Single particle motion and trapped particles

- **Gyromotion of ions and electrons**
- **Drifts in electric fields**
- **Inhomogeneous magnetic fields**
- **Magnetic and general drift motions**
- **Trapped magnetospheric particles**
- **Motions in a magnetic dipole field and planetary radiation belts**

Gyration of ions and electrons I

The equation of motion for a particle in a magnetic field is:

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Taking the dot product with **v** yields (for $\boldsymbol{E} = \mathbf{0}$) the equation:

$$m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{d}{dt}\left(\frac{mv^2}{2}\right) = 0$$

A magnetic field can not change the particle‘s energy. If $\boldsymbol{B} = B\mathbf{e}_z$,and B is uniform, then υ_z is constant; taking the second derivative yields:

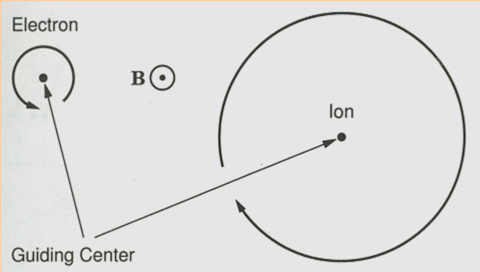
$$\begin{aligned} \ddot{v}_x &= -\omega_g^2 v_x \\ \ddot{v}_y &= -\omega_g^2 v_y \end{aligned}$$

We introduced the gyrofrequency, $\omega_g = qB/m$, with charge q and mass m of the particle.

Gyration of ions and electrons II

The previous equation for a harmonic oscillator has the solution:

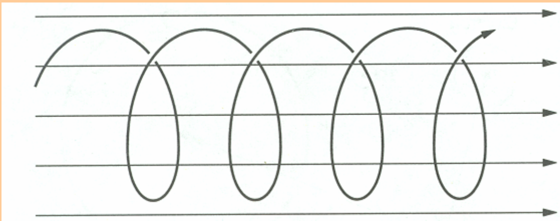
$$\begin{aligned} x - x_0 &= r_g \sin \omega_g t \\ y - y_0 &= r_g \cos \omega_g t \end{aligned}$$



The equation describes a circular orbit around the field with gyroradius, r_g, and gyrofrequency, ω_g. The orbit's center (x_0,y_0) is called the ***guiding center***. The gyration represents a microcurrent, which creates a field opposite to the background one. This behaviour is called ***diamagnetic effect***.

Gyration of ions and electrons III

Helicoidal ion orbit in a uniform magnetic field



$$r_g = \frac{v_\perp}{|\omega_g|} = \frac{m v_\perp}{|q| B}$$

$$\alpha = \tan^{-1}\left(\frac{v_\perp}{v_\|}\right)$$

If one includes a constant speed parallel to the field, the particle motion is three-dimensional and looks like a ***helix***. The ***pitch angle*** of the helix or particle velocity with respect to the field depends on the ratio of perpendicular to parallel velocity components.

Electric drifts I

Adding an electric field to the magnetic field results in ***electric drift motion***, the nature of which depends on whether the field is nonuniform in space or variable in time. A parallel field component yields straight acceleration along the magnetic field:

$$\boxed{m\dot{v}_{\|} = qE_{\|}}$$

Particles in space plasmas are usually very ***mobile along the magnetic field***. A perpendicular electric field component (in x-axis) leads to the famous $\boldsymbol{E} \times \boldsymbol{B}$ drift:

$$\begin{aligned} \dot{v}_x &= \omega_g v_y + \frac{q}{m} E_x \\ \dot{v}_y &= -\omega_g v_x \end{aligned}$$

$$\begin{aligned} \ddot{v}_x &= -\omega_g^2 v_x \\ \ddot{v}_y &= -\omega_g^2 \left(v_y + \frac{E_x}{B} \right) \end{aligned}$$

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

The $\boldsymbol{E} \times \boldsymbol{B}$ drift does not depend on the charge, thus electrons and ions drift in the same direction!

Electric drifts II

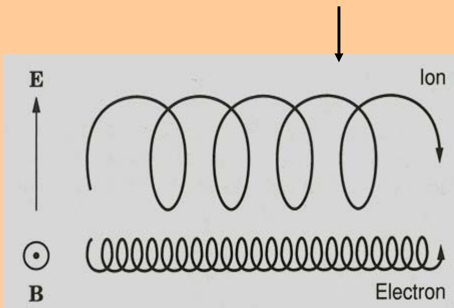
The $\boldsymbol{E} \times \boldsymbol{B}$ drift has a close link to the Lorentz transformation, because a particle can by drifting transform the external electric field away, such that in its rest frame the electric field vanishes:

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \qquad \mathbf{E} = -\mathbf{v} \times \mathbf{B}$$

Solving the last equation for the velocity, yields the previous $\boldsymbol{E} \times \boldsymbol{B}$ drift.

For slowly time-varying electric fields, particles perform a ***polarization drift*** perpendicular to the magnetic field.

$$\mathbf{v}_p = \frac{1}{\omega_g B} \frac{d\mathbf{E}_\perp}{dt}$$



Magnetic drifts I

Inhomogeneity will lead to a drift. A typical magnetic field in space will have gradients, and thus field lines will be curved. We Taylor expand the field:

$$\boxed{\mathbf{B} = \mathbf{B}_0 + (\mathbf{r} \cdot \nabla)\mathbf{B}_0}$$

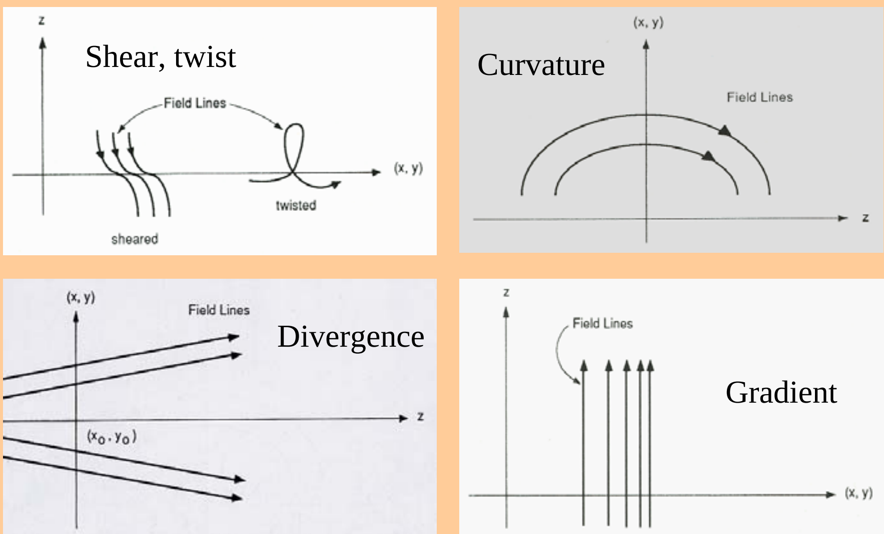
where $\boldsymbol{B}_0$ is measured at the guiding center and **r** is the distance from it. The modified equation of motion then reads:

$$\boxed{m\frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B}_0) + q\left[\mathbf{v} \times (\mathbf{r} \cdot \nabla)\mathbf{B}_0\right]}$$

Expanding the velocity in the small drift plus gyromotion, $\mathbf{v} = \mathbf{v}_g + \mathbf{v}_\nabla$, then we find the stationary drift:

$$\boxed{\mathbf{v}_\nabla = \frac{1}{B_0^2}\langle(\mathbf{v}_g \times (\mathbf{r} \cdot \nabla)\mathbf{B}_0) \times \mathbf{B}_0\rangle}$$

Nonuniform magnetic fields in space

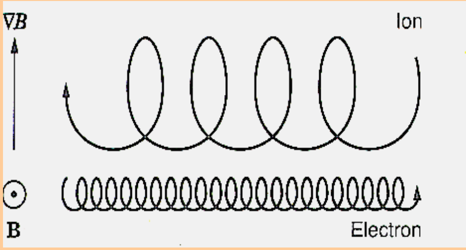


Magnetic drifts II

By inserting the previous analytic solution for the helical gyration orbit, we can time average over a gyroperiod and thus obtain the expression:

$$\boxed{\mathbf{v}_\nabla = \frac{mv_\perp^2}{2qB^3}(\mathbf{B} \times \nabla B)}$$

showing that the non-uniform magnetic field **B** leads to a ***gradient drift*** perpendicular to both, the field and its gradient, as sketched below:



The ratio in front of the gradient term is the particle's **magnetic moment**, i.e. the ratio between the kinetic energy and the magnetic field:

$$\boxed{\mu = \frac{mv_\perp^2}{2B} = \frac{W_\perp}{B}}$$

General force drifts

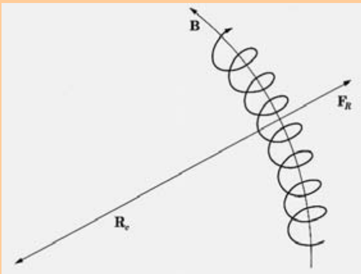
By replacing the electric field ***E*** in the drift formula by any field exerting a force ***F***/*q*, we obtain the general ***guiding-center drift***:

$$\boxed{\mathbf{v}_F = \frac{1}{\omega_g}\left(\frac{\mathbf{F}}{m} \times \frac{\mathbf{B}}{B}\right)}$$

In particular when the field lines are curved, the centrifugal force is

$$\boxed{\mathbf{F}_R = mv_\parallel^2 \frac{\mathbf{R}_c}{R_c^2}}$$

where $\mathbf{R}_c$ is the local radius of curvature. The particle kinematics is illustrated on the right.

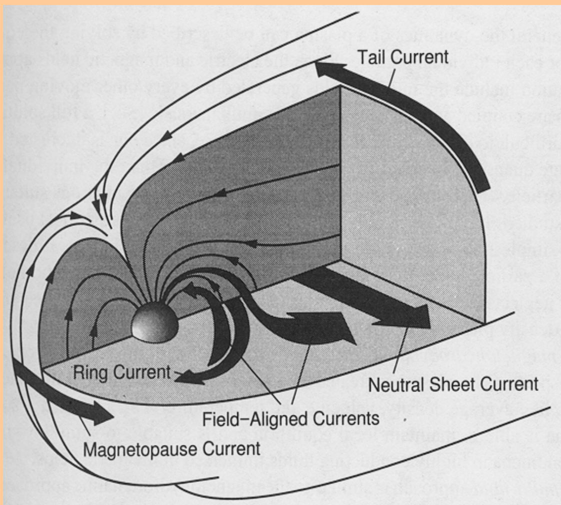


Summary of guiding center drifts

E×B Drift:	$\mathbf{v}_E = \dfrac{\mathbf{E}\times\mathbf{B}}{B^2}$	
Polarization Drift:	$\mathbf{v}_P = \dfrac{1}{\omega_g B}\dfrac{d\mathbf{E}_\perp}{dt}$	$\mathbf{j}_P = \dfrac{n_e(m_i+m_e)}{B^2}\dfrac{d\mathbf{E}_\perp}{dt}$
Gradient Drift:	$\mathbf{v}_\nabla = \dfrac{mv_\perp^2}{2qB^3}(\mathbf{B}\times\nabla B)$	$\mathbf{j}_\nabla = \dfrac{n_e(\mu_i+\mu_e)}{B^2}(\mathbf{B}\times\nabla B)$
Curvature Drift:	$\mathbf{v}_R = \dfrac{mv_\parallel^2}{qR_c^2B^2}(\mathbf{R}_c\times\mathbf{B})$	$\mathbf{j}_R = \dfrac{2n_e(W_{i\parallel}+W_{e\parallel})}{R_c^2B^2}(\mathbf{R}_c\times\mathbf{B})$

Associated with all these drift are corresponding drift currents.

Synopsis of the magnetospheric current system



The distortion of the Earth‘s dipole field is accompanied by a current system. The currents can be guided by the strong background field, so-called ***field-aligned currents*** (like in a wire), which connect the polar cap with the magnetotail regions. The compression of the dayside field leads to the ***magnetopause current***. A ***tail current*** flows on the tail surface and as a ***neutral sheet current*** in the interior. The ***ring current*** is carried by radiation belt particles flowing around the Earth in east-west direction.

Magnetic mirror

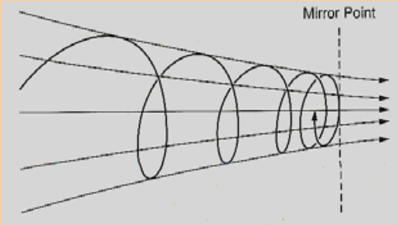
Let us follow the guiding center of a particle moving along an inhomogeneous magnetic field by considering the magnetic moment:

$$\mu = \frac{mv^2 \sin^2 \alpha}{2B}$$

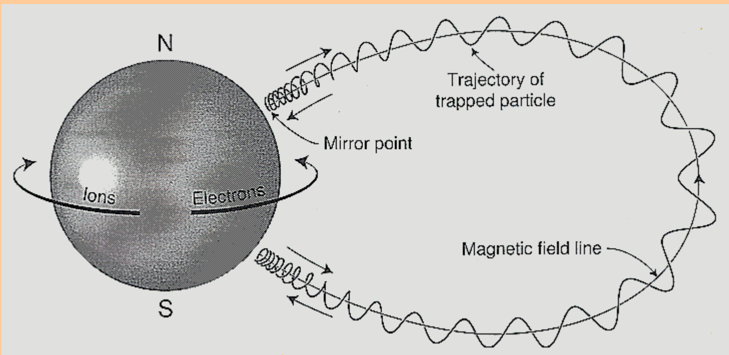
where we used the ***pitch angle*** α. Apparently, pitch angles at different locations are related by the corresponding magnetic field strengths:

$$\frac{\sin^2 \alpha_2}{\sin^2 \alpha_1} = \frac{B_2}{B_1}$$

The point where the angle reaches 90° is called the ***mirror point***.

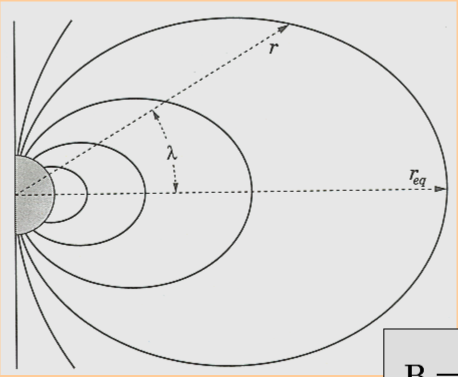


Trajectories of particles confined in a dipole field



A ***dipole magnetic field*** has a field strength minumum at the equator and converging field lines at the polar regions (mirrors). Particles can be ***trapped*** in such a field. ***They perform gyro, bounce and drift motions***.

Magnetic dipole field



At distances not too far from the surface the Earth's magnetic field can be approximated by an azimuthally symmetric ***dipole field*** with a moment:

$\boldsymbol{M_E} = \mathbf{8.05\ 10^{22}\ Am^2}$

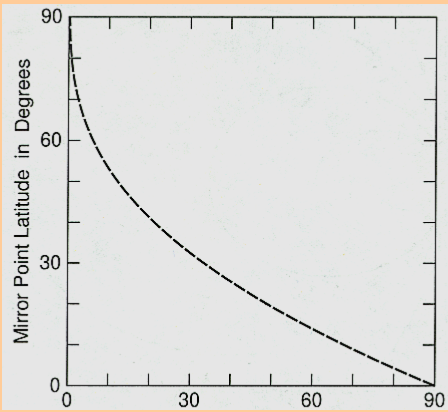
$$\mathrm{B} = \frac{\mu_0}{4\pi}\frac{M_E}{r^3}(-2\sin\lambda\hat{\mathrm{e}}_r + \cos\lambda\hat{\mathrm{e}}_\lambda)$$

Measuring the distance in units of the Earth's radius, R_E, and using the equatorial surface field, B_E (= 0.31 G), yields with the so-called ***L-shell*** parameter ($L=r_{eq}/R_E$) the field strength as a function of latitude, λ, and of L as:

$$B(\lambda, L) = \frac{B_E}{L^3}\frac{(1+3\sin^2\lambda)^{1/2}}{\cos^6\lambda}$$

Dipole latitudes of mirror points

Magnetic latitude λ_m of particle's mirror point.

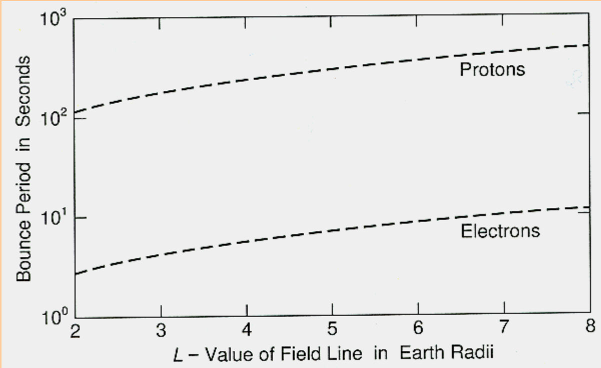


Equatorial pitch angle in degrees

Latitude of mirror point depends only on pitch angle but not on L shell value.

$$\sin^2\alpha_{eq} = \frac{B_{eq}}{B_m} = \frac{\cos^6\lambda_m}{(1+3\sin^2\lambda_m)^{1/2}}$$

Bounce period as function of L shell

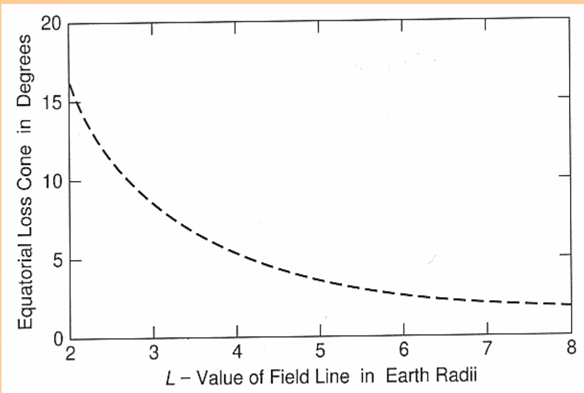


Energy, W, is here 1 keV and α_{eq} = 30°.

Bounce period, τ_b, is the time it takes a particle to move back and forth between the two mirror points (s is the path length along a given field line).

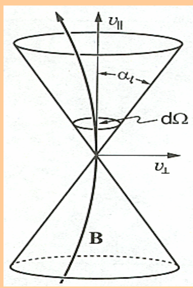
$$\tau_b = 4 \int_0^{\lambda_m} \frac{ds}{v_\parallel} = 4 \int_0^{\lambda_m} \frac{ds}{d\lambda} \frac{d\lambda}{v_\parallel}$$

Equatorial loss cone for different L-values



If the mirror point lies too deep in the atmosphere (below 100 km), particles will be absorbed by collisions with neutrals.

Equatorial loss cone

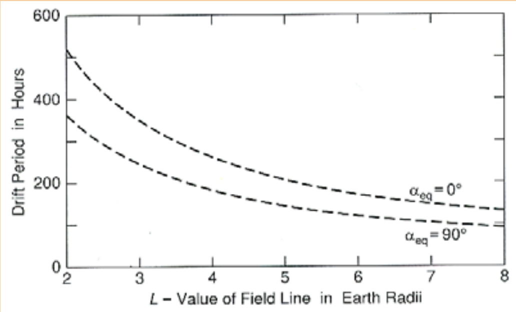


$$\sin^2 \alpha_\ell = (4L^6 - 3L^5)^{-1/2}$$

The loss-cone width depends only on L but not on the particle mass, charge or energy.

Period of azimuthal magnetic drift motion

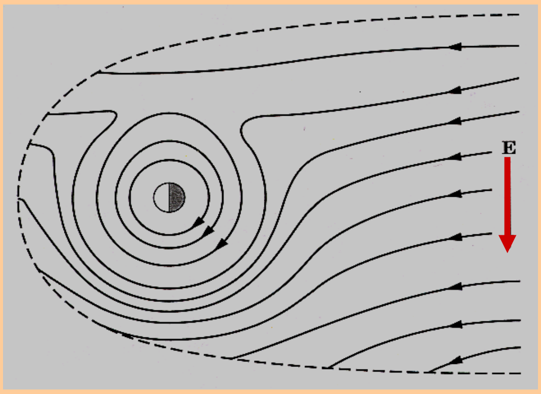
Here the energy, W, is 1 keV and the pitch angle: α_{eq} = 30° and 90°.



$$\langle \tau_d \rangle \approx \frac{\pi q B_E R_E^2}{3LW}(0.35 + 0.15 \sin \alpha_{eq})^{-1}$$

Drift period is of order of several days. Since the magnetospheric field changes on smaller time scales, it is unlikely that particles complete an undisturbed drift orbit. Radiation belt particles will thus undergo **radial (*L*-shell) diffusion**!

Ion paths due to electric drifts (magnetospheric convection)



Solar wind generates an electric field from dawn to dusk in the equatorial plane. Particles will ***drift sunward*** in this field.

$$v_E = \frac{E_{eq}}{B_{eq}} = \frac{E_{eq}L^3}{B_E}$$

- *Close to Earth magnetic drift prevails -> symmetric ring current*
- *Intermediate region -> partial ring current*
- *Far from Earth particles are dominated by* ***E*** × ***B*** *drift.*

Sources and sinks of ring current

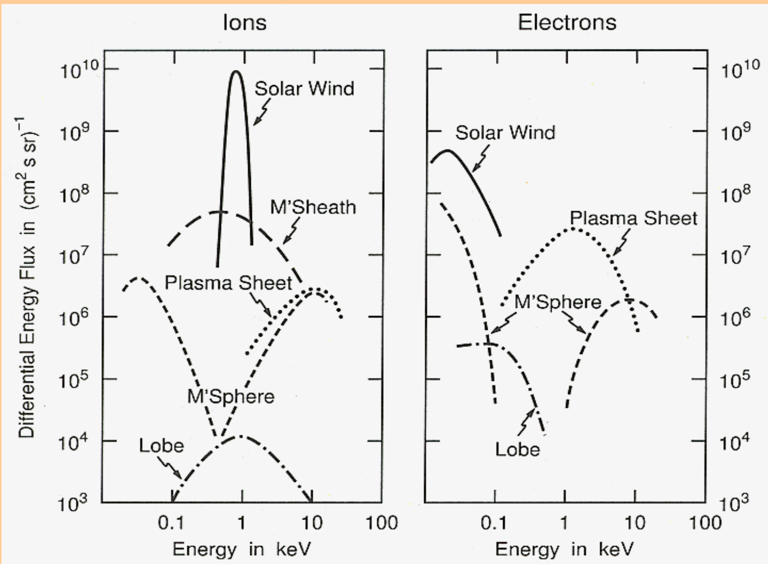
The major source of the ring current is the tail plasma sheet, from which particles are brought in by the electric drift.

Adiabatic heating: While drifting inwards particles conserve their magnetic moments, thus their energy increases according to:

$$\boxed{\frac{W_\perp}{W_{\perp 0}} = \left(\frac{L_0}{L}\right)^3}$$

The major ***sink*** of the ring current is the loss of energetic particles undergoing charge exchange (liftime hours to days). Other loss mechanism: ***Pitch-angle scattering*** into the loss cone in the neutral lower atmosphere.

Particle fluxes in near Earth space



Adiabatic invariants of motion

In classical Hamiltonian mechanics the action integral

$$J_i = \oint p_i dq_i$$

is an invariant of motion for any change that is slow as compared to the oscillation frequency associated with that motion. Three invariants related to:

- ***Gyromotion*** about the local field

$$\Phi_\mu = \frac{2\pi m}{q^2}\mu = \mathrm{const}$$

- ***Bounce motion*** between mirror points

$$J = \oint m v_\parallel ds$$

- ***Drift motion*** in azimuthal direction, with planetary magnetic moment *M*

$$\Phi = \frac{2\pi m}{q^2} M = \mathrm{const}$$

Magnetic flux, $\Phi_\mu = B\pi r_g^2$, through surface encircled by the gyro orbit is constant.