Magnetohydrodynamic waves

- Ideal MHD equations
- Linear perturbation theory
- The dispersion relation
- Phase velocities
- Dispersion relations (polar plot)
- Wave dynamics
- MHD turbulence in the solar wind
- Geomagnetic pulsations

Ideal MHD equations

Plasma equilibria can easily be perturbed and small-amplitude waves and fluctuations can be excited. Conveniently, while considering waves one starts from ideal plasmas. The damping of the waves requires consideration of some kind of dissipation, which will not be done here. MHD (with ideal Ohm's law and no space charges) in the standard form (and with P(n) given) reads:

\[
\frac{\partial n}{\partial t} + \nabla \cdot (nv) = 0
\]

\[
\frac{\partial(nmv)}{\partial t} + \nabla \cdot (nmvv) = -\nabla \cdot \left( P + \frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} \nabla \cdot (BB)
\]

\[
\frac{\partial B}{\partial t} = \nabla \times (v \times B)
\]

\[
\nabla \cdot B = 0
\]
MHD equilibrium and fluctuations

We assume stationary ideal homogeneous conditions as the initial state of the single-fluid plasma, with vanishing average electric and velocity fields, overall pressure equilibrium and no magnetic stresses. These assumptions yield:

\[
\begin{align*}
\nu_0 &= 0 \\
E_0 &= 0 \\
\nabla \left( p_0 + B_0^2 / 2\mu_0 \right) &= 0 \\
(B_0 \cdot \nabla) B_0 &= 0
\end{align*}
\]

These fields are decomposed as sums of their background initial values and space- and time-dependent fluctuations as follows:

\[
\begin{align*}
n &= n_0 + \delta n \\
\nu &= \delta \nu \\
E &= \delta E \\
B &= B_0 + \delta B
\end{align*}
\]

Linear perturbation theory

Because the MHD equations are nonlinear (advection term and pressure/stress tensor), the fluctuations must be small.

-> Arrive at a uniform set of linear equations, giving the dispersion relation for the eigenmodes of the plasma.

-> Then all variables can be expressed by one, say the magnetic field.

Usually, in space plasma the background magnetic field is sufficiently strong (e.g., a planetary dipole field), so that one can assume the fluctuation obeys:

\[ |\delta B| \ll B_0 \]

In the uniform plasma with straight field lines, the field provides the only symmetry axis which may be chosen as \(z\)-axis of the coordinate system such that: \(B_0 = B_0 \hat{e}_z\).
Linearized MHD equations I

Linarization of the MHD equations leads to three equations for the three fluctuations, $\delta n$, $\delta v$, and $\delta B$:

\[
\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \delta v = 0
\]
\[
m_i n_0 \frac{\partial \delta v}{\partial t} = -\nabla \left( \delta p + \frac{1}{\mu_0} B_0 \cdot \delta B \right) + \frac{1}{\mu_0} (B_0 \cdot \nabla) \delta B
\]
\[
\frac{\partial \delta B}{\partial t} = (B_0 \cdot \nabla) \delta B - B_0 (\nabla \cdot \delta v)
\]

Using the adiabatic pressure law, and the derived sound speed, $c_s^2 = p_0/(m_i n_0)$, leads to an equation for $\delta p$ and gives:

\[
\frac{\partial \delta p}{\partial t} = m_i c_s^2 \frac{\partial \delta n}{\partial t} = -n_i \rho_0 c_s^2 \nabla \cdot \delta v
\]

Linearized MHD equations II

Inserting the continuity and pressure equations, and using the Alfvén velocity, $v_A = B_0 / (\mu_0 n m_i)^{1/2}$, two coupled vector equations result:

\[
\frac{\partial \delta v}{\partial t} = v_A^2 \nabla \parallel \left( \frac{\delta B \perp}{B_0} \right) - \nabla \left( \frac{\delta p}{m_i n_0} \right)
\]
\[
\frac{\partial}{\partial t} \left( \frac{\delta B}{B_0} \right) = \nabla \parallel \delta v \perp - \delta \parallel (\nabla \perp \cdot \delta v \perp)
\]

Time differentiation of the first and insertion of the second equation yields a second-order wave equation which can be solved by Fourier transformation.

\[
\frac{\partial^2 \delta v}{\partial t^2} = c_{ms}^2 \nabla \cdot \delta v
\]
\[
+ v_A^2 \left( \nabla \parallel^2 \delta v - \nabla \parallel \delta v \parallel - \delta \parallel \nabla \parallel \cdot \delta v \perp \right)
\]
Dispersion relation

The ansatz of travelling plane waves,

\[ \delta \mathbf{v} = \delta \mathbf{v}_0 \exp(i \mathbf{k} \cdot \mathbf{x} - i \omega t) \]

with arbitrary constant amplitude, \( \delta \mathbf{v}_0 \), leads to the system,

\[
\begin{bmatrix}
(\omega^2 - k_{||}^2 v_A^2) & 0 & 0 \\
0 & (\omega^2 - c_{ms}^2 k_{\perp}^2) & 0 \\
0 & 0 & (\omega^2 - c_s^2 k_{\perp}^2)
\end{bmatrix}
\begin{bmatrix}
\delta v_{0x} \\
\delta v_{0y} \\
\delta v_{0z}
\end{bmatrix} = 0
\]

To obtain a nontrivial solution the determinant must vanish, which means

\[ \omega^2 - k_{||}^2 v_A^2 = \omega^2 - c_{ms}^2 k_{\perp}^2 = \omega^2 - c_s^2 k_{\perp}^2 \]

Here the magnetosonic speed is given by \( c_{ms}^2 = c_s^2 + v_A^2 \). The wave vector component perpendicular to the field is oriented along the x-axis, \( k = k_{||} \hat{e}_z + k_{\perp} \hat{e}_x \).

Alfvén waves

Inspection of the determinant shows that the fluctuation in the y-direction decouples from the other two components and has the linear dispersion

\[ \omega_A = \pm k_{||} v_A \]

This transverse wave travels parallel to the field. It is called shear Alfvén wave. It has no density fluctuation and a constant group velocity, \( \mathbf{v}_{gr,A} = v_A e_A \), which is always oriented along the background field, along which the wave energy is transported.

The transverse velocity and magnetic field components are (anti)-correlated according to: \( \delta v_y / v_A = \pm \delta B_y / B_0 \), for parallel (anti-parallel) wave propagation. The wave electric field points in the x-direction: \( \delta E_x = \delta B_y / v_A \).
Magnetosonic waves

The remaining four matrix elements couple the fluctuation components, $\delta v_\parallel$ and $\delta v_\perp$. The corresponding determinant reads:

$$\omega^4 - \omega^2 c_{ms}^2 k^2 + c_s^2 v_A^2 k^2 k_\parallel^2 = 0$$

This bi-quadratic equation has the roots:

$$\omega_{ms}^2 = \frac{k^2}{2} \left\{ c_{ms}^2 \pm \left[ (c_A^2 - c_s^2)^2 + 4 c_A^2 c_s^2 k_\perp^2 k^2 \right]^{1/2} \right\}$$

which are the phase velocities of the compressive fast and slow magnetosonic waves. They depend on the propagation angle $\theta$, with $k_\perp^2/k^2 = \sin^2 \theta$. For $\theta = 90^0$ we have: $\omega = k c_{ms}$, and $\theta = 0^0$:

$$\omega^2 = \frac{1}{2} k^2 \left[ c_s^2 + v_A^2 \pm (c_s^2 - v_A^2) \right]$$

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Phase-velocity polar diagram of MHD waves
Dependence of phase velocity on propagation angle

Magnetosonic wave dynamics

In order to understand what happens physically with the dynamic variables, $\delta v_x$, $\delta B_x$, $\delta B_{||}$, $\delta v_{||}$, $\delta p$, and $\delta n$, inspect again the equation of motion written in components:

$\omega \delta v = \frac{k}{m_i n_0} \left( \delta p + \frac{1}{\mu_0} B_0 \cdot \delta B \right) - \frac{k \cdot B_0}{\mu_0 m_i n_0} \delta B$

Parallel direction: $\omega v_{||} = \frac{k_{||} \delta p}{m_i n_0}$  \hspace{1cm} \text{Parallel pressure variations cause parallel flow.}$

Oblique direction: $\omega \left( k_{||} v_{||} + k_{\perp} v_{\perp} \right) = \frac{k^2 \delta p_{\text{tot}}}{m_i n_0}$

Total pressure variations ($p_{\text{tot}} = p + B^2/2\mu_0$) accelerate (or decelerate) flow, for in-phase (or out-of-phase) variations of $\delta p$ and $\delta B$, leading to the fast and slow mode waves.
Magnetohydrodynamic waves

- Magnetosonic waves
  - compressible
  - parallel slow and fast
  - perpendicular fast
  \[ c_{\text{ms}} = (c_s^2 + v_A^2)^{1/2} \]

- Alfvén wave
  - incompressible
  - parallel and oblique
  \[ v_A = B/(4\pi\rho)^{1/2} \]

Alfvén waves in the solar wind

\[ \delta v = \pm \delta v_A \]
Alfvénic fluctuations

Ulysses observed many such waves (4-5 per hour) in fast wind over the poles:

- Arc-polarized waves
- Phase-steepened

Rotational discontinuity:

\[ \Delta V = \pm \Delta V_A \]

Finite jumps in velocities over gyrokinetic scales

Tsurutani et al., 1997

Compressive fluctuations in the solar wind

Marsch and Tu, JGR, 95, 8211, 1990

Kolmogorov-type turbulence
Magnetohydrodynamic turbulence in space plasmas

Integral invariants of ideal MHD

\[ E = \frac{1}{2} \int d^3x \left( V^2 + V_A^2 \right) \quad \text{Energy} \]
\[ H_c = \int d^3x \left( V \cdot V_A \right) \quad \text{Helicity} \]
\[ H_m = \int d^3x \left( A \cdot B \right) \quad \text{Magnetic helicity} \]
\[ B = \nabla \times A \]

Elsässer variables:
\[ Z^\pm = V \pm V_A \]
\[ E^\pm = \frac{1}{2} \int d^3x (Z^\pm)^2 = \int d^3x e^\pm(x) \]

Geomagnetic pulsations

Fast magnetic fluctuations of the Earth’s surface magnetic field (few mHz up to a few Hz), corresponding to oscillation periods from several hundreds to fractions of seconds. The pulsating quasi-sinusoidal (continuous pulsations) disturbances are observed globally and associated with Alfvén (and magneto-acoustic) waves. The irregular pulsations are short-lived and localized.
Field line resonances

The shown Pc5 pulsations are caused by oscillations of the Earth magnetic field and explained as standing single-fluid shear Alfvén waves, whose wavelengths must fit the geometry. The length, $l$, of the fieldline between two reflection point must be a multiple of half the wavelength, $\lambda$, implying: $\nu_h \lambda = 2l$, with $\nu_h = 1,2,3,...$. From the dispersion relation, with the average Alfvén velocity, $\langle v_A \rangle$, along the fieldline one finds:

$$\omega_h = \frac{\nu_h \pi \langle v_A \rangle}{l}$$

Fundamental poloidal field-line resonances

Magnetosheath turbulence as source of pulsations

Excitation of global magnetospheric pulsations through driver at matching frequency: $\omega_{ex} = \omega_{res}$.

Excitation scenarios:

- Surface waves excited by Kelvin-Helmholtz instability driven by flow around magnetopause
- Compressional waves leaking from magnetosheath through polar cusp into magnetosphere