# Plasma waves in the fluid picture I

- Langmuir oscillations and waves
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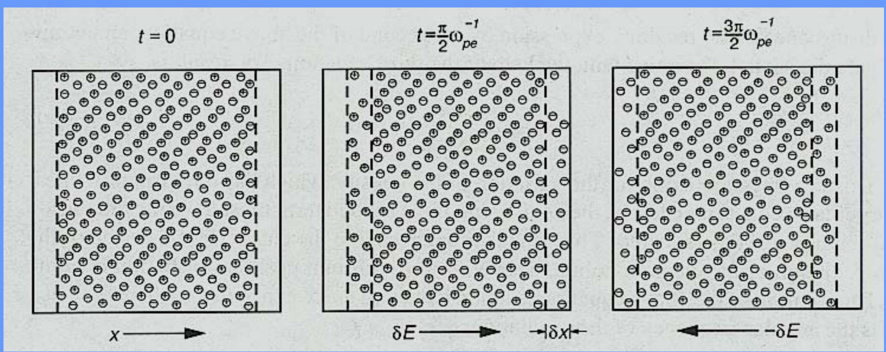
- Dispersion in a cold electron plasma

## Langmuir oscillations I

Consider high-frequency electron oscillations (ions remain at rest) with displacement $\delta x$, causing an electric field, $\delta E$, and corresponding force, $-e\delta E$, on each electron, and thus leading to a ***density perturbation***, $\delta n$, as shown below:



## Langmuir oscillations II

The electron fluid equations read:

***Continuity***:

$$\frac{\partial \delta n}{\partial t} = -n_e \frac{\partial \delta v_{e,x}}{\partial x}$$

***Momentum***:

$$\frac{\partial \delta v_{e,x}}{\partial t} = -\frac{e}{m_e}\delta E$$

***Poisson***:

$$\frac{\partial \delta E}{\partial x} = -\frac{e}{\epsilon_0}\delta n$$

Taking the time derivative yields the linear wave equation:

$$\frac{\partial^2 \delta n}{\partial t^2} + \frac{n_e e^2}{m_e \epsilon_0}\delta n = 0$$

It can easily be solved by a simple temporal oscillation with the angular frequency:

$$\omega_{pe}^2 = \frac{n_e e^2}{m_e \epsilon_0}$$

***Electron plasma frequency***

## Langmuir waves

The plasma oscillations are somewhat artificial, since the thermal electrons can move and change their (adiabatic) pressure such that the force balance is:

$$\frac{\partial \delta v_{e,x}}{\partial t} = -\frac{e}{m_e}\delta E - \frac{\gamma_e k_B T_e}{m_e n_e}\frac{\partial \delta n}{\partial x}$$

The wave equation contains now spatial dispersion as well and reads:

$$\frac{\partial^2 \delta n}{\partial t^2} - \frac{\gamma_e k_B T_e}{m_e}\frac{\partial^2 \delta n}{\delta x^2} + \omega_{pe}^2 \delta n = 0$$

A plane wave ansatz yields the dispersion relation with a lower cutoff at the electron plasma frequency. The electron thermal speed is $\upsilon_{the}=(k_B T_e/m_e)^{1/2}$.

$$\omega_l^2 = \omega_{pe}^2 + k^2 \gamma_e \upsilon_{the}^2$$

***-> Langmuir oscillations become travelling electrostatic waves for $k \neq 0$.***

## Ion acoustic waves I

At frequencies below the electron oscillations the ions come into play. They contribute their own plasma frequency:

$$\omega_{pi} = \left(\frac{n_i Z^2 e^2}{m_i \epsilon_0}\right)^{1/2}$$

which is for protons ($Z$=1) by a large factor, $(m_i/m_e)^{1/2} = 43$, smaller than $\omega_{pe}$. When we assume quasineutrality, $n_e \approx n_i$, the electron dynamics reduces to

$$e\delta E = -\gamma_e k_B T_e \frac{\partial \ln n_e}{\partial x}$$

Upon linearization, $n_e = n_0 + \delta n_e$, and with the electric field, $\delta E = -\partial \delta\Phi/\partial x$, the density fluctuation becomes proportional to the potential fluctuation:

$$n_e = n_0 \exp\left(\frac{e\delta\phi}{\gamma_e k_B T_e}\right)$$

The linearized ion equations of motion:

$$\frac{\partial \delta n_i}{\partial t} = -n_i \frac{\partial \delta v_{i,x}}{\partial x}$$
$$\frac{\partial \delta v_{i,x}}{\partial t} = \frac{e}{m_i}\delta E$$

## Ion acoustic waves II

We neglect the ion pressure, assuming $T_i << T_e$, and exploit charge-neutrality of the fluctuations, $\delta n_e = \delta n_i = \delta n$, and thus arrive at a single wave equation:

$$\frac{\partial^2 \delta n}{\partial t^2} - \frac{\gamma_e k_B T_e}{m_i}\frac{\partial^2 \delta n}{\partial x^2} = 0$$

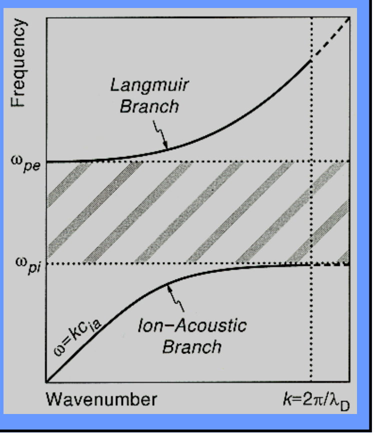
Its plane wave solution gives the dispersion of ***ion acoustic waves***.

$$\omega_{ia}^2 = \frac{\gamma_e k_B T_e}{m_i} k^2$$

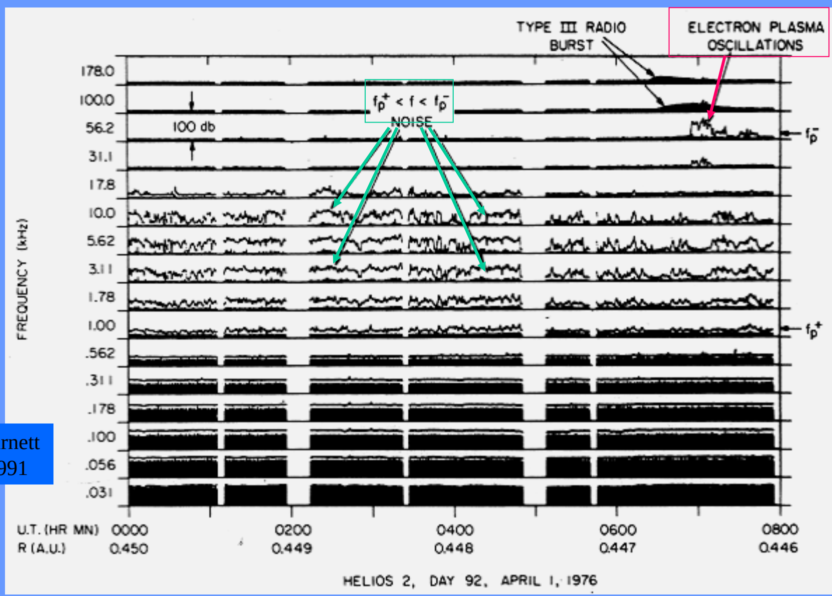
Upon division by $k^2$ one finds the phase velocity, the ion acoustic speed, in which ions provide inertia and electrons the restoring force.

$$c_{ia} = \left(\frac{\gamma_e k_B T_e}{m_i}\right)^{1/2}$$

***No electrostatic wave can propagate between $\omega_{pe}$ and $\omega_{pi}$ in an unmagnetized plasma.***



## Ion acoustic and Langmuir waves




Gurnett, 1991

## Ordinary electromagnetic waves I

The occurence of electrostatic waves is a particular property of a plasma with free charges, which can also contribute to ***current oscillations***. These become the source of electromagnetic waves, of which the magnetized plasma can carry a large variety. An electromagnetic wave of frequency $\omega$ will set an electron in motion, creating a current density:

$$\delta \mathbf{j}_{em} = -en_0 \delta \mathbf{v}_e$$

The velocity disturbance follows from the equation of motion in the electromagnetic plane wave field, δ***E***:

$$\delta \mathbf{v}_e = -\frac{ie}{\omega m_e} \delta \mathbf{E}$$

From this we read of the current density, $\delta \boldsymbol{j}_{em} = \sigma_{em} \delta \boldsymbol{E}$, resulting in the conductivity:

$$\sigma_{em} = \frac{i\epsilon_0 \omega_{pe}^2}{\omega}$$

The dispersion relation in terms of the ***refractive index***, $N$, is given by:

$$N^2 = \frac{k^2 c^2}{\omega^2}$$

In vacuo, $N^2 = 1$. In plasma, unity is replaced by the ***dielectric constant***:

$$\frac{k^2 c^2}{\omega^2} = \epsilon(\omega, \mathbf{k})$$

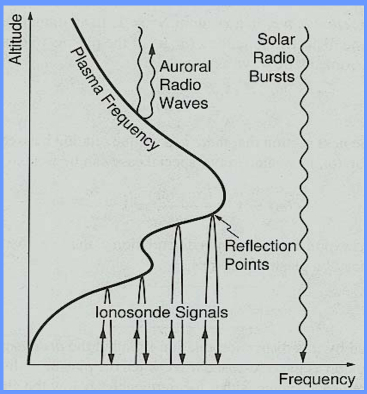
## Ordinary electromagnetic waves II

There is a unique relation between conductivity and dielectric constant, which in our case can be written as:

$$\epsilon(\omega) = 1 + \frac{i\sigma_{em}(\omega)}{\epsilon_0 \omega} = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

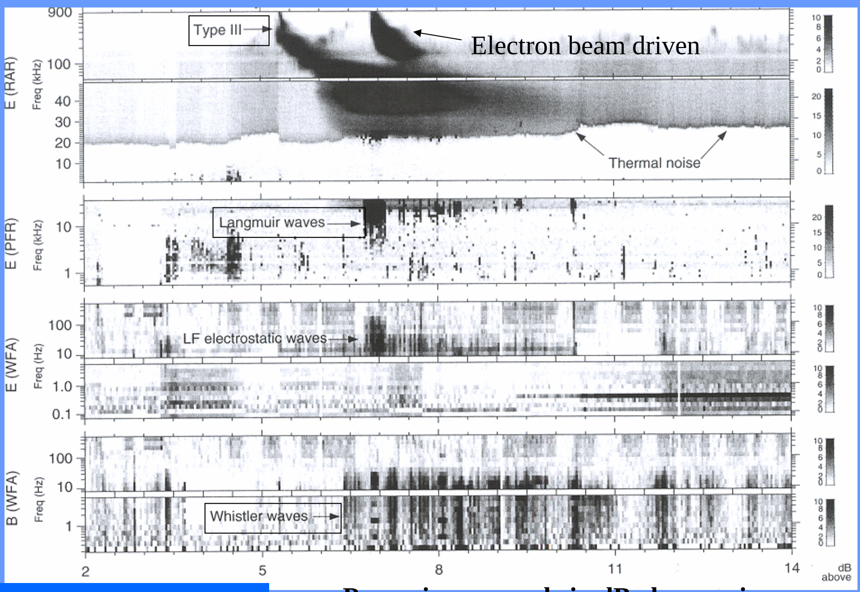
With the help of this expression, we find the dispersion relation (of the ordinary mode):

$$\omega_{om}^2 = \omega_{pe}^2 + c^2 k^2$$




The wave number vanishes at the plasma frequency, which is a ***cut-off*** for the ordinary mode. Here $N^2$ becomes formally negative, the wave is ***reflected***. See on the right side the ionospheric reflection of radio waves.

## Ulysses wave data - day 73 in 1995




McDowell and Kellog, 2001

**Power in grey scale in dB above noise**

## General wave equation

Maxwell's equations including *external* sources, $\boldsymbol{j}_{ex}$ and $\rho_{ex}$, read:

$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 (\mathbf{j} + \mathbf{j}_{ex})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} (\rho + \rho_{px})$$

Taking the time derivative of the first and replacing the magnetic field with the second, yields a general wave equation for the electric field:

$$\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \left( \frac{\partial \mathbf{j}}{\partial t} + \frac{\partial \mathbf{j}_{ex}}{\partial t} \right)$$

## The conductivity tensor

We have included $\boldsymbol{j}_{ex}$ and $\rho_{ex}$ explicitly, which may be imposed from outside on the plasma, but since the equations are linear in charge density and current can simply be added to the *internal* induced ones. They may be assumed to be given in linear response to the total field by Ohm's law:

$$\mathbf{j} = \int d^3x' \int_{-\infty}^{t} dt' \sigma \left( \mathbf{x}, \mathbf{x}', t, t' \right) \cdot \mathbf{E}(t', \mathbf{x}')$$

*The integration from -∞ to t reflects* ***causality*** *(no effect before a cause).*

The relation is constitutive for the ***material properties*** of a plasma and involves all microscopic particle motions. If the medium is *stationary* and *uniform* only the coordinate differences enter, and thus Ohm's law reduces then to:

$$\mathbf{j}(t, \mathbf{x}) = \int d^3x' \int_{-\infty}^{t} dt' \sigma \left( \mathbf{x} - \mathbf{x}', t - t' \right) \cdot \mathbf{E}$$

## General wave dispersion equation I

Interpreting the electric field as a superposition of plane waves (Fourier analysis) with amplitude

$$\delta\mathbf{E}(\omega,\mathbf{k}) = \delta\mathbf{E}_0(\omega,\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$$

Making using of the folding-integral theorem transforms, after substantial formal algebra (left as an exercise), the wave differential equation into an algebraic (dyadic tensor) equation for the electric field amplitude:

$$\left[\left(k^2 - \frac{\omega^2}{c^2}\right)\mathbf{I} - \mathbf{k}\mathbf{k} - i\omega\mu_0\sigma(\omega,\mathbf{k})\right]\cdot\delta\mathbf{E}_0(\omega,\mathbf{k}) = 0$$

The requirement that the electric field be real leads to the symmetry conditions:

$$\begin{aligned}\delta\mathbf{E}^*(\omega,\mathbf{k}) &= \delta\mathbf{E}(-\omega,-\mathbf{k})\\ \sigma^*(\omega,\mathbf{k}) &= \sigma(-\omega,-\mathbf{k})\end{aligned}$$

***Nontrivial solutions require: the determinant of the dynamic matrix vanishes.***

## General wave dispersion equation II

$$D(\omega,\mathbf{k}) = Det\left[\left(k^2 - \frac{\omega^2}{c^2}\right)\mathbf{I} - \mathbf{k}\mathbf{k} - i\omega\mu_0\sigma(\omega,\mathbf{k})\right] = 0$$

In dielectric media it is convenient to use the electric induction and ***dielectric tensor***, $\varepsilon(\mathbf{k},\omega)$, via the relation

$$\delta\mathbf{D} = \epsilon\cdot\delta\mathbf{E}$$

With its help the current density is

$$\delta\mathbf{j}(\omega,\mathbf{k}) = -i\omega\epsilon_0\left[\epsilon(\omega,\mathbf{k}) - \mathbf{I}\right]\cdot\delta\mathbf{E}(\omega,\mathbf{k})$$

Using Ohm's law gives the general dielectric tensor and the dispersion relation defined as

$$\epsilon(\omega,\mathbf{k}) = \mathbf{I} + \frac{i}{\omega\epsilon_0}\sigma(\omega,\mathbf{k})$$

$$Det\left[\frac{k^2c^2}{\omega^2}\left(\frac{\mathbf{k}\mathbf{k}}{k^2} - \mathbf{I}\right) + \epsilon(\omega,\mathbf{k})\right] = 0$$

***Its solutions describe linear eigenmodes at frequeny $\omega$(k) and wavevector k.***

## Dispersion in an isotropic plasma

In this case only k defines a symmetry direction, and thus the unit tensor can be decomposed as follows:

$$\mathbf{I}_L = \frac{\mathbf{kk}}{k^2} \qquad \mathbf{I}_T = \mathbf{I} - \frac{\mathbf{kk}}{k^2}$$

Corresponding to ***longitudinal*** and ***transverse*** components such that

$$\epsilon(\omega, \mathbf{k}) = \epsilon_L(\omega, k)\mathbf{I}_L + \epsilon_T(\omega, k)\mathbf{I}_T$$

$$\begin{aligned} \epsilon_L(\omega, k) &= 0 \\ \epsilon_T(\omega, k) - \frac{k^2c^2}{\omega^2} &= 0 \end{aligned}$$

$$\begin{aligned} \epsilon_L(\omega, k) &= \frac{\mathbf{k}\cdot\epsilon(\omega, \mathbf{k})\cdot\mathbf{k}}{k^2} \\ \epsilon_T(\omega, k) &= \frac{tr\epsilon(\omega, \mathbf{k}) - \epsilon_L(\omega, k)}{2} \end{aligned}$$

***The dispersion relation in isotropic media splits into two separate parts,* electrostatic *and purely* electromagnetic *waves.***

## Dispersion in MHD-fluid theory

One-fluid magnethydrodynamics is only valid at low frequencies, $\omega << (\omega_{gi}, \omega_{pi})$, for long wavelengths, and for small phase speeds, such that $\omega/k << c$.

Near $\omega_{gi}, \omega_{pi}$, $\omega_{ge}$, and $\omega_{pe}$ the ion and electron inertia becomes important. At high frequencies new waves appear which require single- or multi-fluid equations for their adequate description, to account for the natural ***kinetic scales*** (which MHD does not have) in a multi-component plasma.

***To derive the dispersion equation the induced current density must be calculated. The simplest model is the cold electron fluid in a strong field. Each new species introduces new dispersion branches.***

## Dispersion in a cold electron plasma I

For cold (zero pressure) electrons the magnetic field is not affected by the electron motion and can be included in the gyrofrequency vector $\boldsymbol{\omega}_{ge}=e\mathbf{B}_0/m_e$. The equations of motions for the fluctuations read:

$$\frac{d\delta v_{\parallel}}{dt} = -\frac{e}{m_e}\delta E_{\parallel}$$
$$\frac{d\delta \mathbf{v}_{\perp}}{dt} = -\frac{e}{m_e}\delta \mathbf{E}_{\perp} + \omega_{ge} \times \delta \mathbf{v}_{\perp}$$

Time differentiation yields the ***driven oscillator*** equation:

$$\frac{\partial^2 \delta \mathbf{v}_{\perp}}{\partial t^2} + \omega_{ge}^2 \delta \mathbf{v}_{\perp} = -\frac{e}{m_e}\left(\frac{\partial \delta \mathbf{E}_{\perp}}{\partial t} + \omega_{ge} \times \delta \mathbf{E}_{\perp}\right)$$

Assuming a cold plasma means all electrons have the same speed, and thus the current density is simply:

$$\delta \mathbf{j} = -en_0 \delta \mathbf{v} = \sigma \cdot \delta \mathbf{E}$$

## Dispersion in a cold electron plasma II

For vanishing perpendicular electric field, the electrons perform a pure gyromotion. For the inhomogeneous solution part we make the ansatz of a periodic oscillation, $\delta\mathbf{v} \approx exp(-i\omega t)$. After some vector algebra (left as an exercise) one obtains the frequency-dependent ***conductivity tensor***

$$\sigma(\omega) = \epsilon_0 \omega_{pe}^2 \begin{bmatrix} \frac{i\omega}{\omega^2-\omega_{ge}^2} & \frac{\omega_{ge}}{\omega^2-\omega_{ge}^2} & 0 \\ -\frac{\omega_{ge}}{\omega^2-\omega_{ge}^2} & \frac{i\omega}{\omega^2-\omega_{ge}^2} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{bmatrix}$$

From this equation the dielectric tensor follows by definition.

$$\epsilon_{cold}(\omega) = \begin{bmatrix} 1+\frac{\omega_{pe}^2}{\omega_{ge}^2-\omega^2} & -\frac{i\omega_{ge}}{\omega}\frac{\omega_{pe}^2}{\omega_{ge}^2-\omega^2} & 0 \\ \frac{i\omega_{ge}}{\omega}\frac{\omega_{pe}^2}{\omega_{ge}^2-\omega^2} & 1+\frac{\omega_{pe}^2}{\omega_{ge}^2-\omega^2} & 0 \\ 0 & 0 & 1-\frac{\omega_{pe}^2}{\omega^2} \end{bmatrix}$$

## Dispersion in a cold electron plasma III

The cold electron plasma dispersion relation thus reads:

$$Det\left[\frac{k^2c^2}{\omega^2}\left(\mathbf{I}-\frac{\mathbf{kk}}{k^2}\right)-\epsilon_{cold}\right]=0$$

We can write as shorthand for the dielectric tensor elements:

$$\epsilon=\begin{bmatrix}\epsilon_1 & -i\epsilon_2 & 0\\ i\epsilon_2 & \epsilon_1 & 0\\ 0 & 0 & \epsilon_3\end{bmatrix}$$

$$\begin{aligned}\epsilon_1 &= 1-\frac{\omega_{pe}^2}{\omega^2-\omega_{ge}^2}\\ \epsilon_2 &= -\frac{\omega_{ge}}{\omega}\frac{\omega_{pe}^2}{\omega^2-\omega_{ge}^2}\\ \epsilon_3 &= 1-\frac{\omega_{pe}^2}{\omega^2}\end{aligned}$$

## Dispersion in a cold electron plasma IV

By using the vectorial refractive index, $\mathbf{N} = \mathbf{k}\, c/\omega$, with $N^2 = N_\perp^2 + N_\parallel^2$, and without loss of generality, $k_y$=0, and $\mathbf{k}$ in the ($x$, $z$) plane, we obtain:

$$Det\begin{bmatrix}N_\parallel^2-\epsilon_1 & i\epsilon_2 & -N_\parallel N_\perp\\ -i\epsilon_2 & N^2-\epsilon_1 & 0\\ -N_\parallel N_\perp & 0 & N_\perp^2-\epsilon_3\end{bmatrix}=0$$

***Basic dispersion relation for a zero-temperature charge-compensated electron plasma, which is valid only for:*** $k \ll 1/r_{ge}$ and $v_{the} \ll \omega/k$.

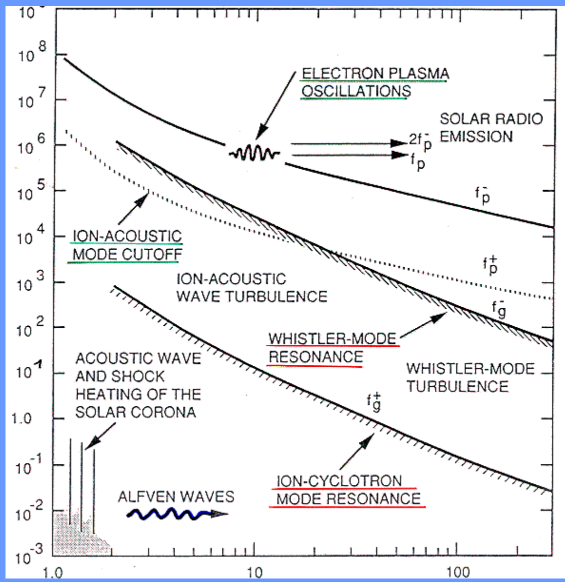
We distinguish between ***parallel***, $N_\perp = 0$, and ***perpendicular***, $N_\parallel = 0$, propagation, in which cases the dispersion relation factorises. The electric field has the components: $E_\parallel = E_z$ and $\mathbf{E}_\perp = E_x\hat{\mathbf{e}}_x + E_y\hat{\mathbf{e}}_y$, which suggests to use instead right-hand (R) and left-hand (L) circularly polarised components:

$$\sqrt{2}\delta E_{R,L}=(\delta E_x \mp i\delta E_y)$$

## Plasma waves and frequencies in the solar wind

Frequency/Hz

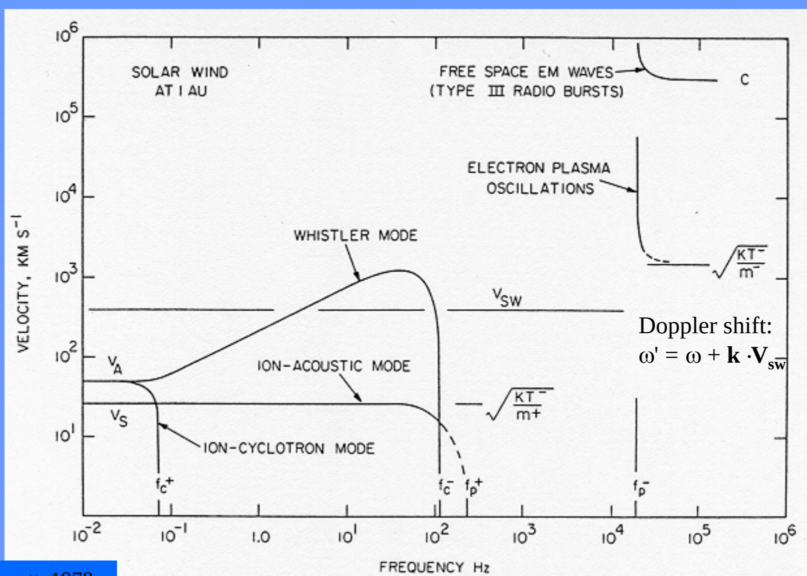
**Non-uniformity leads to strong radial variations of the plasma parameters!**




Gurnett, 1978

Solar distance /$R_S$

## Wave phase velocities in the supersonic solar wind




Doppler shift:
$\omega' = \omega + \mathbf{k} \cdot \mathbf{V}_{sw}$

Gurnett, 1978