Elements of kinetic theory

- **Introduction**
- **Phase space density**
- **Equations of motion**
- **Average distribution function**
- **Boltzmann-Vlasov equation**
- **Velocity distribution functions**
- **Moments and fluid variables**
- **The kinetic plasma temperature**

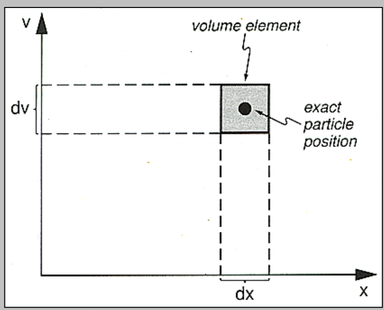
Introduction

Kinetic theory describes the plasma statistically, i.e. the collective behaviour of the various particles under the influence of their self-generated electromagnetic fields.

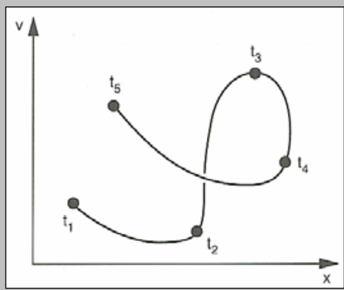
Collective behaviour and complexity arises from:

- Many particles (species: electrons, protons, heavy ions)
- Long-range self-consistent fields, $\mathbf{B}(\mathbf{x},t)$ and $\mathbf{E}(\mathbf{x},t)$
- Fields are averages over the microscopic fields and generated by all particles together
- Strong mutual interactions between fields and particles may lead to nonlinearities

Phase space



Six-dimensional *phase space* with coordinates axes **x** and **v** and volume element d**x**d**v**



Many particles (i=1, N) having time-dependent position $\mathbf{x}_i(t)$ and velocity $\mathbf{v}_i(t)$. The particle path at subsequent times (t_1,.., t_5) is a curve in phase space (see illustration right figure).

Phase space density

For each individual particle (index *i*) we may define the exact density in phase space through sharp three-dimensional delta functions ($\delta(\mathbf{x}) = \delta(x)\,\delta(y)\,\delta(z)$) as follows:

$$\mathcal{F}_i(\mathbf{x},\mathbf{v},t) = \delta(\mathbf{x}-\mathbf{x}_i(t))\delta(\mathbf{v}-\mathbf{v}_i(t))$$

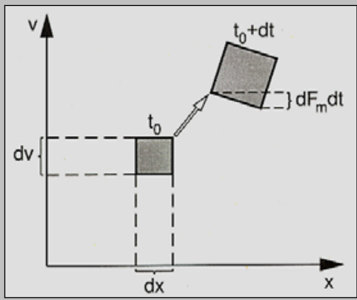
The multi-particle density is simply obtained by summation over all particles (of all components). The geometrical content is that the phase-space volume occupied consists of the sum of all individual phase-space volume elements.

Since particles are subject to the action of forces (different for different particles), the total phase-space volume will deform but remain constant (particle number conservation).

Equation of motion with electromagnetic forces

Deformation of d**x**d**v** due to microscopic force.

(3-d: d**v**=d^{3}v=$dv_x dv_y dv_z$)



The instantaneous velocity is $\mathbf{v}_i = d\mathbf{x}_i(t)/dt$, with the total derivate with respect to time. Denoting the microscopic field by index m, the equation of motion reads:

$$\frac{d}{dt}\mathbf{v}_i(t) = \frac{q}{m}\left[\mathbf{E}_m(\mathbf{x}_i(t),t) + \mathbf{v}_i(t) \times \mathbf{B}_m(\mathbf{x}_i(t),t)\right]$$

Maxwell equations

$$\nabla \times \mathbf{B}_m(\mathbf{x},t) = \mu_0 \mathbf{j}_m(\mathbf{x},t) + \epsilon_0\mu_0 \frac{\partial}{\partial t}\mathbf{E}_m(\mathbf{x},t)$$

$$\nabla \times \mathbf{E}_m(\mathbf{x},t) = -\frac{\partial}{\partial t}\mathbf{B}_m(\mathbf{x},t)$$

$$\nabla \cdot \mathbf{E}_m(\mathbf{x},t) = \frac{1}{\epsilon_0}\rho_m(\mathbf{x},t)$$

$$\nabla \cdot \mathbf{B}_m(\mathbf{x},t) = 0$$

Ampère, Faraday, Gauß

Microscopic electromagnetic fields

$$\rho_m(\mathbf{x},t) = \sum_s q_s \int \mathcal{F}_s(\mathbf{x},\mathbf{v},t)\, d^3 \upsilon$$

$$\mathbf{j}_m(\mathbf{x},t) = \sum_s q_s \int \mathcal{F}_s(\mathbf{x},\mathbf{v},t)\,\mathbf{v}\, d^3 \upsilon$$

Microscopic ***charge*** and ***current*** densities

Klimontovich equation

If no particles are lost from or added to the plasma the exact phase space density is conserved. Thus the total time derivative

$$\frac{d}{dt}\mathcal{F}(\mathbf{x},\mathbf{v},t) = 0$$

vanishes, and is in 6-d phase space given after the chain rule of differentiation as follows:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} + \frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}}$$

$$\frac{\partial \mathcal{F}}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}}\mathcal{F} + \frac{q}{m}(\mathbf{E}_m + \mathbf{v}\times\mathbf{B}_m)\cdot\nabla_{\mathbf{v}}\mathcal{F} = 0$$

This still describes the plasma state fully at all times.

Boltzmann equation

We now define an *ensemble averaged* phase space density, the distribution function, through the decomposition:

$$\mathcal{F}(\mathbf{x},\mathbf{v},t) = f(\mathbf{x},\mathbf{v},t) + \delta\mathcal{F}(\mathbf{x},\mathbf{v},t)$$

with vanishing fluctuations: $\langle\delta\mathcal{F}\rangle$ = 0.
Similarly, the microscopic field is decomposed:

$$\begin{aligned}\mathbf{E}_m(\mathbf{x},t) &= \mathbf{E}(\mathbf{x},t) + \delta\mathbf{E}(\mathbf{x},t)\\ \mathbf{B}_m(\mathbf{x},t) &= \mathbf{B}(\mathbf{x},t) + \delta\mathbf{B}(\mathbf{x},t)\end{aligned}$$

Inserting these decompositions into the Klimontovich equation yields after ensemble averaging the Boltzmann equation.

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}}f + \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}}f = \left(\frac{\partial f}{\partial t}\right)_c$$

Models for the collision terms

The second-order term on the right of the Boltzmann equation contains all correlations between fields and particles, due to collisions and (wave-) fluctuation-particle interactions, and is notoriously difficult to evaluate.

$$\left(\frac{\partial f}{\partial t}\right)_c = -\frac{q}{m}\langle(\delta\mathbf{E} + \mathbf{v}\times\delta\mathbf{B})\cdot\nabla_{\mathbf{v}}\delta\mathcal{F}\rangle$$

Concerning neutral-ion collisions a simple relaxation approach is sometimes applied, with f_n being the velocity distribution function (VDF) of the neutrals, and ν_n is their collision frequency:

$$\left(\frac{\partial f}{\partial t}\right)_c = \nu_n(f_n - f)$$

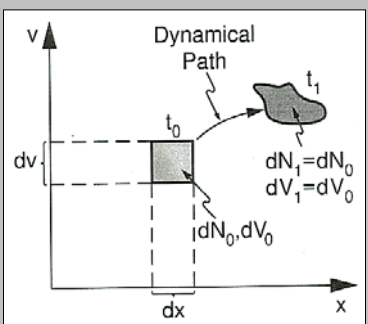
Collisions (Landau or Fokker-Planck) and wave-particle interactions can often be described as a diffusion process:

$$\left(\frac{\partial f}{\partial t}\right)_c = \nabla_{\mathbf{v}}\cdot(\mathbf{D}\cdot\nabla_{\mathbf{v}} f)$$

Vlasov equation

Since most space plasmas are collisionless, we neglect the right-hand side in the Boltzmann equation and thus obtain the simplest kinetic equation named after Vlasov:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f = 0$$



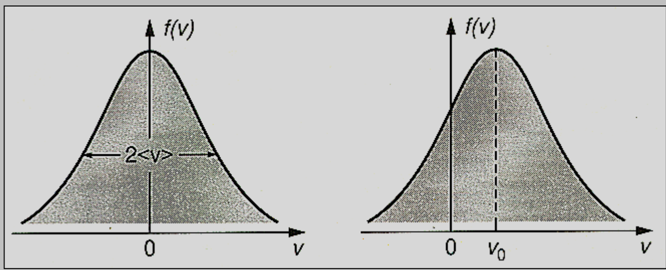
This equation expresses phase space density conservation (Liouville theorem) visualised in the left figure. A volume element evolves under the Lorentz force like in an incompressible fluid and remains constant as the number of particles contained in it.

The Vlasov equation is still highly nonlinear via closure with Maxwell's equations.

Maxwellian velocity distribution function

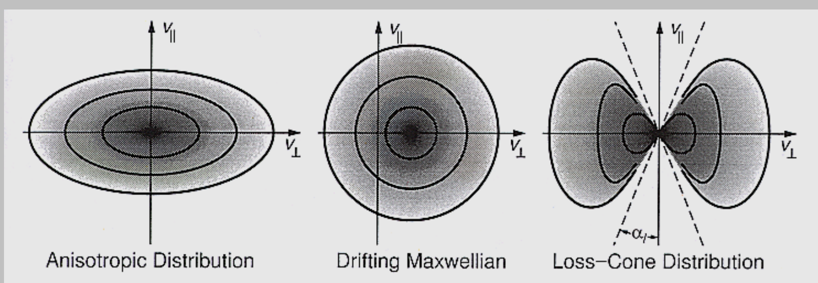
The general equilibrium VDF in a uniform thermal plasma is the *Maxwellian* (Gaussian) distribution.

The average velocity spread (variance) is, <v> = $(2k_BT/m)^{1/2}$, and the mean drift velocity, $\mathbf{v}_0$.



$$f(\mathbf{v}) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m(\mathbf{v}-\mathbf{v}_0)^2}{2k_B T}\right)$$

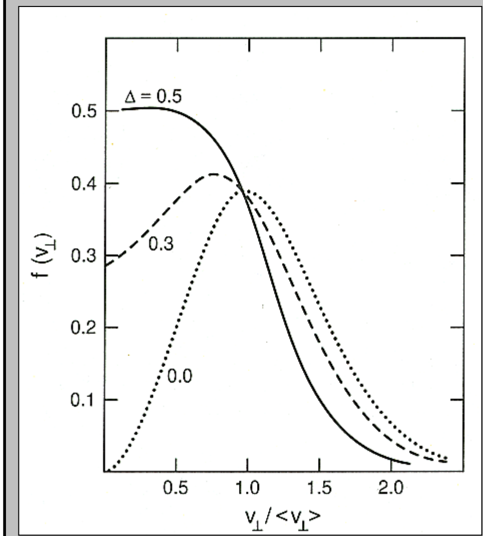
Anisotropic model velocity distributions



The most common anisotropic VDF in a uniform thermal plasma is the *bi-Maxwellian* distribution. Left figure shows a sketch of it, with $T_\perp > T_{||}$.

$$f(v_\perp, v_\|) \quad = \quad \frac{n}{T_\perp T_\|^{1/2}}\left(\frac{m}{2\pi k_B}\right)^{3/2} \exp\left(-\frac{mv_\perp^2}{2k_B T_\perp} - \frac{mv_\|^2}{2k_B T_\|}\right)$$

Loss-cone model distribution function



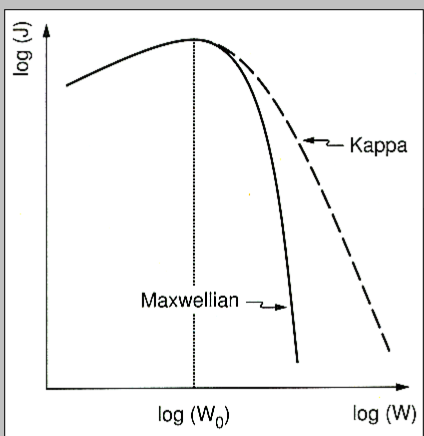
$$f(v_\parallel, v_\perp) = \frac{n}{(\pi^3 \langle v_\parallel \rangle^2 \langle v_\perp \rangle^4)^{1/2}}$$

$$\times \exp\left(-\frac{v_\parallel^2}{\langle v_\parallel \rangle^2}\right) G(v_\perp, \Delta, \beta)$$

$$G = \Delta \exp\left(-\frac{v_\perp^2}{\langle v_\perp \rangle^2}\right) + \frac{1-\Delta}{1-\beta} \times \left[\exp\left(-\frac{v_\perp^2}{\langle v_\perp \rangle^2}\right) - \exp\left(-\frac{v_\perp^2}{\beta \langle v_\perp \rangle^2}\right)\right]$$

Here Δ and β are parameters to fit the loss cone. Δ=0 gives an empty loss cone, and Δ=1 reproduces a simple Maxwellian. β allows to change the slope of *f* inside the loss cone.

Kappa and power-law distribution function

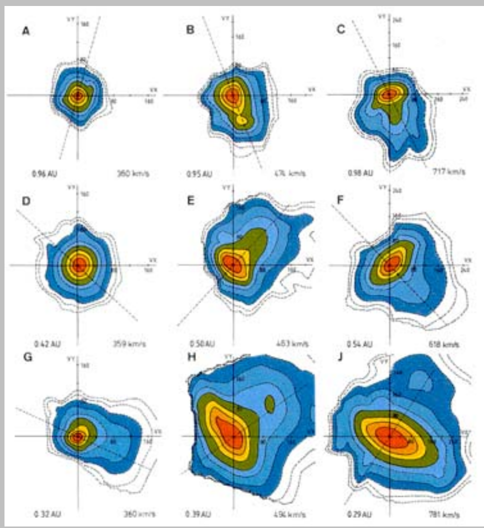


$$f_\kappa(W) = n \left(\frac{m}{2\pi\kappa W_0}\right)^{3/2} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)} \times \left(1 + \frac{W^*}{\kappa W_0}\right)^{-(\kappa+1)}$$

Differential particle flux function, J(W) ~ v^2 f(v), with W=$mv^2/2$.

Here κ is a shape parameter. If κ >> 1, the distribution approaches a Maxwellian, κ = 2 is a Lorentzian, and for small κ > 2 the VDF has a power-law tail in proportion to $(W/W_0)^{-\kappa}$, with the average thermal energy $W_0 = k_B T\,(1-3/(2\kappa))$.

Measured solar wind proton velocity distributions



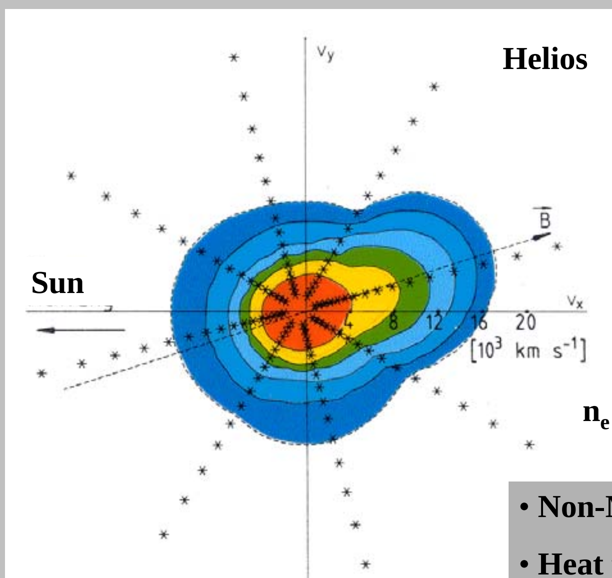
Helios

- **Temperature anisotropies**
- **Ion beams**
- **Plasma instabilities**
- **Interplanetary heating**

Plasma measurements made at 10 s resolution (> 0.29 AU from the Sun)

Marsch et al., JGR, **87**, 52, 1982

Measured solar wind electrons



n_e = 3 -10 cm^{-3}

- **Non-Maxwellian**
- **Heat flux tail**

Pilipp et al., JGR, **92**, 1075, 1987

Velocity moments I

The microsopic distribution depends on **v**, **x**, and *t*. The macroscopic physical parameters, like density or temperature, depend only on **x** and *t* and thus are obtained by integration over the entire velocity space as so-called *moments*. The *i*-th moment is the following integral:

$$\mathcal{M}_i(\mathbf{x},t) = \int f(\mathbf{v},\mathbf{x},t)\mathbf{v}^i d^3v$$

Where $\mathbf{v}^i = \mathbf{vv...v}$ (i-fold) denotes an *i*-fold dyadic product, i.e. a tensor of rank *i*.

Velocity moments II

The *number density* is defined as 0-th order moment:

$$n = \int f(\mathbf{v}) d^3v$$

The bulk *flow velocity* is defined as 1-st order moment:

$$\mathbf{v}_b = \frac{1}{n}\int \mathbf{v} f(\mathbf{v}) d^3v$$

The *pressure tensor* is defined as the fluctuation of the velocities of the ensemble from the mean velocity, i.e. as the 2-nd order moment:

$$\mathbf{P} = m\int (\mathbf{v}-\mathbf{v}_b)(\mathbf{v}-\mathbf{v}_b) f(\mathbf{v}) d^3v$$

Velocity moments III

The trace-less parts of the *pressure* tensor **P** correspond to the stresses in the plasma.

The *heat flux tensor* is used to describe the multi-directional flow of internal energy and defined as 3-rd order moment:

$$\mathbf{Q} = m \int (\mathbf{v} - \mathbf{v}_b)(\mathbf{v} - \mathbf{v}_b)(\mathbf{v} - \mathbf{v}_b) f(\mathbf{v}) d^3 v$$

More relevant to decribe deviations from thermal equilibrium is half the trace of **Q**, the *heat flux vector,* **q**, that is defined as:

$$\mathbf{q} = \frac{m}{2} \int (\mathbf{v} - \mathbf{v}_b)^2 (\mathbf{v} - \mathbf{v}_b) f(\mathbf{v}) d^3 v$$

Concept of temperature

The isotropic scalar *pressure* is defined as a third of the trace of **P**, i.e. $p = 1/3\, P_{ii}$, which leads through the ideal gas law, $p = nk_B T$, to the *kinetic temperature* defined as 2-nd moment:

$$T = \frac{m}{3k_B n} \int (\mathbf{v} - \mathbf{v}_b) \cdot (\mathbf{v} - \mathbf{v}_b) f(\mathbf{v}) d^3 v$$

This temperature can formally be calculated for any VDF and thus is not necessarily identical with the thermodynamic temperature. To demonstrate its meaning, calculate the kinetic temperature for the Maxwellian at rest:

$$f(v) = \frac{n}{(\pi \langle v \rangle^2)^{3/2}} \exp\left(-\frac{v^2}{\langle v \rangle^2} \right)$$

Note that by integration, with the volume element $d^3v = 4\pi v^2 dv$, one finds (exercise!) that

$$T = \frac{m \langle v \rangle^2}{2k_B}$$