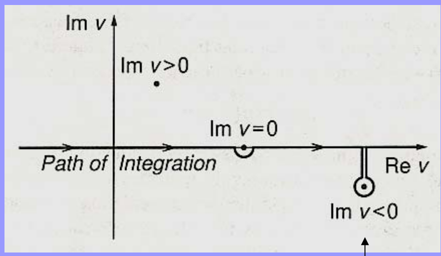
Then the dielectric function (after partial) integration reads:

$$\epsilon(k,p) = 1 - \frac{\omega_{pe}^2}{n_0 k^2} \int_{-\infty}^{\infty} dv \frac{\partial f_0(v)/\partial v}{v - ip/k} = 1 + \frac{\omega_{pe}^2}{n_0} \int_{-\infty}^{\infty} dv \frac{f_0(v)}{(p+ikv)^2}$$

Landau damping II

The Laplace integral may have poles where $\boldsymbol{\varepsilon(k, p)}$ **=** 0. Note that this is a complex function. The solutions may be called $ip_i(k) = \omega_i + i\gamma_i$. The integration is carried out in the complex v-plane. Integration contours for three possible positions of the pole:

$$\int_{-\infty}^{\infty} dv \frac{\partial f_0(v)/\partial v}{(v - ip/k)}$$



$$= \begin{cases} \int_{-\infty}^{\infty} dv_r \frac{\partial f_0(v)/\partial v}{(v_r - ip/k)} & \gamma > 0 \\ \int_{-\infty}^{\infty} dv_r \frac{\partial f_0(v)/\partial v}{(v_r - ip/k)} + 2\pi i \frac{\partial f_0(v)}{\partial v}|_{v=ip/k} & \gamma < 0 \end{cases}$$

Contribution from negative pole

General damping rate

Let us split the dielectric function $\varepsilon(k, \omega, \gamma)$ in its real and imaginary part and expand about the real axis, assuming $\omega >> \gamma$. This gives:

$$\begin{aligned} \epsilon(k,\omega,\gamma) &= \epsilon_r(k,\omega,0) + i\gamma \frac{\partial \epsilon_r(k,\omega,\gamma)}{\partial \omega}|_{\gamma=0} \\ &+ i\epsilon_i(k,\omega,0) = 0 \end{aligned}$$

Setting the real and imaginary parts separately equal to zero leads to the general solution for electrostatic waves:

$$\epsilon_r(k,\omega,0) = 0$$

$$\gamma(k,\omega) = -\frac{\epsilon_i(k,\omega,0)}{\partial \epsilon_r(k,\omega,\gamma)/\partial \omega|_{\gamma=0}}$$

The first equation gives the real ***frequeny of the eigenmode***, the second the ***damping rate*** of any weakly damped mode.

Damped Langmuir waves

Expanding $(p+ik\upsilon)^{-2}$ in the real part of the dielectric function $\varepsilon(k, p)$ gives:

$$\epsilon(k,p) = 1 + \frac{\omega_{pe}^2}{n_0 p^2} \int_{-\infty}^{\infty} d\upsilon_r f_0(\upsilon_r) \left(1 - \frac{2ik\upsilon_r}{p} - \frac{3k^2\upsilon_r^2}{p^2}\right) - 2\pi i \frac{\omega_{pe}^2}{n_0 k^2} \frac{\partial f_0(\upsilon)}{\partial \upsilon}\Big|_{\upsilon = ip/k}$$

Exercise: Carry out the three integrations (first moments of the Maxwellian), a procedure which yields the dispersion of Langmuir waves:

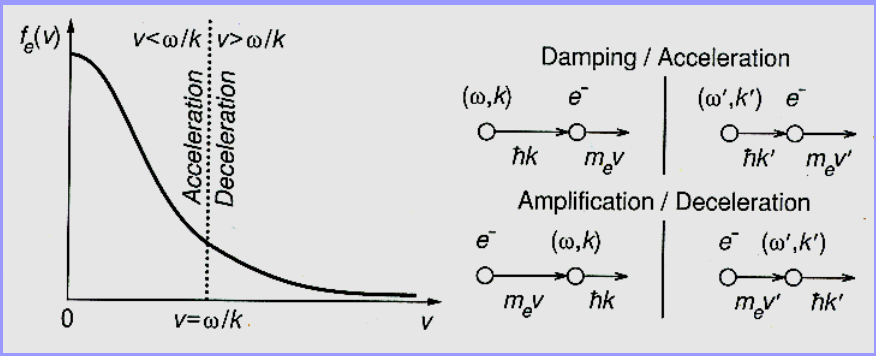
$$\omega_l = \pm\omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_D^2\right) + i\gamma_l(k)$$

$$\gamma_l(k) = -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_{pe}}{k^3\lambda_D^3} \exp\left(-\frac{1}{2k^2\lambda_D^2} - \frac{3}{2}\right)$$

The first equation gives the frequeny of the ***Langmuir mode***, the second is the ***Landau damping*** term due to thermal **decorrelation** effects. Note that for T_e -> 0, λ_D -> 0, and thus γ -> 0.

Physics of Landau damping I

The ***collisionless dissipation*** of plasma oscillations is due to the subtle effects of the few particles being in resonance with the waves, i.e. with speeds close to the phase speed: $\upsilon = v_{ph} = \omega/k$.



Maxwellian (left) and schematic wave-electron interaction (wave as a quantum of momentum and energy)

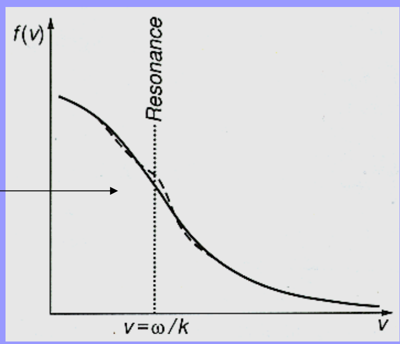
Physics of Landau damping II

Individual wave-particle interaction is considered as an elastic collision conserving energy and momentum. ***Why then wave damping?*** The reason is the asymmetry of the Maxwellian VDF at $v_{ph} = \omega/k$; there are more slow than fast particles.

- ***Wave looses more momentum/energy to slow particles***
- ***Wave gains less momentum/energy from fast particles***

The retarded and accelerated particles, right and left of the resonance, are accumulated at ω/k. The VDF deforms and flattens, so as to locally balance gain and loss,

-> ***plateau formation***.



Ion acoustic waves I

Landau damping effects all wave modes in a thermal plasma. In addition, there are new modes owing their existance to the finite temperature. Consider an ion-electron plasma. The dispersion equation (with $ip = \omega + i\gamma$) reads:

$$\epsilon(k,p) = 1 + \frac{\omega_{pe}^2}{n_{0e}k^2}\int_{-\infty}^{\infty}\frac{dv\, f_{0e}(v)}{(v-ip/k)^2} + \frac{\omega_{pi}^2}{n_{0i}k^2}\int_{-\infty}^{\infty}\frac{dv\, f_{0i}(v)}{(v-ip/k)^2} = 0$$

Exercise: Expand the electron and ion integrals such that the inequalities are fulfilled:

$$\frac{k_B T_i}{m_i} \ll \frac{\omega^2}{k^2} \ll \frac{k_B T_e}{m_e}$$

Such an expansion of $(v - ip/k)^{-2}$ in the dielectric function $\varepsilon(k, p)$ gives the approximate real part of the dispersion relation:

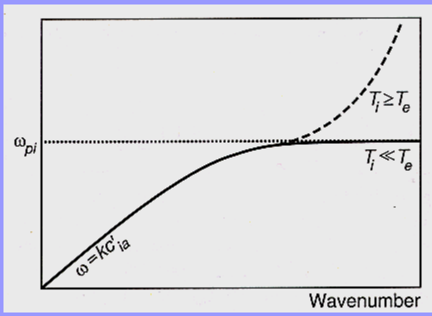
$$\epsilon(k,\omega) = 1 + \frac{1}{k^2\lambda_D^2} - \frac{\omega_{pi}^2}{\omega^2}\left(1 + \frac{3k^2}{\omega^2}\frac{k_B T_i}{m_i}\right)$$

Ion acoustic waves II

Solving the previous equation interatively gives the modified ion acoustic dispersion containing finite ion temperature effects:

$$\omega_{ia}^2 = \frac{\omega_{pi}^2}{1 + 1/k^2\lambda_D^2}\left[1 + \frac{3T_i}{T_e}\left(1 + k^2\lambda_D^2\right)\right]$$

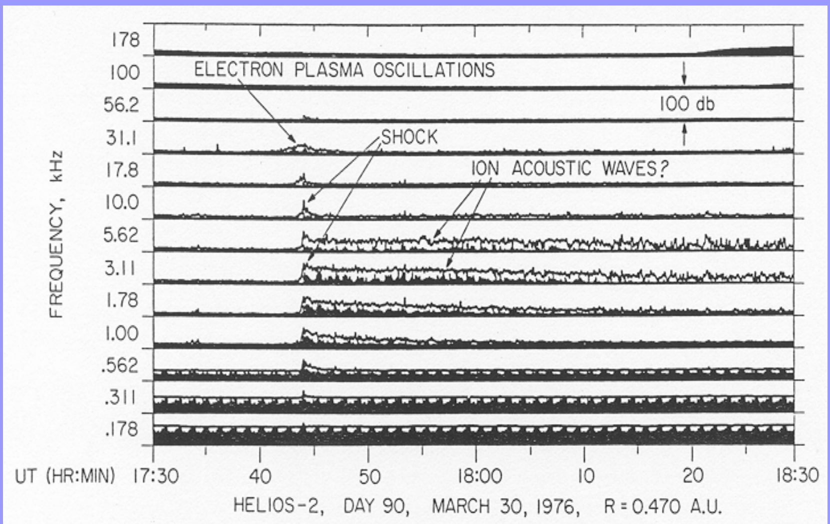
In the long-wavelength limit, $(k\lambda_D)^2 << 1$, this yields the dispersionless ion acoustic wave, $\omega = \pm\ kc_{ia}'$, with a slightly modified ion acoustic speed.



In the long-wavelength limit and for cold ions ($T_i << T_e$) the damping is only small.

$$\gamma_{ia} \approx \omega_{ia}(\pi/8)^{1/2}(m_e/m_i)^{1/2}$$

Ion acoustic waves at a shock



Gurnett et al., JGR **84**, 541, 1979

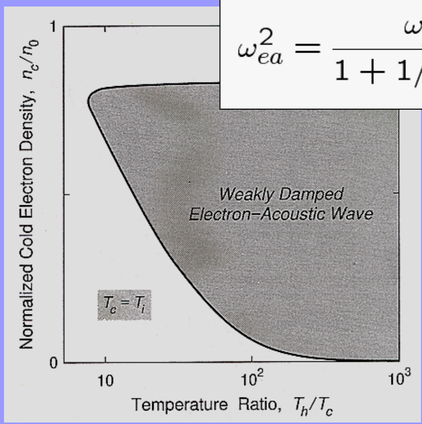
$\omega = \omega_s + \mathbf{k}\cdot\mathbf{V}$

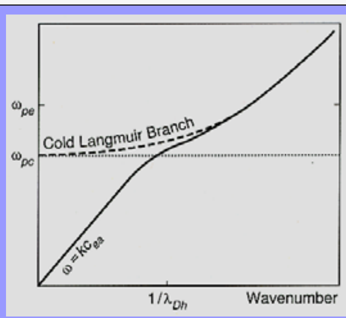
$\boxed{\omega_s = c_s k/(1+k^2\lambda_D^2)^{1/2}}$

Electron acoustic waves

Consider an electron plasma with two component, a hot (n_h) and cold (n_c) one, with $T_c << T_h$, such as core and halo in the solar wind electron VDF. Then an ***electron acoustic wave*** may exist, with the dispersion:

$$\omega_{ea}^2 = \frac{\omega_{pc}^2}{1 + 1/k^2\lambda_{Dh}^2}\left(1 + 3k^2\lambda_{Dc}^2 + 3\frac{n_{0h}}{n_{0c}}\frac{T_c}{T_h}\right)$$





Electromagnetic waves in unmagnetized plasma

In previous lectures we derived the general wave and dispersion equations. What needs to be calculated kinetically is the ***induced current density****, by means of the perturbed VDF. Since we are interested in the final oscillating state, we can simply use a* ***plane wave ansatz*** *in space and time and* ***Fourier transform*** *the perturbed Vlasov equation. This gives:*

$$\mathbf{j}(\mathbf{k},\omega) = -\sum_{s=e,i}\frac{q_s}{m_s}\int d^3v\,\mathbf{v}\frac{\partial f_{s0}(v)/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}\cdot\delta\mathbf{E}(\mathbf{k},\omega)$$

The resulting dispersion relation for a warm unmagnetized plasma reads:

$$\left(\omega^2 - k^2c^2\right)\mathbf{I} = -\sum_{s=e,i}\frac{\omega_{ps}^2}{n_0}\int\frac{d^3v\,\omega}{\omega - \mathbf{k}\cdot\mathbf{v}}\mathbf{v}\frac{\partial f_{s0}(v)}{\partial\mathbf{v}}$$

Result: Dispersion of a free ordinary wave mode for large phase velocities (ω >> **k**·**v**). It is practically ***undamped*** as long as relativistic particle effects do not matter.

$$\omega_{om}^2 = k^2c^2 + \omega_{pe}^2$$

The plasma dispersion function

In the calculation of the warm plasma dispersion relations one continuously encounters singular integrals of the kind:

$$Z(\zeta) = \int_{-\infty}^{\infty} \frac{dx\, f_0(x)}{x-\zeta}$$

where $f_0(x)$ is some equilibrium function, which is usually an analytic function of its arguments, x, that is interpreted as the real part of a complex variable, $z=x+iy$. The integral is taken along the entire real axis. For a Maxwellian this function is called the ***plasma dispersion function***, which is related to the complex error function, $Z(\zeta)=i\sqrt{\pi}\,\mathrm{erf}(\zeta)$.

$$Z(\zeta) = \pi^{-1/2} \int_{-\infty}^{\infty} \frac{dx\, \exp(-x^2)}{x-\zeta}$$

For ions (electrons) and electrostatic waves the argument is: $\zeta_{i,e} = \omega / k v_{thi,e}$.

Dispersion relation for a magnetized plasma

What we have to calculate here kinetically is the ***induced current density***, by means of the perturbed VDF. The linearized Vlasov equation reads:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{q}{m}\mathbf{v}\times\mathbf{B}_0\cdot\frac{\partial}{\partial \mathbf{v}}\right)\delta f(\mathbf{v},\mathbf{x},t) = -\frac{q}{m}(\delta\mathbf{E}+\mathbf{v}\times\delta\mathbf{B})\cdot\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}}$$

One can integrate this Vlasov equation in time over the ***unperturbed helical particle orbits*** to obtain $\delta f(\mathbf{v})$, and then sum over the current contributions of the various species (left as a tedious exercise.....) with a ***gyrotropic*** VDF. After considerable algebra, the full dielectric tensor is:

$$\epsilon(\omega,\mathbf{k}) = \left(1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\right)\mathbf{I} - \sum_s \sum_{l=-\infty}^{l=\infty} \frac{2\pi\omega_{ps}^2}{n_{0s}\omega^2} \int_0^{\infty}\int_{-\infty}^{\infty} v_\perp dv_\perp dv_\parallel \left(k_\parallel \frac{\partial f_{0s}}{\partial v_\parallel} + \frac{l\omega_{gs}}{v_\perp}\frac{\partial f_{0s}}{\partial v_\perp}\right)\frac{\mathbf{S}_{ls}(v_\parallel,v_\perp)}{k_\parallel v_\parallel + l\omega_{gs} - \omega}$$

Particle resonances

As in the discussion of the Landau method for electrostatic modes, the ***damping of the eigenmodes*** of a magnetized plasma is largely determined by the ***poles*** in the integrand of the dielectric tensor. They are at the ***resonance*** positions, where

$$\omega - k_{||}v_{||} - l\omega_{gs} = 0$$

This corresponds to ***cyclotron resonance*** or in case, $l = 0$, to the Landau resonance, where the particle speed matches the phase speed.

The Doppler-shifted frequency of a resonant particle is a multiple harmonic of their gyrofrequency --> **constant electric field** (in a circularly polarized wave). --> **acceleration or deceleration**

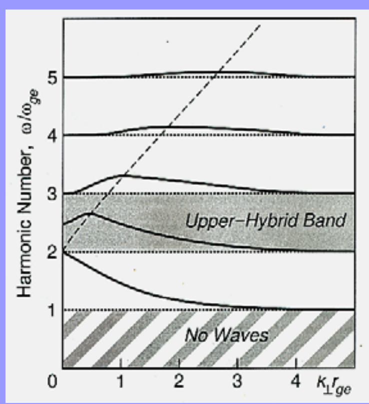
*The **resonant** particles are responsible for the kinetic effects (**wave damping and growth**) in a magnetized warm plasma.*

Electrostatic plasma waves

- Magnetized Langmuir and ion-acoustic waves
- Electron and ion Bernstein waves
- Lower- and upper-hybrid waves

Electron-cyclotron or Bernstein wave dispersion for $k_{||} = 0$. Here Λ_l is the modified Bessel function, with $\eta_e = 0.5(k_\perp v_{the\perp}/\omega_{ge})^2$.

$$\omega_{ec}^2 = l^2\omega_{ge}^2\left[1 + \frac{2\omega_{pe}^2}{\omega_{ge}^2}\frac{\Lambda_l(\eta_e)}{\eta_e}\right]$$



Electromagnetic plasma waves

$$Det\left[\frac{k^2c^2}{\omega^2}\left(\frac{\mathbf{kk}}{k^2}-\mathbf{I}\right)+\epsilon(\omega,\mathbf{k})\right]=0$$

- Whistler mode waves
- Ion cyclotron waves $\longrightarrow$
- Kinetic Alfvén waves

$$\frac{k_\parallel^2 v_A^2}{\omega_{gi}^2}=\frac{\omega^2}{\omega_{gi}(\omega_{gi}-\omega)}$$

Kinetic Alfvén waves propagate across the magnetic field and obey $k_\parallel << k_\perp \approx 1/r_{gi}$. They contain thermal effects of the ions. For cold electrons the dispersion is:

$$\omega_{ska}^2 = k_\parallel^2 v_A^2 \frac{1+k_\perp^2 r_{gi}^2}{1+k_\perp^2 c^2/\omega_{pe}^2}$$