Space Plasma Physics

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1. Basic Plasma Physics concepts
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Plasma Models

3. Single particle motion, Test particle model
4. Statistic description of plasma, BBGKY-Hierarchy and kinetic equations
5. Fluid models, Magneto-Hydro-Dynamics
6. Magneto-Hydro-Statics
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Multi-fluid theory

- Full plasma description in terms of particle distribution functions (VDFs), $f_s(\mathbf{v},\mathbf{x},t)$, for species, s.
- For slow large-scale variations, a description in terms of moments is usually sufficient -> **multi-fluid** (density, velocity and temperature) description.
- Fluid theory is looking for evolution equations for the basic macroscopic moments, i.e. number density, $n_s(\mathbf{x},t)$, velocity, $\mathbf{v}_s(\mathbf{x},t)$, pressure tensor, $\mathbf{P}_s(\mathbf{x},t)$, and kinetic temperature, $T_s(\mathbf{x},t)$. For a two fluid plasma consisting of electrons and ions, we have s=e,i.

Magneto-hydro-dynamics (MHD)

- Fluid equations reduce from the 6D space from kinetic theory to our usual 3D configuration space (and the time).
- In Fluid theory, we miss some important physics, however. With single fluid MHD even more than multi fluid (2 Fluid: electrons and ions).
- 3 fluid (+neutral particles) and 4 fluid models are popular in space plasma physics, too, also one can use different fluids for different Ion-species.
- Hybrid-model: Ions treated as particles and electrons as fluid.

Fluid models, Magneto-Hydro-Dynamics

- If we solved the kinetic equations (Vlasov or Fokker-Planck-equation) we derive macroscopic variables by taking velocity-moments of the distribution functions.

$$n_\alpha(\mathbf{x},t) = \overline{n}_\alpha \int f_\alpha(\mathbf{x},\mathbf{v},t)\, d\mathbf{v}$$

$$\mathbf{V}_\alpha(\mathbf{x},t) = \frac{\int \mathbf{v}\, f_\alpha(\mathbf{x},\mathbf{v},t)\, d\mathbf{v}}{\int f_\alpha(\mathbf{x},\mathbf{v},t)\, d\mathbf{v}}$$

$$P_\alpha(\mathbf{x},t) = \overline{n}_\alpha m_\alpha \int (\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)\; f_\alpha(\mathbf{x},\mathbf{v},t)\, d\mathbf{v}$$

Fluid models, Magneto-Hydro-Dynamics

- Solving the kinetic equations is, however, very difficult, even numerically. One has to resolve all spatial and temporal plasma scales like Debey-length/frequency, Gyro-radius/frequency etc.
- In space plasmas these scales are often orders of magnitudes smaller as the macroscopic scales we are interested in (like size of solar active regions or planetary magnetospheres).
- Solution: We take velocity-moments over the kinetic equations (Vlasov-equation), instead of (generally unknown) solution of the kinetic equation.

Continuity equation

Evolution equations of moments are obtained by taking the corresponding moments of the Vlasov equation:

$$\int \left[\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\;\cdot\nabla_{\mathbf{v}} f_s\right] d^3v = 0$$

Taking the zeroth moment yields for the first term:

$$\frac{\partial}{\partial t}\int f_s d^3v = \frac{\partial n_s}{\partial t}$$

In the second term, the velocity integration and spatial differentiation can be interchanged which yields a divergence:

$$\nabla_{\mathbf{x}}\cdot\int \mathbf{v} f_s d^3v = \nabla\cdot(n_s\mathbf{v}_s)$$

In the force term, a partial integration leads to a term, which does not contribute.

$$\int f_s \nabla_{\mathbf{v}}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B}) d^3v$$

Continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) = 0$$

We have again (like in BBGKY-hierarchy) the problem, that the equation for the 0-moment (density) contains the first moment (velocity **v** or momentum n**v**).

Momentum equation

The evolution equation for the flow velocity/momentum is obtained by taking the first moment of the Vlasov equation:

$$\int \mathbf{v}\Big[\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_s\Big] d^3v = 0$$

Since the phase space coordinate **v** does not depend on time, the first term yields the time derivative of the flux density:

$$\frac{\partial}{\partial t}\int \mathbf{v} f_s d^3v = \frac{\partial}{\partial t}(n_s \mathbf{v}_s)$$

In the second term, velocity integration and spatial differentiation can be exchanged, and $\mathbf{v}(\mathbf{v}\cdot\nabla_x) = \nabla_x \cdot(\mathbf{vv})$ be used. With the definition of the pressure tensor we get:

$$\nabla_{\mathbf{x}} \cdot \int \mathbf{vv} f_s d^3v = \nabla\cdot(n_s \mathbf{v}_s \mathbf{v}_s) + \frac{1}{m_s}\nabla\cdot\mathbf{P}_s$$

In the third term, a partial integration with respect to the velocity gradient operator ∇_v gives the remaining integral:

$$\int f_s(\nabla_{\mathbf{v}}\mathbf{v})\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B})d^3v = n_s(\mathbf{E}+\mathbf{v}_s\times\mathbf{B})$$

We can now add up all terms and obtain the final result:

$$\frac{\partial(n_s\mathbf{v}_s)}{\partial t} + \nabla\cdot(n_s\mathbf{v}_s\mathbf{v}_s) + \frac{1}{m_s}\nabla\cdot\mathbf{P}_s - \frac{q_s}{m_s}n_s(\mathbf{E}+\mathbf{v}_s\times\mathbf{B}) = 0$$

This momentum density conservation equation for species s corresponds to the Navier-Stokes equation for neutral fluids. In a plasma the equation becomes more complicated due to coupling with Maxwell-equations via the Lorentz-force.

Momentum equation or equation of motion

$$\frac{\partial(n_s\mathbf{v}_s)}{\partial t} + \nabla\cdot(n_s\mathbf{v}_s\mathbf{v}_s) + \frac{1}{m_s}\nabla\cdot\mathbf{P}_s - \frac{q_s}{m_s}n_s(\mathbf{E}+\mathbf{v}_s\times\mathbf{B}) = 0$$

This equation for the first-moment (velocity, momentum) contains the second moment (pressure tensor).

And so on .....

Equation for moment k, contains always the moment k+1 => **Hierarchy of moments**

Hierarchy of moments

- As in kinetics (BBGKY) we have to cut the hierarchy somewhere by making suitable approximations.
- Number of (scalar) variables increase by taking higher orders of moments v^n.
- MHD, 5 moments: density, 3 components of velocity and a scalar (isotropic) pressure (or temperature). These 5 moments are called plasmadynamic variables.
- Fluid equations with a higher number of moments (13, 21, 29, 37, ...) are possible and take anisotropy and approximated correction terms (for cutting the infinite chain of hierarchy-equations) into account.

Energy equation

The equations of motion do not close, because at any order a new moment of the next higher order appears (closure problem), leading to a chain of equations. In the momentum equation the pressure tensor, $\mathbf{P}_s$, is required, which can be obtained from taking the seond-order moment of Vlasov's equation. The results become complicated. Often only the trace of $\mathbf{P}_s$, the isotropic pressure, p_s, is considered, and the traceless part, $\mathbf{P}'_s$, the stress tensor is separated, which describes for example the shear stresses.

Full energy (temperature, heat transfer) equation:

$$\frac{3}{2}n_s k_B\left(\frac{\partial T_s}{\partial t} + \mathbf{v}_s\cdot\nabla T_s\right) + p_s\nabla\cdot\mathbf{v}_s = -\nabla\cdot\mathbf{q}_s - \left(\mathbf{P}'_s\cdot\nabla\right)\cdot\mathbf{v}_s$$

The sources or sinks on the right hand side are related to heat conduction, $\mathbf{q}_s$, or mechanical stress, $\mathbf{P}'_s$.

Equation of state

A truncation of the equation hierarchy can be achieved by assuming an equation of state, depending on the form of the pressure tensor.

If it is isotropic, $\mathbf{P}_s = p_s\mathbf{1}$, with the unit dyade, **1**, and ideal gas equation, $p_s = n_s k_B T_s$, then we have a diagonal matrix:

$$\mathbf{P}_s = \begin{pmatrix} p_s & 0 & 0 \\ 0 & p_s & 0 \\ 0 & 0 & p_s \end{pmatrix}$$

- Isothermal plasma: T_s = const
- Adiabatic plasma: $T_s = T_{s0}\,(n_s/n_{s0})^{\gamma-1}$, with the adiabatic index $\gamma = c_P/c_V = 5/3$ for a mono-atomic gas.
- Incompressible plasma Div **v** =0

Equation of state

Due to strong magnetization, the plasma pressure is often anisotropic, yet still gyrotropic, which implies the form:

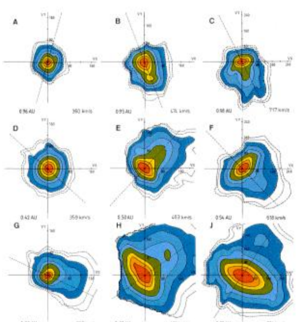
$$\mathbf{P}_s = p_{s\perp}\mathbf{I} + \left(p_{s\parallel} - p_{s\perp}\right)\frac{\mathbf{BB}}{B^2}$$

$$\mathbf{P}_s = \begin{pmatrix} p_{s\perp} & 0 & 0 \\ 0 & p_{s\perp} & 0 \\ 0 & 0 & p_{s\parallel} \end{pmatrix}$$

with a different pressure (temperature) parallel and perpendicular to the magnetic field. Then one has two energy equations, which yield (without sinks and sources) the double-adiabatic equations of state:

- $T_\perp \propto B$ -> perpendicular heating in increasing field
- $T_{//} \propto (n/B)^2$ -> parallel cooling in declining density

Measured solar wind proton velocity distributions



Helios

$$\mathbf{P}_s = \begin{pmatrix} p_{s\perp} & 0 & 0 \\ 0 & p_{s\perp} & 0 \\ 0 & 0 & p_{s\parallel} \end{pmatrix}$$

Non-isotropic particle distribution functions lead to pressure tensor (instead of scalar pressure) in fluid approach.

Marsch et al., JGR, **87**, 52, 1982

One-fluid theory

Consider simplest possible plasma of fully ionized hydrogen with electrons with mass m_e and charge $q_e = -e$, and ions with mass m_i and charge $q_i = e$. We define charge and current density by:

$$\rho = e(n_i - n_e) \qquad \mathbf{j} = e(n_i\mathbf{v}_i - n_e\mathbf{v}_e)$$

Usually ***quasineutrality*** applies, $n_e = n_i$, and space charges vanish, $\rho = 0$, but the plasma carries a finite current, i.e. we still need an equation for **j**. We introduce the *mean mass*, *density* and *velocity* in the single-fluid description as

$$m = m_e + m_i = m_i\left(1 + \frac{m_e}{m_i}\right)$$
$$n = \frac{m_e n_e + m_i n_i}{m_e + m_i}$$

$$\mathbf{v} = \frac{m_i n_i \mathbf{v}_i + m_e n_e \mathbf{v}_e}{m_e n_e + m_i n_i}$$

One-fluid momentum equation

Constructing the equation of motion is more difficult because of the nonlinear advection terms, $n_s\mathbf{v}_s\mathbf{v}_s$.

$$\frac{\partial(n_e\mathbf{v}_e)}{\partial t} + \nabla\cdot(n_e\mathbf{v}_e\mathbf{v}_e) = -\frac{1}{m_e}\nabla\cdot\mathbf{P}_e - \frac{n_e e}{m_e}(\mathbf{E} + \mathbf{v}_e\times\mathbf{B}) + \frac{\mathbf{R}}{m_e}$$
$$\frac{\partial(n_i\mathbf{v}_i)}{\partial t} + \nabla\cdot(n_i\mathbf{v}_i\mathbf{v}_i) = -\frac{1}{m_i}\nabla\cdot\mathbf{P}_i + \frac{n_i e}{m_i}(\mathbf{E} + \mathbf{v}_i\times\mathbf{B}) - \frac{\mathbf{R}}{m_i}$$

The equation of motion is obtained by adding these two equations and exploiting the definitions of ρ, m, n, **v** and **j**. When multiplying the first by m_e and the second by m_i and adding up we obtain:

$$\begin{aligned} & -\nabla\cdot(\mathbf{P}_e + \mathbf{P}_i) + e(n_i - n_e)\mathbf{E} + e(n_i\mathbf{v}_i - n_e\mathbf{v}_e)\times\mathbf{B} \\ & = -\nabla\cdot\mathbf{P} + \rho\mathbf{E} + \mathbf{j}\times\mathbf{B} \end{aligned}$$

Here we introduced the total pressure tensor, $\mathbf{P} = \mathbf{P}_e + \mathbf{P}_i$. In the nonlinear parts of the advection term we can neglect the light electrons entirely.

Magnetohydrodynamics (MHD)

With these approximations, which are good for many quasineutral space plasmas, we have the MHD momentum equation, in which the space charge (electric field) term is also mostly disregarded.

$$\frac{\partial(nm\mathbf{v})}{\partial t} + \nabla\cdot(nm\mathbf{v}\mathbf{v}) = -\nabla\cdot\mathbf{P} + \rho\mathbf{E} + \mathbf{j}\times\mathbf{B}$$

Note that to close the full set, an equation for the current density is needed. For negligable displacement currents, we simply use Ampere's law in magnetohydrodynamics and **B** as a dynamic variable, and replace then the Lorentz force density by:

$$\mathbf{j}\times\mathbf{B} = -\frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{B})$$

Generalized Ohm‘s law

The evolution equation for the current density, **j**, is derived by use of the electron equation of motion and called generalized Ohm‘s law. It results from a subtraction of the ion and electron equation of motion. The nonlinear advection terms cancel in lowest order. The result is:

$$\frac{m_e}{e}\frac{\partial \mathbf{j}}{\partial t} = \nabla\cdot\left(\mathbf{P}_e - \frac{m_e}{m_i}\mathbf{P}_i\right) - \left(1+\frac{m_e}{m_i}\right)\mathbf{R} + n_e e\left(1+\frac{m_e n_i}{m_i n_e}\right)\left[\mathbf{E} + \left(\mathbf{v}_e + \frac{m_e n_i}{m_i n_e}\mathbf{v}_i\right)\times\mathbf{B}\right]$$

The right hand sides still contain the individual densities, masses and speeds, which can be eliminated by using that $m_e/m_i \ll 1$, $n_e \approx n_i$

Generalized Ohm‘s law

We obtain a simplified equation:

$$\frac{m_e}{e}\frac{\partial \mathbf{j}}{\partial t} = \nabla\cdot\mathbf{P}_e + ne\left(\mathbf{E} + \mathbf{v}_e\times\mathbf{B}\right) - \mathbf{R}$$

Key features in single-fluid theory: Thermal effects on **j** enter only via, $\mathbf{P}_e$, i.e. the electron pressure gradient modulates the current. The Lorentz force term contains the electric field as seen in the electron frame of reference.

Generalized Ohm‘s law

Omitting terms of the order of the small mass ratio, the fluid bulk velocity is, $\mathbf{v}_i = \mathbf{v}$. Using this and the quasineutrality condition yields the electron velocity as: $\mathbf{v}_e = \mathbf{v} - \mathbf{j}/ne$. Finally, the collision term with frequency ν_c can be assumed to be proportional to the velocity difference, and use of the resistivity, $\eta = m_e\nu_c/ne^2$, permits us to write:

$$\mathbf{R} = m_e n \nu_c(\mathbf{v}_i - \mathbf{v}_e)$$

$$\mathbf{R} = \eta n e \mathbf{j}$$

Generalized Ohm‘s law

The resulting Ohm‘s law can then be written as:

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{j} + \frac{1}{ne}\mathbf{j}\times\mathbf{B} - \frac{1}{ne}\nabla\cdot\mathbf{P}_e + \frac{m_e}{ne^2}\frac{\partial \mathbf{j}}{\partial t}$$

The right-hand side contains, in a plasma in addition to the resistive term, three new terms: **Hall term,** ***electron pressure, contribution of electron inertia to current flow.***

Popular further simplifications are:

- Resistive MHD: $\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{J}$
- Ideal MHD: $\mathbf{E} + \mathbf{v}\times\mathbf{B} = 0$

Magnetic tension

The Lorentz force or Hall term introduces a new effect in a plasma which is specific for magnetohydrodynamics: ***magnetic tension***, giving the conducting fluid stiffness. For slow variations Ampere‘s law can be used to derive:

$$\mathbf{j}\times\mathbf{B} = -\frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{B})$$

Applying some vector algebra to the right hand side gives:

$$\mathbf{j}\times\mathbf{B} = -\nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\nabla\cdot(\mathbf{B}\mathbf{B})$$

magnetic pressure $p_B = \frac{B^2}{2\mu_0}$

Plasma beta

Starting from the MHD equation of motion for a plasma at rest in a steady quasineutral state, we obtain the simple force balance:

$$\nabla\cdot\mathbf{P} = -\frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{B})$$

which expresses ***magnetohydrostatic equilibrium***, in which thermal pressure balances magnetic tension. If the particle pressure is nearly isotropic and the field uniform, this leads to the total pressure being constant:

$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) = 0$$

The ratio of these two terms is called the ***plasma beta***: $\beta = \frac{2\mu_0 p}{B^2}$

Requirements for the validity of MHD

Variations must be large and slow, $\omega < \omega_{gi}$ and $k < 1/r_{gi}$, which means fluid scales must be much larger than gyro-kinetic scales. Consider Ohm's law:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v} \times \mathbf{B} - \frac{1}{ne} \mathbf{j} \times \mathbf{B} + \frac{1}{ne} \nabla \cdot \mathbf{P}_e - \frac{m_e}{ne^2} \frac{\partial \mathbf{j}}{\partial t} - \eta \mathbf{j} \right)$$

Convection, Hall effect, thermoelectricity, polarization, resistivity

ω_{gi}/ν_c ω/ν_c

Only in a strongly collisional plasma can the Hall term be dropped.

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}_e \times \mathbf{B})$$

In collisionless MHD the electrons are frozen to the field.

MHD-equations (fluid part, including gravity)

a) Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \text{(mass conservation)}$$

b) Momentum conservation equation (equation of motion)

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{j} \times \mathbf{B} - \nabla p - \rho \nabla \psi$$

c) Energy equation (various different forms possible)

$$\rho^\gamma \frac{\partial}{\partial t} \left(\frac{p}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left(\frac{p}{\rho^\gamma} \right) = -(\gamma - 1) \mathcal{L}$$

where

$$\mathcal{L} = \underbrace{\nabla \cdot \mathbf{q}}_{\text{heat flux}} + \overbrace{L_r}^{\text{radiative losses}} - \underbrace{\frac{\mathbf{j}^2}{\sigma}}_{\text{Ohmic heating}} - \overbrace{H}^{\text{everything else}}.$$

source: Neukirch 1998

MHD-equations (Maxwell-part + Ohm's law)

d) Ampère's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

(displacement current neglected).

e) Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

f) no magnetic monopoles

$$\nabla \cdot \mathbf{B} = 0$$

g) Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{R}$$

Summary: Fluid equations, MHD

- Multi-fluid theory
- Equation of state
- Single-fluid theory
- Generalised Ohm's law
- Magnetic tension and plasma beta
- Validity of magnetohydrodynamics

How to proceed?

- We will look at example solution of the MHD, which are relevant for space plasmas.
- First we compute static equilibria (no plasma-flow, no time dependence) => Magneto-statics
- Stationary solutions (no time dependence, but including stationary plasma flows)
- Sequences of equilibria (slow time dependence)
- Time dependent solutions, waves, instabilities and magnetic reconnection.