Boundaries, shocks and discontinuities

- **Fluid boundaries**
- **General jump conditions**
- **Rankine-Hugoniot conditions**
- **Set of equations for jumps at a boundary**
- **Discontinuities**
- **Shock types**
- **Bow shock geometry**

Fluid boundaries

Equilibria between plasmas with different properties give rise to the evolution of boundaries, which take the form of narrow (gyrokinetic scales) layers called ***discontinuities***. Conveniently, one starts from ideal plasmas (without dissipation) on either side. The transition from one side to the other requires some ***disspation***, which is concentrated in the layer itself but vanishes outside. MHD (with ideal Ohm's law and no space charges) in conservation form reads:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

$$\frac{\partial (nm\mathbf{v})}{\partial t} + \nabla \cdot (nm\mathbf{v}\mathbf{v}) = -\nabla \cdot \left(\mathbf{P} + \frac{B^2}{2\mu_0}\mathbf{I}\right) + \frac{1}{\mu_0}\nabla \cdot (\mathbf{B}\mathbf{B})$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

$$\nabla \cdot \mathbf{B} = 0$$

Definitions, normal and jumps

Changes occur perpendicular to the discontinuity, parallel the plasma is uniform. The normal vector, **n**, to the surface *S*(**x**) is defined as:

$$\mathbf{n} = -\frac{\nabla \mathcal{S}}{|\nabla \mathcal{S}|}$$

Any closed line integral (along a rectangular box tangential to the surface and crossing *S* from medium 1 to 2 and back) of a quantity *X* reduces to

$$\oint_S \frac{dX}{dn} dn = 2\int_1^2 \frac{dX}{dn} dn = 2\,(X_2 - X_1) = 2[X]$$

Since an integral over a conservation law vanishes, the gradient operation can be replaced by

$$\begin{array}{ccc} \nabla X & \rightarrow & \mathbf{n}[X] \\ \nabla \cdot \mathbf{X} & \rightarrow & \mathbf{n} \cdot [\mathbf{X}] \\ \nabla \times \mathbf{X} & \rightarrow & \mathbf{n} \times [\mathbf{X}] \end{array}$$

Transform to a frame moving with the discontinuity at local speed, **U**. Because of ***Galilean invariance***, the time derivative becomes:

$$\partial/\partial t = -\mathbf{U} \cdot \nabla = -U \cdot \mathbf{n}(\partial/\partial n)$$

Discontinuities and shocks

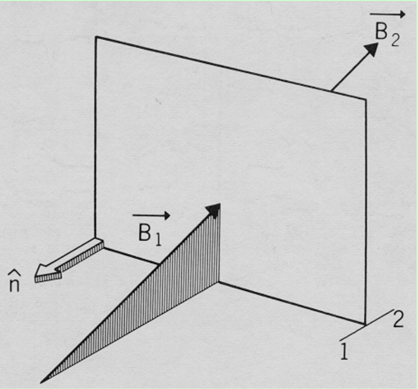
Continuity of the mass flux and magnetic flux:

$B_n = B_{1n} = B_{2n}$

$G_n = \rho_1(V_{1n} - U) = \rho_2(V_{2n} - U)$

U is the speed of surface in the normal direction; **B** magnetic field vector; **V** the flow velocity. ***Mach number***, M = V/C. Here C is the wave phase speed.

Shock:	$G \neq 0$
Discontinuity:	$G = 0$



Contact discontinuity (CD)

Index 1 ***upstream*** and 2 ***downstream***;

B does not change across the surface of the CD, but $\rho_1 \neq \rho_2$ and $T_1 \neq T_2$.

Rankine-Hugoniot conditions I

In the ***comoving frame*** (**v**' = **v** - **U**) the discontinuity (D) is stationary so that the time derivative can be dropped. We ***skip the prime*** and consider the situation in a frame where D is at rest. We assume an isotropic pressure, **P**=p**1**. Conservation laws transform into the ***jump conditions*** across D, reading:

$$
\begin{aligned}
\mathbf{n}\cdot[n\mathbf{v}] &= 0 \\
\mathbf{n}\cdot[nm\mathbf{v}\mathbf{v}] + \mathbf{n}\left[p + \frac{B^2}{2\mu_0}\right] - \frac{1}{\mu_0}\mathbf{n}\cdot[\mathbf{B}\mathbf{B}] &= 0 \\
[\mathbf{n}\times\mathbf{v}\times\mathbf{B}] &= 0 \\
\mathbf{n}\cdot[\mathbf{B}] &= 0
\end{aligned}
$$

An additional equation expresses conservation of total energy across the D, whereby w denotes the specific internal energy in the plasma, $w=c_v T$.

$$\left[nm\mathbf{n}\cdot\mathbf{v}\left\{\frac{v^2}{2} + w + \frac{1}{nm}\left(p + \frac{B^2}{\mu_0}\right)\right\} - \frac{1}{\mu_0}(\mathbf{v}\cdot\mathbf{B})\mathbf{n}\cdot\mathbf{B}\right] = 0$$

Rankine-Hugoniot conditions II

The normal component of the magnetic field is continuous: $\boxed{[B_n] = 0}$

The ***mass flux*** across D is a constant: $\boxed{[nv_n] = 0}$

Using these two relations and splitting **B** and **v** into their normal (index n) and tangential (index t) components gives three remaining ***jump conditions***:

$$
\begin{aligned}
nmv_n\,[\mathbf{v}_t] &= \frac{B_n}{\mu_0}\,[\mathbf{B}_t] && \text{stress balance} \\
B_n\,[\mathbf{v}_t] &= [v_n\mathbf{B}_t] && \text{tangential electric field} \\
nmv_n\,[\mathbf{v}_n] &= -\left[p + \frac{B^2}{2\mu_0}\right] && \text{pressure balance}
\end{aligned}
$$

These equations contain all basic information about any D in ideal MHD.

Mass-flux classification of D's

Introduce the average of X across a D by the mean value:

$$\langle X \rangle = \frac{1}{2}(X_1 + X_2)$$

New variables: ***Specific volume***, $V = (nm)^{-1}$ and constant ***normal mass flux***, $F = nm\upsilon_n$. After tedious algebra (***left as a hard exercise***) the determinant for the system of RHC can be written as a seventh-order equation in F, reading:

$$F\left(F^2 - \frac{B_n^2}{\mu_0\langle\mathcal{V}\rangle}\right)\left\{F^4 + F^2\left(\frac{[p]}{[\mathcal{V}]} - \frac{\langle\mathbf{B}\rangle^2}{\mu_0\langle\mathcal{V}\rangle}\right) - \frac{B_n^2}{\mu_0\langle\mathcal{V}\rangle}\frac{[p]}{[\mathcal{V}]}\right\} = 0$$

Tangential, contact D

Rotational D

Shocks

Contact and tangential discontinuity

CD and TD are characterized by a zero normal mass flow, and thus $\upsilon_n = 0$.

$$F_I = 0$$

$B_n \neq 0$ -> Contact D

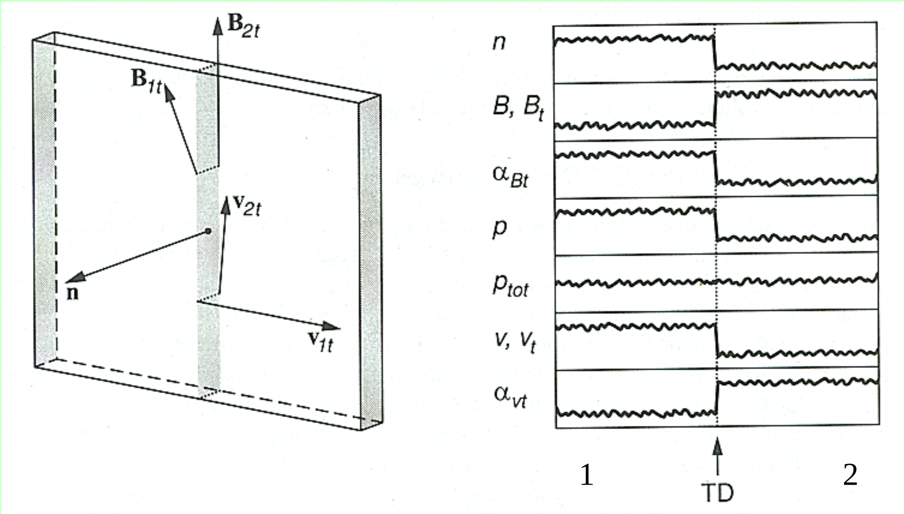
$$\begin{aligned} [p] &= 0 \\ [\mathbf{v}_t] &= 0 \\ [B_n] &= 0 \\ [\mathbf{B}_t] &= 0 \end{aligned}$$

Since in a CT the thermal pressure remains constant, any change in density must be compensated by a change in temperature. However, a temperature jump is quickly ironed out by electron heat conduction -> ***CD do not persist long***.

$B_n = 0$ -> Tangential D

$$\left[p + \frac{B^2}{2\mu_0}\right] = 0$$

Schematic parameter changes across a TD



Total pressure is constant. TD's are often observed in the solar wind.

Rotational discontinuity (RD)

RDs are characterized by a finite normal mass flow, but a continuous υ_n.

$$F_{II} = \pm \frac{B_n}{\sqrt{\mu_0 \langle \mathcal{V} \rangle}}$$

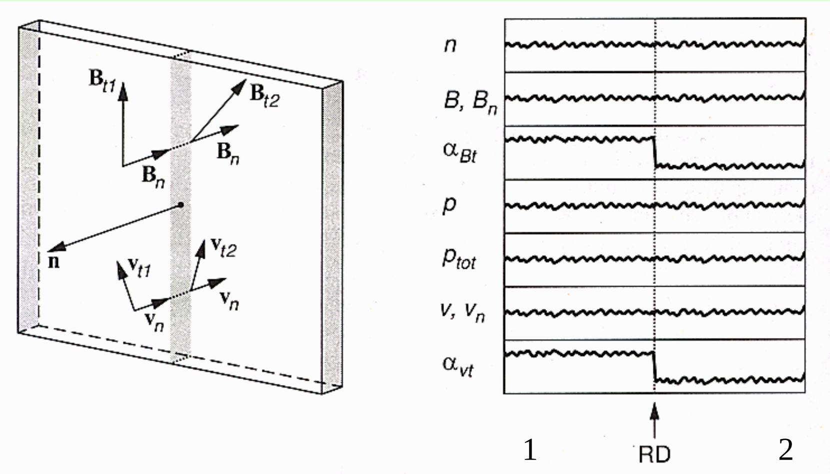
Because of the continuity of F and υ_n, there can be no jump in density. Since B_n and υ_n are constant, the tangential components must rotate together!

Constant normal υ_n leads naturally to a constant v_{An}. Thus the so-called ***Walen relation*** holds:

$$\upsilon_n = \frac{B_n}{\sqrt{n m \mu_0}}$$

$$\begin{aligned}
[n] &= 0 \\
[p] &= 0 \\
[\upsilon_n] &= 0 \\
[B_n] &= 0 \\
\left[B^2\right] &= 0 \\
\left[\mathbf{v}_t - \frac{\mathbf{B}_t}{\sqrt{n m \mu_0}}\right] &= 0
\end{aligned}$$

Schematic parameter changes across a RD



At a RD the jump in tangential flow velocity is exactly equal to the jump in tangential Alfvén velocity. RD's occur frequently in the fast solar wind.

Entropy changes at discontinuities

Instead of the energy conservation equation one may use the entropy equation, which can without dissipation be written as:

$$\frac{d\mathcal{S}}{dt} = \frac{\partial \mathcal{S}}{\partial t} + \mathbf{v} \cdot \nabla \mathcal{S} = 0$$

For an ideal isotropic gas we have (with gas constant R_0 and polytropic index γ=5/3) the ***entropy change***:

$$\Delta \mathcal{S} = R_0 ln \left[\frac{T^{1/(\gamma-1)}}{n} \right]$$

In a steady state incompressible medium:

$$\nabla \cdot (\mathbf{v}\mathcal{S}) = 0$$

Or written as a jump condition:

$$[\upsilon_n \mathcal{S}] = 0$$

- ***Discontinuity (D) with*** $[\upsilon_n] = 0$ ***conserves entropy***
- ***D in compressible medium with*** $[\upsilon_n] \neq 0$ ***leads to increase of entropy***

Shocks

This third type of D is characterised by a ***non-vanishing normal mass flux***, $F = nm\upsilon_n \neq 0$. F is a solution of the bi-quadratic equation:

$$F_{III}^4 + F_{III}^2\left(\frac{[p]}{[\mathcal{V}]} - \frac{\langle \mathbf{B}\rangle^2}{\mu_0\langle\mathcal{V}\rangle}\right) - \frac{B_n^2}{\mu_0\langle\mathcal{V}\rangle}\frac{[p]}{[\mathcal{V}]} = 0$$

Shock solutions are obtained if the specific volume $V = (nm)^{-1}$ jumps, and since F^2 must be positive, the following inequality holds:

$$\frac{\langle \mathbf{B}\rangle^2}{\mu_0\langle\mathcal{V}\rangle} > \frac{[p]}{[\mathcal{V}]}$$

This condition is easily satisfied if the right ratio is negative, hence when pressure and specific volume vary oppositely across the D; --> shock

Machnumbers

We may multiply the previous equation for the normal mass flux, $F = nmv_n$, by $\langle\mathcal{V}\rangle^2 = <(nm)^{-1}>^2$, and introduce the following two speeds:

Effective Alfvén speeds:	$U_A^2 = <\mathbf{B}>^2\langle\mathcal{V}\rangle/\mu_0$; $U_{An}^2 = <B_n>^2\langle\mathcal{V}\rangle/\mu_0$
Effective sound speed:	$C_S^2 = -([p]/[\mathcal{V}])\langle\mathcal{V}\rangle^2$
Effective shock speed:	$U = F\langle\mathcal{V}\rangle$

In terms of these speeds the shock „dispersion equation“ can be written as:

$$U^4 - U^2(U_A^2+C_S^2) + C_S^2 U_{An}^2 = 0$$

This yields the fast and slow magnetoacoustic velocities, U_F and U_S, as possible solutions. The corresponding Machnumbers are: $M_{F,S} = U/U_{F,S}$.

Coplanarity

Knowing that for the shock $\upsilon_n \neq 0$ and $[\upsilon_n] \neq 0$, we can eliminate $[\upsilon_t]$ from the RHCs and obtain:

$$[\upsilon_n \mathbf{B}_t] = \frac{B_n^2}{\mu_0 F_{III}} [\mathbf{B}_t]$$

Hence the cross product of the left with the right hand side must vanish:

$$[\mathbf{B}_t] \times [\upsilon_n \mathbf{B}_t] = 0$$

When resolving these brackets one obtains the condition:

$$(\upsilon_{n1} - \upsilon_{n2}) \left(\mathbf{B}_{t1} \times \mathbf{B}_{t2} \right) = 0$$

***The resulting coplanarity theorem implies that the magnetic field across the shock has a 2-D geometry: upstream and dowstream tangential fields are parallel to each other and coplanar with the shock normal* n.**

Shock jump conditions

We use for the sake of simplicity the internal energy for a monoatomic gas under adiabatic conditions: *w=p/(nm(*γ*-1))*. With this the energy conservation can be brought in a form using jumps and averages:

$$[p]\langle \upsilon_n \rangle + [\upsilon_n]\gamma\langle p \rangle + \frac{\gamma - 1}{4\mu_0}[\upsilon_n][B_t^2] = 0$$

Exploiting the coplanarity condition, we can write the tangential components in scalar form as follows:

$$[\upsilon_n]\langle B_t \rangle + [B_t]\langle \upsilon_n \rangle = \frac{B_n^2}{\mu_0 F_{III}}[B_t]$$

$$\begin{aligned} B_n [\upsilon_t] &= \frac{B_n^2}{nm\mu_0}[B_t] \\ F_{III} [\upsilon_n] &= -\left[p + \frac{B_t^2}{2\mu_0} \right] \end{aligned}$$

These equations form a closed set for the jumps in shocks (general RHCs).

Fast and slow shocks

By eliminating the jump in the normal velocity $[v_n]$ one can obtain a relation between the jumps in the thermal and tangential magnetic pressure (where the quantity H is defined below):

$$\left(\frac{\langle v_n \rangle}{\gamma - 1} - H\right)[p] = \frac{H}{\mu_0 F_{III}}\left[B_t^2\right]$$

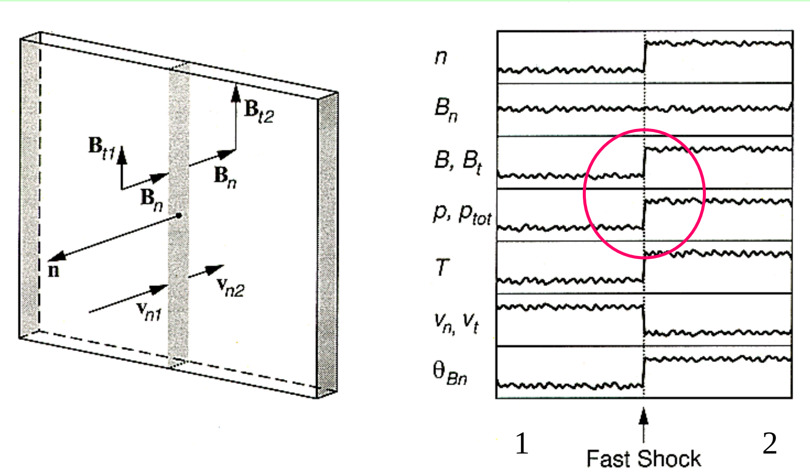
$$F_{III}\mathcal{H} = \frac{[B_t^2]}{4\mu_0} + \frac{\gamma\langle p \rangle}{\gamma - 1}$$

Since always $[p] > 0$ for shocks, because the disturbed plasma is compressed and heated, one can distinguish between two shock types:

Fast shocks with increasing magnetic pressure, $[B_t^2] > 0$, satisfying $\langle v_n \rangle > (\gamma - 1)H$ and $M_F > 1$

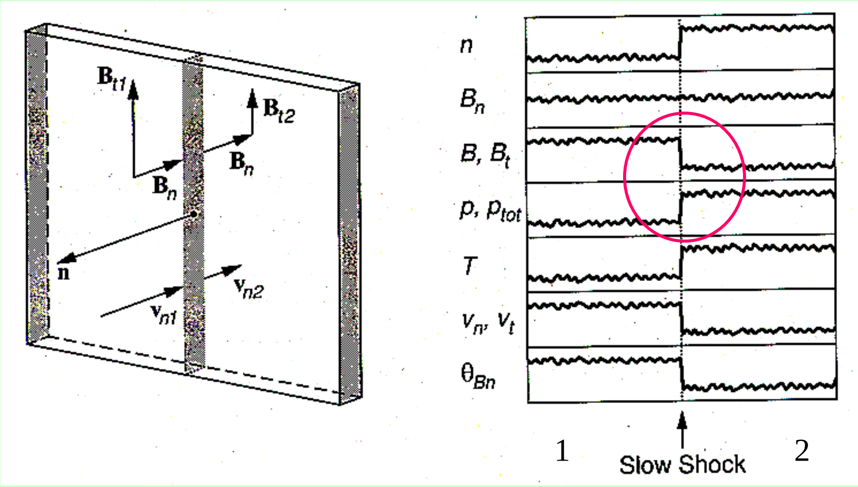
Slow shocks with decreasing magnetic pressure, $[B_t^2] < 0$, satisfying $\langle v_n \rangle < (\gamma - 1)H$ and $M_S > 1$

Schematic parameter changes across a fast shock



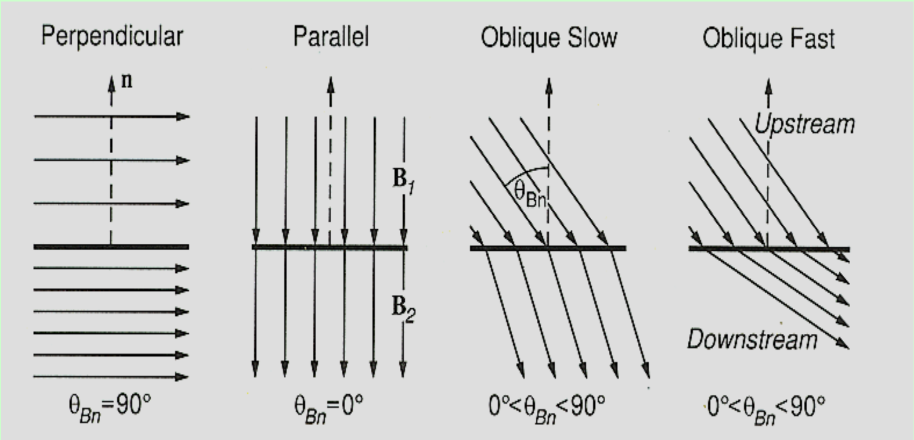
In a fast shock the magnetic field increases. It is tilted toward the surface and bends away from the normal. Fast shocks may evolve from fast mode waves.

Schematic parameter changes across a slow shock

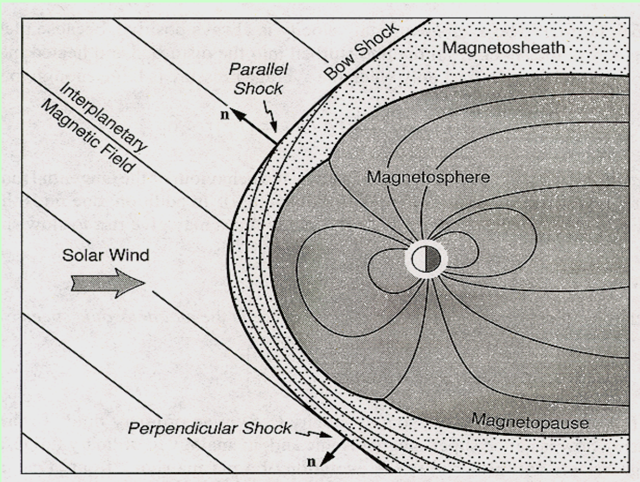


In a slow shock the magnetic field decreases. It is tilted away from the surface and bends toward the normal. Slow shocks may evolve from slow mode waves.

Four possible geometries of shock normals



Bow shock



The most famous and mostly researched shock is the ***bow shock*** standing in front of the Earth as result of the interaction of the *magnetosphere* with the *supersonic solar wind*, with a high Machnumber, $M_F \approx 8$. Solar wind density and field jump by about a factor of 4 into the *magnetosheath*.