# Space Plasma Physics

Thomas Wiegelmann, 2012

1. Basic Plasma Physics concepts
2. Overview about solar system plasmas

## Plasma Models

3. Single particle motion, Test particle model
4. Statistic description of plasma, BBGKY-Hierarchy and kinetic equations
5. Fluid models, Magneto-Hydro-Dynamics
6. Magneto-Hydro-Statics
7. Stationary MHD and Sequences of Equilibria

## Stationary MHD and Sequences of Equilibria

- Slowly varying sequences of equilibria.
- Topology conserving sequences of equilibria and formation of thin current sheets.
- Comparison: Stationary incompressible Hydro-dynamics and Magneto-hydro-statics
- Transformation from static MHD-equilibria to equilibria with plasma flow.

## Sequences of Equilibria

- So far we studied static and stationary states as independent equilibria, which do not depend on time.
- Equilibria do, however, often depend on boundary conditions (like magnetic field in solar photosphere for coronal modelling or solar wind pressure for magnetotail models) which vary slowly in time.
- => We get a time-sequence of equilibria.
- We do not, however, understand how the transition between different equilibria takes place physically.
- For some cases (e.g. magnetospheric convection) we know that the plasma is ideal (no topology changes) in quiet times.

## Sequences of Equilibria

- In the equilibrium theory developed so far, the different equilibria are not constraint.
- => For different boundary conditions we get different magneto-static equilibria, which might have a different magnetic topology.
- Such a sequence of equilibria CANNOT be considered as a physical meaningful slow evolution within ideal MHD.
- Can we constrain a sequence of static equilibria in a way that the ideal MHD-equation are obeyed?

## Ideal MHD

Source: Schindler & Birn, JGR 1982

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$ Continuity Eq.

$$\cancel{\rho \frac{\partial \underline{v}}{\partial t}} + \cancel{\rho \underline{v} \cdot \nabla \underline{v}} = -\nabla p + \underline{j} \times \underline{B}$$ Force-balance, Momentum equation

$$\underline{E} + \underline{v} \times \underline{B} = 0$$ Ideal Ohms Law

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right) + \underline{v} \cdot \nabla\left(\frac{p}{\rho^{\gamma}}\right) = 0$$ Equation of state, Adiabatic convection

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{B} = \mu_0 \underline{j}$$

$$\nabla \cdot \underline{B} = 0$$

We often can neglect the terms on the left side of the momentum equation (when they small compared to pressure gradient and Lorentz-force) We must still solve, however continuity equation, Ohms law and Eq. of state.

## Sequences of Equilibria

- For special configurations (liked magnetospheric convection) it is possible to reformulate the ideal MHD equations in order to compute sequence of static equilibria under constrains of field line conservation.
- A principle way is:
  -compute an initial static equilibria
  -solve the time dependent ideal MHD-equations numerically and change the boundary conditions slowly in time.
  -If a nearby equilibrium exists, the MHD-code will very likely find it.
- The code will also find out if the configuration becomes unstable. One has to take care about algorithm to avoid artificial numerical diffusion.

## Stationary incompressible MHD

$$\nabla \cdot (\rho \vec{v}) = 0$$
$$\rho (\vec{v} \cdot \nabla) \vec{v} = \vec{j} \times \vec{B} - \nabla P$$
$$\nabla \times \left( \vec{v} \times \vec{B} \right) = \vec{0}$$
$$\nabla \times \vec{B} = \mu_0 \vec{j}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \cdot \vec{v} = 0$$

Subsets are
1.) No plasma flow: Magneto-Hydro-Statics (MHS)
2.) No magnetic field: Stationary Incom-pressibel Hydro-Dynamics (SIHD)

## SIHD

$$\rho (\vec{v} \cdot \nabla) \vec{v} = -\nabla P$$
$$\nabla \cdot \vec{v} = 0$$

vector identity:

$$(\vec{v} \cdot \nabla) \vec{v} = \nabla \left( \frac{1}{2} v^2 \right) - \vec{v} \times (\nabla \times \vec{v})$$

$$-\rho \vec{v} \times (\nabla \times \vec{v}) + \nabla \left( \frac{\rho}{2} v^2 \right) = -\nabla P$$
$$\nabla \cdot \vec{v} = 0$$

## SIHD

$$-\rho \underbrace{(\nabla \times \vec{v})}_{\text{vorticity}} \times \vec{v} - \nabla \left( \frac{\rho}{2} v^2 + P \right) = 0$$
$$\nabla \cdot \vec{v} = 0$$

## Magneto-hydro-statics

$$\frac{1}{\mu_0} \underbrace{(\nabla \times \vec{B})}_{\text{current density}} \times \vec{B} - \nabla P = 0$$
$$\nabla \cdot \vec{B} = 0$$

## SIHD and MHS have same mathematical structure (Gebhardt&Kiessling 92):

$$(\nabla \times \vec{c}) \times \vec{c} = \nabla \Pi$$
$$\nabla \cdot \vec{c} = 0$$

| MHS | SIHD |
|---|---|
| $\vec{c} = \vec{B}$ | $\vec{c} = \sqrt{\rho}\, \vec{v}$ |
| $\Pi = \mu_0 \, p$ | $\Pi = -\frac{\rho}{2} v^2 - p$ |

## Stationary incompressible MHD

$$\nabla \cdot (\rho \vec{v}) = 0$$
$$\rho (\vec{v} \cdot \nabla) \vec{v} = \vec{j} \times \vec{B} - \nabla P$$
$$\nabla \times \left( \vec{v} \times \vec{B} \right) = \vec{0}$$

If plasma flow parallel to magnetic field

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \cdot \vec{v} = 0$$

## Stationary incompressible MHD

- Mathematical structure of hydro-dynamics and magneto-static is similar.
- We assume, that we found solution of a magneto-static equilibrium (Grad-Shafranov Eq. in 2D = > Flux-function or Euler-potentials in 3D).
- We use the similar mathematical structure to find transformation equations (different flux-function) to solve stationary MHD.
- We introduce the Alfven velocity $v_A = B/\sqrt{\mu_0 \rho}$ and the Alfven Machnumber $M_A = v/v_A$
- We limit to sub-Alfvenic flows here. Pure super-Alfvenic flows can be studied similar. Somewhat tricky are trans-Alfvenic flows.

## Stationary incompressible MHD

- With this approach we can eliminate the plasma velocity in the SMHD equations and get:

$\vec{B}\cdot\vec{\nabla}M_A = 0$ => Alfven Machnumber is constant on field lines

$$\vec{\nabla}\Pi = \frac{(1-M_A^2)\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}}{\mu_0} - \frac{|\vec{B}|^2}{2\mu_0}\vec{\nabla}\left(1-M_A^2\right)$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

- Obviously the equations reduce to magnetostatics for M_A=0. The second equation changes it's sign from sub- to super-Alfvenic flows.

## Stationary incompressible MHD in 2D

- Similar as in the magneto-static case we represent the magnetic field with a flux-function: $\alpha(x,z)$

$$\vec{B} = \vec{\nabla}\alpha\times\vec{e}_y$$

- Flux functions are not unique and we can transform to another flux function $A(x,z)$
- The Alfven mach number is constant on field lines:

$$M_A = M_A(\alpha) = M_A(A)$$

- We can now eliminate terms containing the Alfven Mach number by choosing:

$$(1-M_A^2)\left(\frac{\partial\alpha}{\partial A}\right)^2 \equiv 1$$

## Grad-Shafranov-Equation with flow

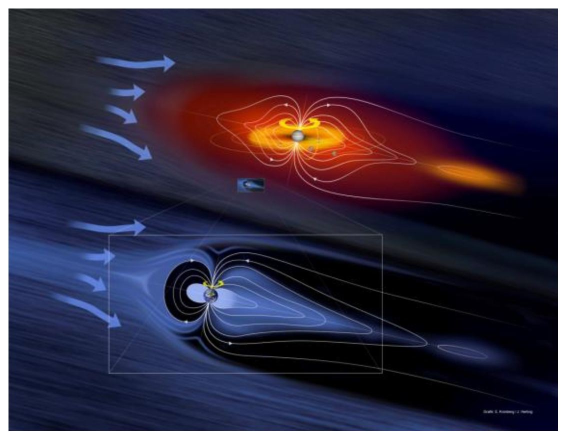
- The stationary incompressible MHD-equations reduce to a Grad-Shafranov equation

$$-\frac{1}{\mu_0}\Delta A = \frac{\partial\Pi}{\partial A}$$

- Any solution we found for the static case can be used to find a solution for equilibria with flow by:

$$\alpha = \pm\int\frac{dA}{\sqrt{1-M_A(A)^2}}$$

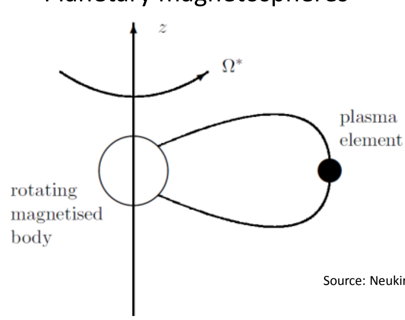
- Transformation can become complicated, however.



## Jovian Magnetosphere

- Jupiter: fast rotation 10 h, mass-loading 1000 kg/s
- Dynamics driven largely by internal sources.
- Planetary rotation coupled with internal plasma loading from the moon Io may lead to additional currents, departure from equilibrium, magnetospheric instabilities and substorm-like processes.
- Stationary states of a fast rotating magnetosphere cannot be modeled with a magneto-static model.
- We have to incorporate the rotation => Equilibria with centrifugal force.

## Planetary magnetospheres




Source: Neukirch 1998

We use a cylinder geometry and derive the corresponding Grad-Shafranov Equation.

## Grad-Shafranov Eq. in cylinder-geometry

This section is based on a lecture by Neukirch 1998

- Rotational invariance without additional forces

$$\begin{aligned}\mathbf{B} &= \nabla A \times \nabla \phi + B_\phi \mathbf{e}_\phi \\ &= \frac{1}{\varpi} \nabla A \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi \\ &= -\frac{1}{\varpi}\frac{\partial A}{\partial z}\mathbf{e}_\varpi + \frac{1}{\varpi}\frac{\partial A}{\partial \varpi}\mathbf{e}_z + B_\phi \mathbf{e}_\phi\end{aligned}$$

Here $A$ is *not* the $\phi$-component of the vector potential, but $A/\varpi$ is !

$$\mathbf{B} \cdot \nabla p = 0$$

$$p(\varpi, z) = F(A(\varpi, z)).$$

- The current density becomes:

$$\nabla \times \mathbf{B} = -\frac{\partial B_\phi}{\partial z}\mathbf{e}_\varpi - \left[\frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2} + \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial A}{\partial \varpi}\right)\right]\mathbf{e}_\phi + \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi B_\phi)\mathbf{e}_z$$

- An for the Lorentz-force we get:

$$\begin{aligned}\mathbf{j}\times\mathbf{B} &= \mathbf{j}\times\left(\frac{1}{\varpi}\nabla A \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi\right) \\ &= \frac{1}{\varpi} j_\phi \nabla A - \frac{1}{\varpi}(\mathbf{j}\cdot\nabla A)\mathbf{e}_\phi + B_\phi \mathbf{j}\times\mathbf{e}_\phi \\ &= -\frac{1}{\mu_0 \varpi}\left\{\left[\frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2} + \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial A}{\partial \varpi}\right)\right]\nabla A - \right. \\ &\qquad \left[\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi B_\phi)\frac{\partial A}{\partial z} - \frac{\partial B_\phi}{\partial z}\frac{\partial A}{\partial \varpi}\right]\mathbf{e}_\phi + \\ &\qquad \left. B_\phi \nabla(\varpi B_\phi)\right\} \\ &= \frac{dp}{dA}\nabla A.\end{aligned}$$

Looking at the $\phi$-component of this equation we see that

$$\begin{aligned}\frac{1}{\varpi}\left[\frac{\partial}{\partial \varpi}(\varpi B_\phi)\frac{\partial A}{\partial z} - \frac{\partial}{\partial z}(\varpi B_\phi)\frac{\partial A}{\partial \varpi}\right] &= \frac{1}{\varpi}\mathbf{B}\cdot\nabla(\varpi B_\phi) \\ &= 0.\end{aligned}$$

As the rotational gradient vanishes

It follows that

$$b_\phi(\varpi, z) = \varpi B_\phi(\varpi, z) = G(A(\varpi, z)).$$

This allows us to write the rest of the equations as

$$-\frac{1}{\mu_0 \varpi}\left[\frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial A}{\partial \varpi}\right) + \frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2}\right]\nabla A - \frac{1}{\mu_0 \varpi^2} b_\phi \frac{db_\phi}{dA}\nabla A = \frac{dp}{dA}\nabla A$$

- For non-vanishing gradient of A we get the Grad-Shafranov-Eq. in cylinder geometry:

$$-\nabla\cdot\left(\frac{1}{\varpi^2}\nabla A\right) = \mu_0 \frac{dp}{dA} + \frac{1}{\varpi^2} b_\phi \frac{db_\phi}{dA}$$

## Including the centrifugal force

- We now include a strictly rotational plasma flow

$$\mathbf{v} = \varpi \Omega \mathbf{e}_\phi$$

a) Continuity equation

$$\nabla\cdot(\rho\mathbf{v}) = \frac{1}{\varpi}\frac{\partial}{\partial \phi}(\varpi \rho \Omega) = 0$$

because of axisymmetry.

b) Ohm's law and Faraday's law

From Faraday's law

$$\nabla \times \mathbf{E} = \mathbf{0}$$

we conclude that

$$\mathbf{E} = -\nabla \Phi.$$

With

$$\mathbf{B} = \frac{1}{\varpi}\nabla A \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi$$

(from $\nabla\cdot\mathbf{B} = 0$) we find that

$$\begin{aligned}\mathbf{v}\times\mathbf{B} &= \varpi\Omega\mathbf{e}_\phi \times \left(\frac{1}{\varpi}\nabla A \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi\right) \\ &= \Omega \nabla A\end{aligned}$$

so that the ideal Ohm's law

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \mathbf{0}$$

acquires the form

$$-\nabla\Phi + \Omega\nabla A = \mathbf{0}.$$

Taking the curl of this equation results in

$$\nabla\Omega \times \nabla A = \mathbf{0}$$

leading to the conclusion that

$$\Omega = H(A)$$

Ferraro's law of isorotation: The angular velocity is constant on magnetic field lines

Since $\Omega$ is a function of $A$, we also get

$$-\nabla\Phi + \Omega(A)\nabla A = \mathbf{0}$$

and find that the electric potential is a function of A as well.

## We write the velocity more convenient

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\left(\frac{1}{2}|\mathbf{v}|^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v})$$

$$\begin{aligned}\nabla\times\mathbf{v} &= \nabla\times(\varpi\Omega\mathbf{e}_\phi)\\ &= -\varpi\frac{\partial\Omega}{\partial z}\mathbf{e}_\varpi + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi^2\Omega)\mathbf{e}_z\end{aligned}$$

$$\begin{aligned}\mathbf{v}\times(\nabla\times\mathbf{v}) &= \varpi\Omega\mathbf{e}_\phi\times\left(-\varpi\frac{\partial\Omega}{\partial z}\mathbf{e}_\varpi + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi^2\Omega)\mathbf{e}_z\right)\\ &= \varpi^2\Omega\frac{\partial\Omega}{\partial z}\mathbf{e}_z + \Omega\frac{\partial}{\partial\varpi}(\varpi^2\Omega)\mathbf{e}_\varpi\\ &= \nabla(\varpi^2\Omega^2) - \varpi^2\Omega\frac{d\Omega}{dA}\nabla A.\end{aligned}$$

## Momentum Balance Equation

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{j}\times\mathbf{B} - \nabla p$$

$$\frac{1}{2}\rho\nabla(\varpi^2\Omega^2) - \rho\nabla(\varpi^2\Omega^2) + \rho\varpi^2\Omega\frac{d\Omega}{dA}\nabla A = \left[-\frac{1}{\mu_0}\nabla\cdot\left(\frac{1}{\varpi^2}\nabla A\right) - \frac{1}{\mu_0\varpi^2}b_\phi\frac{db_\phi}{dA}\right]\nabla A - \nabla p$$

$$\left[-\frac{1}{\mu_0}\nabla\cdot\left(\frac{1}{\varpi^2}\nabla A\right) - \frac{1}{\mu_0\varpi^2}b_\phi\frac{db_\phi}{dA} - \rho\varpi^2\Omega\frac{d\Omega}{dA}\right]\nabla A - \nabla p + \rho\nabla\eta = \mathbf{0}$$

with $\eta = \varpi^2\Omega^2/2$ the centrifugal potential

## Grad-Shafranov-eq. for rotating systems

With the same arguments as before we conclude that

$$p = F(A,\eta)$$

and the partial differential equations to solve are

$$\begin{aligned}-\nabla\cdot\left(\frac{1}{\varpi^2}\nabla A\right) &= \mu_0\left(\frac{\partial p}{\partial A}\right)_\eta + \frac{1}{\varpi^2}b_\phi\frac{db_\phi}{dA} + \mu_0\varpi^2\rho\Omega\frac{d\Omega}{dA}\\ \left(\frac{\partial p}{\partial\eta}\right)_A &= \rho.\end{aligned}$$

Again we have to provide information on $\rho$ in the same way as before and to integrate the second equation first. When substituting $p$ into the first equation one has to keep $\eta$ constant although $\Omega$ can depend on $A$ !

## Grad-Shafranov-eq. for rotating systems

- Ferraro's law of isorotation restricts the angular velocity.

Imagine a rigidly rotating magnetised body, e.g. a star or a planet, which has a surface into which the field lines are frozen, i.e. $\Omega$ is fixed at the surface of the star. Then by Ferraro's law we have $\Omega(A) = \Omega^*$ for every field line touching the surface in at least one point. This can cause problems for field lines extending very far out because $v_\phi = \varpi\Omega^*$ will become very large. Of course this means that the centrifugal force becomes large and the plasma will be accelerated outwards: a plasma flow along the the field lines starts leading e.g. to a stellar wind and the field lines become open.

## Sequences of equilibria

- One should not naively consider every sequences of static equilibria as a physical reasonable temporal evolution.
- Magnetostatic means, that velocity and time-dependence are small (and can be neglected) in the momentum transport.
- We still have to solve continuity equation, ideal Ohm's law and an equation of state to obtain a physical meaningful time-sequence of equilibria.
- This can become involved.

## Stationary MHD

- Magnetostatics and stationary Hydrodynamics are mathematical similar, also the terms have different physical meaning.
- We can use this property to transform solutions of MHS to stationary MHD for incompressible field line parallel plasma flows.
- Rotating systems are restricted by Ferraros law of isoration and we have to solve two coupled differential equations.

## How to proceed?

- Study time dependent system:
- Plasma Waves
- Instabilities
- Discontinuities
- Waves and instabilities occur in MHD as well as in a kinetic model.
- In discontinuities the Fluid approach often breaks down and one has to apply kinetic models.