

## Magnetohydrodynamic waves

- Ideal MHD equations
- Linear perturbation theory
- The dispersion relation
- Phase velocities
- Dispersion relations (polar plot)
- Wave dynamics
- MHD turbulence in the solar wind
- Geomagnetic pulsations

## Ideal MHD equations

Plasma equilibria can easily be perturbed and small-amplitude *waves and fluctuations* can be excited. Conveniently, while considering waves one starts from ideal plasmas. The damping of the waves requires consideration of some kind of dissipation, which will not be done here. **MHD (with ideal Ohm's law and no space charges) in the standard form (and with  $\mathbf{P}(n)$  given) reads:**

$$\begin{aligned}\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) &= 0 \\ \frac{\partial(nm\mathbf{v})}{\partial t} + \nabla \cdot (nm\mathbf{v}\mathbf{v}) &= -\nabla \cdot \left( \mathbf{P} + \frac{B^2}{2\mu_0} \mathbf{I} \right) + \frac{1}{\mu_0} \nabla \cdot (\mathbf{B}\mathbf{B}) \\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}) \\ \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

## MHD equilibrium and fluctuations

We assume *stationary* ideal *homogeneous* conditions as the initial state of the single-fluid plasma, with vanishing average electric and velocity fields, overall *pressure equilibrium* and no magnetic stresses. These assumptions yield:

$$\begin{aligned} \mathbf{v}_0 &= 0 \\ \mathbf{E}_0 &= 0 \\ \nabla (p_0 + B_0^2/2\mu_0) &= 0 \\ (\mathbf{B}_0 \cdot \nabla) \mathbf{B}_0 &= 0 \end{aligned}$$

These fields are decomposed as sums of their background initial values and space- and time-dependent *fluctuations* as follows:

$$\begin{aligned} n &= n_0 + \delta n \\ \mathbf{v} &= \delta \mathbf{v} \\ \mathbf{E} &= \delta \mathbf{E} \\ \mathbf{B} &= \mathbf{B}_0 + \delta \mathbf{B} \end{aligned}$$

## Linear perturbation theory

Because the MHD equations are nonlinear (advection term and pressure/stress tensor), the fluctuations must be small.

-> *Arrive at a uniform set of linear equations, giving the dispersion relation for the eigenmodes of the plasma.*

-> *Then all variables can be expressed by one, say the magnetic field.*

Usually, in space plasma the background magnetic field is sufficiently strong (e.g., a planetary dipole field), so that one can assume the fluctuation obeys:

$$|\delta \mathbf{B}| \ll B_0$$

In the uniform plasma with straight field lines, the field provides the only *symmetry axis* which may be chosen as  $z$ -axis of the coordinate system such that:  $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_{\parallel}$ .

## Linearized MHD equations I

Linearization of the MHD equations leads to three equations for the three fluctuations,  $\delta n$ ,  $\delta \mathbf{v}$ , and  $\delta \mathbf{B}$ :

$$\begin{aligned}\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \delta \mathbf{v} &= 0 \\ m_i n_0 \frac{\partial \delta \mathbf{v}}{\partial t} &= -\nabla \left( \delta p + \frac{1}{\mu_0} \mathbf{B}_0 \cdot \delta \mathbf{B} \right) + \frac{1}{\mu_0} (\mathbf{B}_0 \cdot \nabla) \delta \mathbf{B} \\ \frac{\partial \delta \mathbf{B}}{\partial t} &= (\mathbf{B}_0 \cdot \nabla) \delta \mathbf{B} - \mathbf{B}_0 (\nabla \cdot \delta \mathbf{v})\end{aligned}$$

Using the adiabatic pressure law, and the derived sound speed,  $c_s^2 = p_0 / (m_i n_0)$ , leads to an equation for  $\delta p$  and gives:

$$\frac{\partial \delta p}{\partial t} = m_i c_s^2 \frac{\partial \delta n}{\partial t} = -m_i n_0 c_s^2 \nabla \cdot \delta \mathbf{v}$$

## Linearized MHD equations II

Inserting the continuity and pressure equations, and using the Alfvén velocity,  $v_A = \mathbf{B}_0 / (\mu_0 n m_i)^{1/2}$ , two coupled vector equations result:

$$\begin{aligned}\frac{\partial \delta \mathbf{v}}{\partial t} &= v_A^2 \nabla_{\parallel} \left( \frac{\delta \mathbf{B}_{\perp}}{B_0} \right) - \nabla \left( \frac{\delta p}{m_i n_0} \right) \\ \frac{\partial}{\partial t} \left( \frac{\delta \mathbf{B}}{B_0} \right) &= \nabla_{\parallel} \delta \mathbf{v}_{\perp} - \hat{\mathbf{e}}_{\parallel} (\nabla_{\perp} \cdot \delta \mathbf{v}_{\perp})\end{aligned}$$

Time differentiation of the first and insertion of the second equation yields a second-order wave equation which can be solved by *Fourier transformation*.

$$\begin{aligned}\frac{\partial^2 \delta \mathbf{v}}{\partial t^2} &= c_{ms}^2 \nabla (\nabla \cdot \delta \mathbf{v}) \\ &+ v_A^2 \left( \nabla_{\parallel}^2 \delta \mathbf{v} - \nabla \nabla_{\parallel} \delta v_{\parallel} - \hat{\mathbf{e}}_{\parallel} \nabla_{\parallel} \nabla \cdot \delta \mathbf{v} \right)\end{aligned}$$

## Dispersion relation

The ansatz of *travelling plane waves*,

$$\delta \mathbf{v} = \delta \mathbf{v}_0 \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$$

with arbitrary constant amplitude,  $\delta \mathbf{v}_0$ , leads to the system,

$$\left[ (\omega^2 - k_{\parallel}^2 v_A^2) \mathbf{I} - c_{ms}^2 \mathbf{k} \mathbf{k} + (\mathbf{k} \hat{\mathbf{e}}_{\parallel} + \hat{\mathbf{e}}_{\parallel} \mathbf{k}) k_{\parallel} v_A^2 \right] \cdot \delta \mathbf{v}_0 = 0$$

To obtain a nontrivial solution the determinant must vanish, which means

$$\begin{bmatrix} \omega^2 - v_A^2 k_{\parallel}^2 - c_{ms}^2 k_{\perp}^2 & 0 & -c_s^2 k_{\parallel} k_{\perp} \\ 0 & \omega^2 - v_A^2 k_{\parallel}^2 & 0 \\ -c_s^2 k_{\parallel} k_{\perp} & 0 & \omega^2 - c_s^2 k_{\parallel}^2 \end{bmatrix} \begin{bmatrix} \delta v_{0x} \\ \delta v_{0y} \\ \delta v_{0\parallel} \end{bmatrix} = 0$$

Here the *magnetosonic speed* is given by  $c_{ms}^2 = c_s^2 + v_A^2$ . The wave vector component perpendicular to the field is oriented along the x-axis,  $\mathbf{k} = k_{\parallel} \hat{\mathbf{e}}_z + k_{\perp} \hat{\mathbf{e}}_x$ .

## Alfvén waves

Inspection of the determinant shows that the fluctuation in the y-direction decouples from the other two components and has the linear dispersion

$$\omega_A = \pm k_{\parallel} v_A$$

This *transverse wave* travels parallel to the field. It is called *shear Alfvén wave*. It has no density fluctuation and a constant group velocity,  $\mathbf{v}_{gr,A} = \mathbf{v}_A$ , which is always oriented along the background field, along which the wave energy is transported.

*The transverse velocity and magnetic field components are (anti)-correlated according to:  $\delta v_y / v_A = \pm \delta B_y / B_0$ , for parallel (anti-parallel) wave propagation. The wave electric field points in the x-direction:  $\delta E_x = \delta B_y / v_A$*

## Magnetosonic waves

The remaining four matrix elements couple the fluctuation components,  $\delta v_{\parallel}$  and  $\delta v_{\perp}$ . The corresponding determinant reads:

$$\omega^4 - \omega^2 c_{ms}^2 k^2 + c_s^2 v_A^2 k_{\perp}^2 k_{\parallel}^2 = 0$$

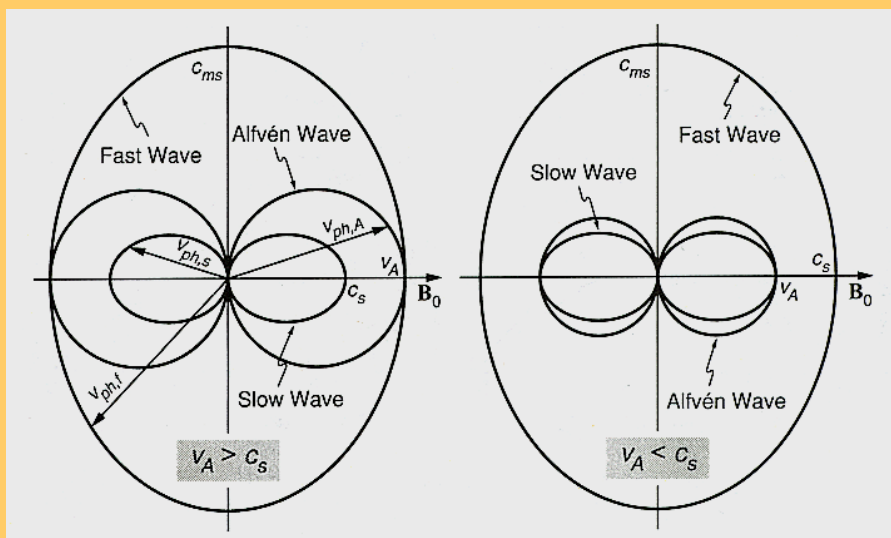
This bi-quadratic equation has the roots:

$$\omega_{ms}^2 = \frac{k^2}{2} \left\{ c_{ms}^2 \pm \left[ (v_A^2 - c_s^2)^2 + 4v_A^2 c_s^2 \frac{k_{\perp}^2}{k^2} \right]^{1/2} \right\}$$

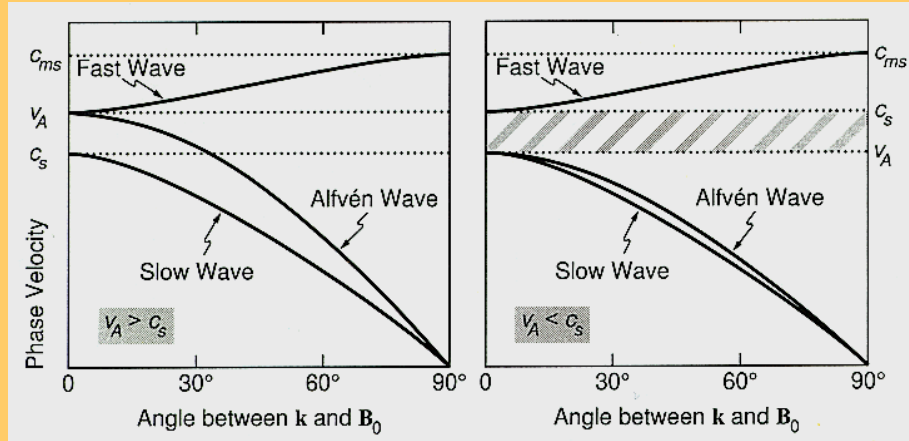
which are the phase velocities of the compressive **fast and slow magnetosonic waves**. They depend on the propagation angle  $\theta$ , with  $k_{\perp}^2/k^2 = \sin^2\theta$ . For  $\theta = 90^\circ$  we have:  $\omega = kc_{ms}$ , and  $\theta = 0^\circ$ :

$$\omega^2 = \frac{1}{2} k^2 [c_s^2 + v_A^2 \pm (c_s^2 - v_A^2)]$$

## Phase-velocity polar diagram of MHD waves



## Dependence of phase velocity on propagation angle



## Magnetosonic wave dynamics

In order to understand what happens physically with the dynamic variables,  $\delta v_x$ ,  $\delta B_x$ ,  $\delta B_{\parallel}$ ,  $\delta v_{\parallel}$ ,  $\delta p$ , and  $\delta n$ , inspect again the equation of motion written in components:

$$\omega \delta \mathbf{v} = \frac{\mathbf{k}}{m_i n_0} \left( \delta p + \frac{1}{\mu_0} \mathbf{B}_0 \cdot \delta \mathbf{B} \right) - \frac{\mathbf{k} \cdot \mathbf{B}_0}{\mu_0 m_i n_0} \delta \mathbf{B}$$

Parallel direction:

$$\omega v_{\parallel} = \frac{k_{\parallel} \delta p}{m_i n_0}$$

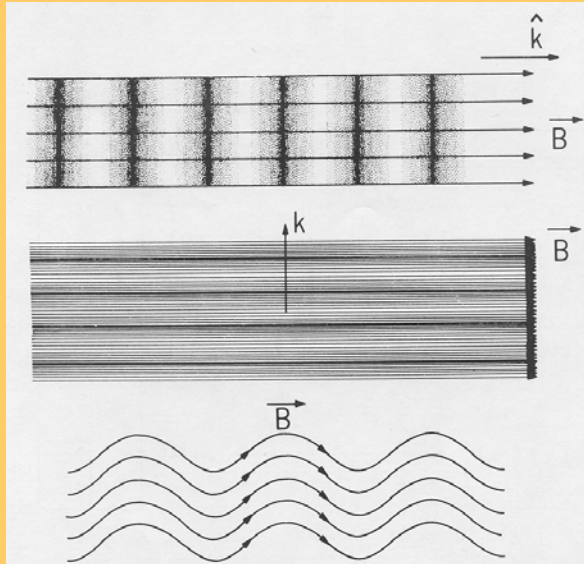
*Parallel pressure variations cause parallel flow.*

Oblique direction:

$$\omega (k_{\parallel} v_{\parallel} + k_{\perp} v_x) = \frac{k^2 \delta p_{tot}}{m_i n_0}$$

*Total pressure variations ( $p_{tot} = p + B^2/2\mu_0$ ) accelerate (or decelerate) flow, for in-phase (or out-of-phase) variations of  $\delta p$  and  $\delta B$ , leading to the **fast and slow mode waves**.*

## Magnetohydrodynamic waves



- Magnetosonic waves

**compressible**

- parallel slow and fast

- perpendicular fast

$$c_{ms} = (c_s^2 + v_A^2)^{1/2}$$

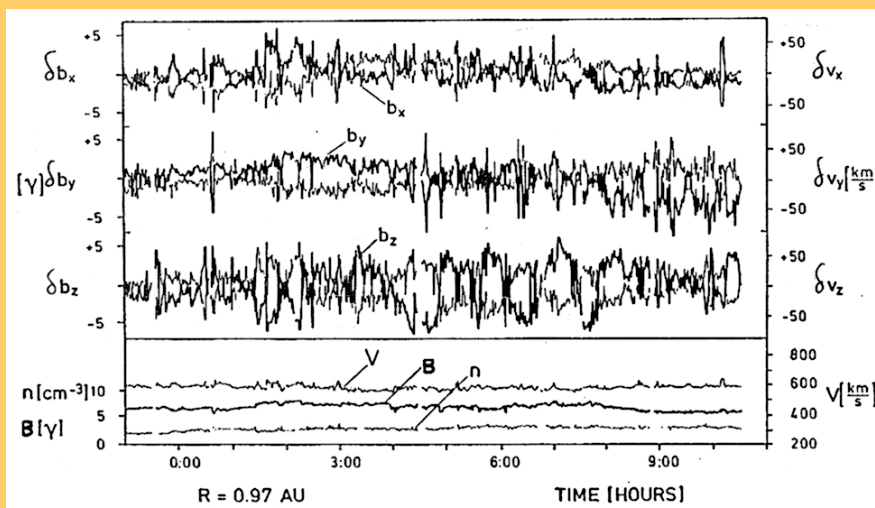
- Alfvén wave

**incompressible**

parallel and oblique

$$v_A = B/(4\pi\rho)^{1/2}$$

## Alfvén waves in the solar wind



Neubauer et al., 1977

Helios

$$\delta v = \pm \delta v_A$$

## Alfvénic fluctuations

Ulysses observed many such waves (4-5 per hour) in fast wind over the poles:

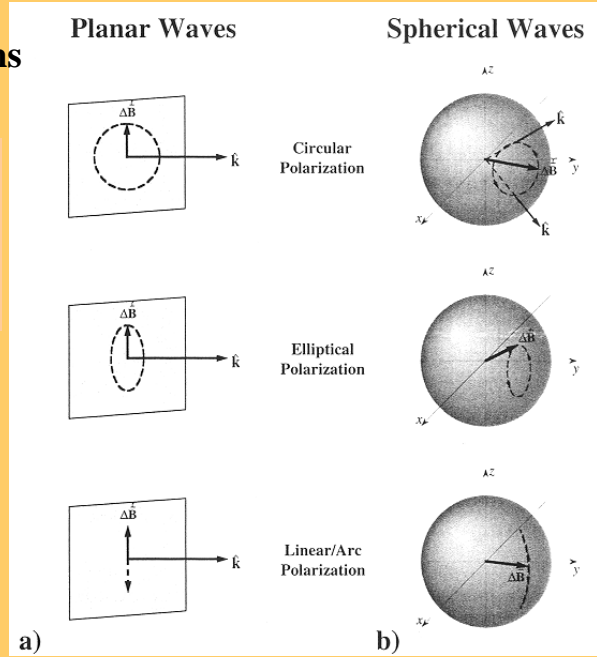
- Arc-polarized waves
- Phase-steepened

*Rotational discontinuity:*

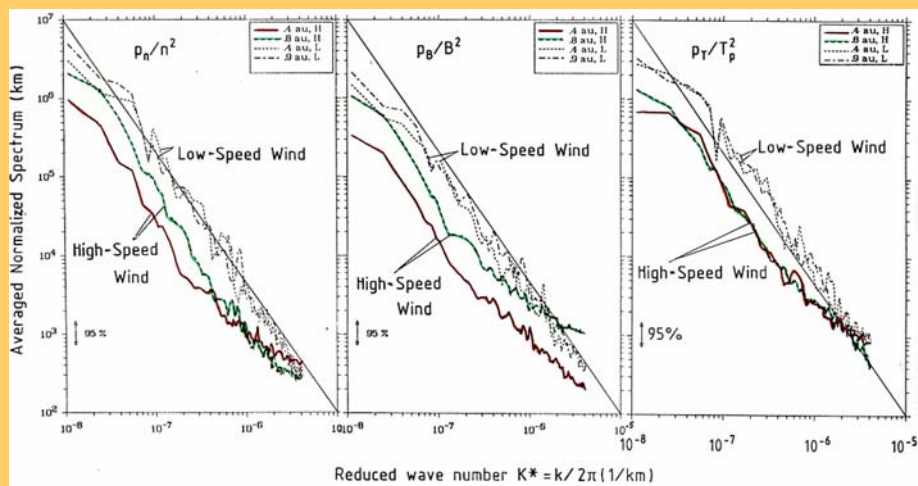
$$\Delta \mathbf{V} = \pm \Delta V_A \hat{\mathbf{k}}$$

Finite jumps in velocities over gyrokinetic scales

Tsurutani et al., 1997



## Compressive fluctuations in the solar wind



Marsch and Tu, JGR, 95, 8211, 1990

Kolmogorov-type turbulence



## Magnetohydrodynamic turbulence in space plasmas

### *Integral invariants of ideal MHD*

$$E = 1/2 \int d^3x (V^2 + V_A^2) \quad \text{Energy}$$

$$H_c = \int d^3x (\mathbf{V} \cdot \mathbf{V}_A) \quad \text{Helicity}$$

$$H_m = \int d^3x (\mathbf{A} \cdot \mathbf{B}) \quad \text{Magnetic helicity}$$

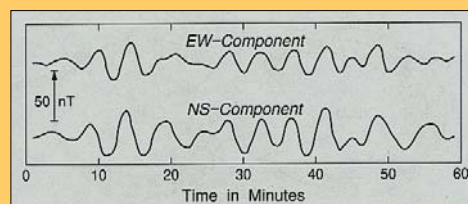
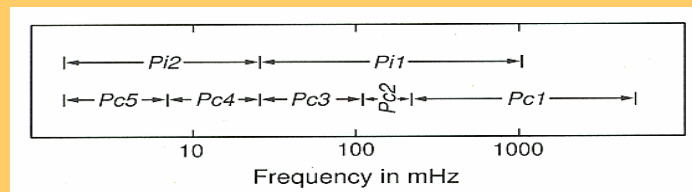
$$\mathbf{B} = \nabla \times \mathbf{A}$$

Elsässer variables:  $\mathbf{Z}^\pm = \mathbf{V} \pm \mathbf{V}_A$

$$E^\pm = 1/2 \int d^3x (\mathbf{Z}^\pm)^2 = \int d^3x e^\pm(\mathbf{x})$$

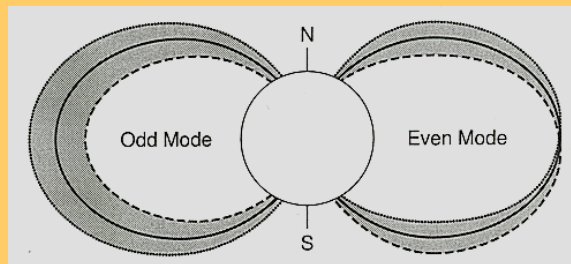
## Geomagnetic pulsations

Fast magnetic fluctuations of the Earth's surface magnetic field (few mHz up to a few Hz), corresponding to oscillation periods from several hundreds to fractions of seconds. The pulsating quasi-sinusoidal (*continuous pulsations*) disturbances are observed globally and associated with Alfvén (and magneto-acoustic) waves. The *irregular pulsations* are short-lived and localized.



## Field line resonances

The shown Pc5 pulsations are caused by oscillations of the Earth magnetic field and explained as standing single-fluid shear Alfvén waves, whose wavelengths must fit the geometry. The length,  $l$ , of the fieldline between two reflection point must be a multiple of half the wavelength,  $\lambda$ , implying:  $v_h \lambda = 2l$ , with  $v_h = 1, 2, 3, \dots$ . From the dispersion relation, with the average Alfvén velocity,  $\langle v_A \rangle$ , along the fieldline one finds:

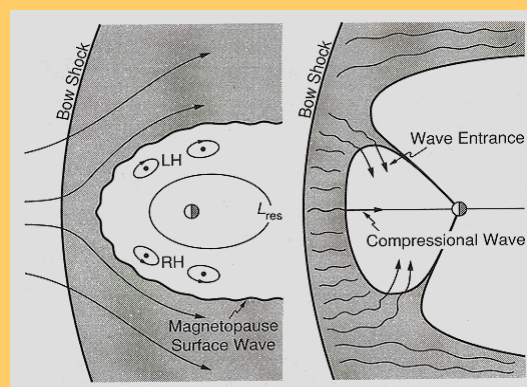


$$\omega_h = \frac{v_h \pi \langle v_A \rangle}{l}$$

*Fundamental poloidal field-line resonances*

## Magnetosheath turbulence as source of pulsations

*Excitation of global magnetospheric pulsations through driver at matching frequency:  $\omega_{ex} = \omega_{res}$ .*



Excitation scenarios:

- Surface waves excited by *Kelvin-Helmholtz instability* driven by flow around magnetopause
- Compressional waves *leaking from magnetosheath* through polar cusp into magnetosphere