Space Plasma Physics

Thomas Wiegelmann, 2012

1. Basic Plasma Physics concepts
2. Overview about solar system plasmas

Plasma Models

3. Single particle motion, Test particle model
4. Statistic description of plasma, BBGKY-Hierarchy and kinetic equations
5. Fluid models, Magneto-Hydro-Dynamics
6. Magneto-Hydro-Statics
7. Stationary MHD and Sequences of Equilibria

Magneto-Hydro-Statics (MHS)

- No time dependence and no plasma flows
- More precise: The dynamic terms in MHD are small compared with static forces (Lorentz-force, plasma pressure gradient, gravity)
- Sequences of equilibria (slow temporal changes) and equilibria with stationary plasma flow are studied in the next lecture.

Parts of this lecture are based on the course INTRODUCTION TO THE THEORY OF MHD EQUILIBRIA given by Thomas Neukirch in St. Andrews 1998.

Magneto-hydro-statics (MHS)

source: Neukirch 1998

a) Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \text{(mass conservation)}$$

b) Momentum conservation equation (equation of motion)

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{j}\times\mathbf{B} - \nabla p - \rho\nabla\psi$$

c) Energy equation (various different forms possible)

$$\rho^\gamma \frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + \mathbf{v}\cdot\nabla\left(\frac{p}{\rho^\gamma}\right) = -(\gamma-1)\mathcal{L}$$

where

$$\mathcal{L} = \underbrace{\nabla\cdot\mathbf{q}}_{\text{heat flux}} + \overbrace{L_r}^{\text{radiative losses}} - \underbrace{\frac{\mathbf{j}^2}{\sigma}}_{\text{Ohmic heating}} - \overbrace{H.}^{\text{everything else}}$$

MHS-equations (Maxwell-part + Ohm's law)

d) Ampère's law

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j}$$

(displacement current neglected).

e) Faraday's law

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

f) no magnetic monopoles

$$\nabla\cdot\mathbf{B} = 0$$

g) Ohm's law

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \mathbf{R}$$

MHS

We finally end up with a set of three equations:

$$\begin{aligned}\mathbf{j}\times\mathbf{B} - \nabla p - \rho\nabla\psi &= \mathbf{0}\\ \nabla\times\mathbf{B} &= \mu_0\mathbf{j}\\ \nabla\cdot\mathbf{B} &= 0\end{aligned}$$

With the vector potential $\mathbf{B} = \nabla\times\mathbf{A}$ one equation (solenoidal condition) is already solved automatically.

Why is it useful to study MHS ?

- From a fundamental point of view we can regard the MHD equations as a set of equations describing extremely complicated dynamical systems. In the study of dynamical systems it is always useful to start with a study of the simplest solutions. These are usually the stationary states and their bifurcation properties, in the MHD case the static equilibria.
- From the point of view of modelling, many physical processes in plasma systems occur slowly, i.e. on time-scales which are much longer than the typical time-scale of the system.

Let L be the length scale of the system, T the slow time scale of evolution and $v_A = B_0/\sqrt{\mu_0\rho_0}$ a typical Alfvén speed. We then define the Alfvén time by $T_A = L/v_A$. The main assumption now is that

$$\frac{T_A}{T} = \frac{v}{v_A} = \varepsilon \ll 1.$$

We now normalize lengths by L, velocities by v, the magnetic field by B_0, the density by ρ_0, the pressure by p_0 and the gravitational potential by ψ_0. Normalised quantities will be denoted by a $\tilde{}$. We obtain

$$\varepsilon^2\tilde\rho\left(\frac{\partial\tilde{\mathbf{v}}}{\partial\tilde t} + \tilde{\mathbf{v}}\cdot\tilde\nabla\tilde{\mathbf{v}}\right) = \tilde{\mathbf{j}}\times\tilde{\mathbf{B}} - 2\beta_p\tilde\nabla\tilde p - 2\beta_g\tilde\rho\tilde\nabla\tilde\psi.$$

Here, β_p is the ratio between plasma pressure and magnetic pressure, the so-called plasma beta, whereas β_g is a similar ratio between the gravitational energy density and the magnetic pressure. Both numbers measure the relative importance of pressure gradient and gravitational force with respect to the $\mathbf{j}\times\mathbf{B}$-force.

Dimensionless Force-balance

Since ε is assume to be small, we obtain to lowest order

$$\mathbf{0} = \tilde{\mathbf{j}}\times\tilde{\mathbf{B}} - 2\beta_p\tilde\nabla\tilde p - 2\beta_g\tilde\rho\tilde\nabla\tilde\psi.$$

- The magnetostatic equations in dimensionless form can be used to evaluate the relative importance of the different terms.
- For a small β_g the gravity force can be neglected. This is usually fulfilled in magnetospheric plasmas.
- If β_p is small, too we can neglect also the plasma-pressure gradient in the force-balance => Force-free fields, a valid assumption in most parts of the solar corona.

- To lowest order we have the MHS force balance equation as fundamental equation and the time appears merely as a parameter.
- The fundamental importance of this quasi-static approximation lies in the fact that sequences of MHS equilibria can be used to model the slow evolution of plasma systems. (Growth phase of magnetospheric substorms, evolution of non-flaring solar active regions)
- Sequences have to satisfy the constraints imposed by the other equations, especially Ohm's law and the continuity equations.
- These constraints usually lead to very complicated integro-differential problems which are difficult to solve.

Systems with symmetries

- In the generic 3D case the MHS-equations are still difficult to solve due to their non-linearities.
- We consider now system with symmetries, e.g. in cartesian geometry (x,y,z) equilibria which are invariant in y. => 2D
- One can use also spherical, cylinder or helical symmetry, but the mathematics become a bit more complicated.
- We represent the magnetic field as:

$$\mathbf{B} = \nabla A \times \mathbf{e}_y + B_y\mathbf{e}_y$$
$$= \nabla \times (A\mathbf{e}_y) + B_y\mathbf{e}_y$$

Grad-Shafranov-Equation

Harold Grad
1923-1986
Source: AIP.org

In 2D the MHS-equations (without gravity)

$$\frac{1}{\mu_0}(\nabla\times\vec B)\times\vec B - \nabla P = 0$$
$$\nabla\cdot\vec B = 0$$

reduce to a single partial differential equation.

Vitalii Dmitrievich Shafranov, born 1929
Source: Physics Uspekhi 53 (1) 101 (2010)

$$-\Delta A = \mu_0\frac{dp}{dA} + B_y\frac{dB_y}{dA}$$

Grad-Shafranov-Equation

- We reduced the MHS to a single partial differential equations.
- The equation still contains the unknown pressure profile A -> p(A) and magnetic shear B_y
- Obviously p(A) has to be positive.
- For p(A)=A^2 the GSE becomes linear (popular choice)
- Within MHD p(A) is arbitrary, but it is possible to derive the pressure profile from kinetic theory.

Linear Grad-Shafranov-Equation

$\mu_0 j_y = k^2 A$

This form of j_y includes linear force-free fields. The Grad-Shafranov equation has the form

$$-\Delta A = k^2 A = \mu_0 \frac{d}{dA}\left(p(A) + \frac{B_y^2(A)}{2\mu_0}\right).$$

We get linear force-free fields if $p =$ constant, i.e.

$$-\Delta A = B_y \frac{dB_y}{dA} = k^2 A.$$

It follows that

$$B_y^2 = k^2 A^2 + B_{y0}^2.$$

Solutions can be obtained for example by separation of variables. In Cartesian coordinates we get

$$A = g(x)h(z)$$

leading to

$$-h\frac{d^2 g}{dx^2} - g\frac{d^2 h}{dz^2} = k^2\, g\, h$$

$$-\frac{1}{g}\frac{d^2 g}{dx^2} = k^2 + \frac{1}{h}\frac{d^2 h}{dz^2} = c^2$$

with c^2 constant. If we choose c^2 to be positive the solutions are

$$\begin{aligned} g &= g_1 \sin(cx) + g_2 \cos(cx) \\ h &= h_1 \exp(\sqrt{c^2 - k^2}\, z) + h_2 \exp(-\sqrt{c^2 - k^2}\, z) \end{aligned}$$

Here g_1, g_2, h_1 and h_2 are constants. In the case of linear force-free fields, we can replace k^2 by α^2. For all linear equations we can superpose different solutions to match boundary conditions for example. That is one of the reasons why the linear Grad-Shafranov equations are so popular.

Non-linear Grad-Shafranov-Equation

$\mu_0 j_y = \lambda \exp(2A)$

This is the first non-linear current density we investigate and is a very popular choice for two reasons:

(a) the complete set of solutions is known explicitly (Liouville, 1853) and the equation has particularly nice properties as conformal invariance;

(b) there is a physical justification for this current profile (!), because it results from a kinetic approach with Maxwellian distribution functions where the particles drift in the y-direction and the plasma is quasi-neutral. The plasma is then in local thermodynamic equilibrium. This argument does only apply, however, to a j_y caused by a pressure gradient and not to magnetic shear !

So the Grad-Shafranov equation is

$$-\Delta A = \lambda \exp(2A).$$

Kinetic theory for p(A)

- With invariance in y we have 2 constants of motion for the particles: Hamiltonian and momentum in y

$$\begin{aligned} H_s &= \frac{1}{2} m_s v^2 + q_s \phi \\ p_{ys} &= m_s v_y + q_s A \end{aligned}$$

- For a local thermic equilibrium the particle distribution function for each species is given by a drifting Maxwellian:

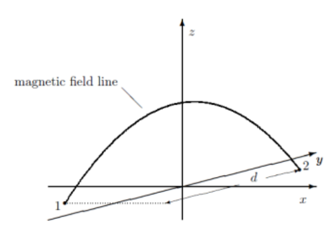
$$f_s(H_s, p_{ys}) \propto \exp\left(-\frac{H_s - u_s p_{ys}}{T_s}\right)$$

- Macroscopic (fluid) quantities like the particle density, current density and pressure for electrons and ions, we get by integration over the velocity space. The plasma pressure is isotropic per construction by a Maxwellian.
- We get the total pressure by adding the partial pressure of electron and ions.
- Assuming quasi-neutrality (n_i=n_e) allows to eliminate the electro static potential and we get:

$$p(A) = \lambda \exp\left(\frac{A}{\hat{A}}\right)$$

- We derived the GSE earlier from MHD, but it is valid also in kinetic theory and can be used to compute Vlasov-equilibria (e.g., intial states for kinetic simulations)

How to get B_y(A) ?



- The magnetic shear of a configuration like a coronal magnetic loop can in principle be computed from the displacement of footpoints.

Non-linear GSE, Liouville solution

$$-\Delta A = \lambda \exp(2A).$$

This equation is sometimes called Liouville's equation. The solutions to this equation are given by

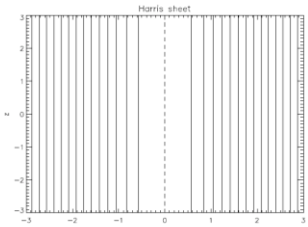
$$A = -\ln\left(\frac{1+\frac{1}{4}\lambda|\psi(u)|^2}{\left|\frac{d\psi}{du}\right|}\right)$$

where $u = x + iz$ and ψ is an analytic function, or written in a slightly different way

$$\lambda \exp(2A) = \frac{\lambda\left|\frac{d\psi}{du}\right|}{\left(1+\frac{1}{4}\lambda|\psi(u)|^2\right)^2}.$$

Non-linear GSE, 1D Harris (1962) Sheet

- We choose : $\psi = \frac{2}{\sqrt{\lambda}}\exp(\sqrt{\lambda}u)$
- And derive the solution: $A \;=\; -\ln\cosh\left(\sqrt{\lambda}x\right)$

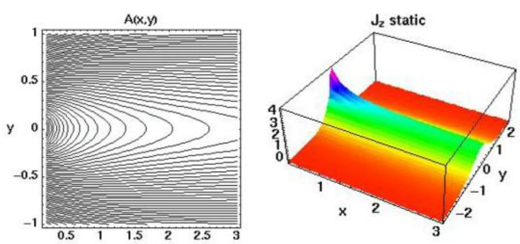


Used to describe current sheets in space plasmas. Valid also in kinetic approach.

Non-linear GSE, Magnetosphere solution
(Schindler & Birn 2004)

$$\Psi(u) = 2\hat{l}\exp\left(i\left(u/\hat{l}+\sqrt{\frac{u/\hat{l}}{\epsilon}}\right)\right) \quad A(x,y)/\hat{A} = \ln\left(\frac{\cosh\left(\frac{y}{\sqrt{2\epsilon}\sqrt{r+x}}+y\right)}{\sqrt{\frac{1}{r}\left(\frac{1}{4\epsilon}+\sqrt{\frac{r+x}{2\epsilon}}\right)+1}}\right)$$

$$r = \sqrt{x^2+y^2}.$$



Source of pictures: Nickeler et al. 2010

Including gravity

- External gravity force: $\mathbf{f} = -\rho\nabla\Psi$
- Force-balance: $\mathbf{j}\times\mathbf{B} - \nabla p - \rho\nabla\Psi = 0$
- Plasma pressure is not constant along field lines
  $$\mathbf{B}\cdot\nabla p = -\rho\mathbf{B}\cdot\nabla\Psi$$
- Pressure gradient needs to be compensated by gravity force, as the Lorentz-force vanishes parallel to the magnetic field

Including gravity

- We use again: $\mathbf{B} = \nabla A \times \mathbf{e}_y + B_y\mathbf{e}_y$
- Similar as for the Grad-Shafranov (GS) equation we get
  $$\mathbf{j}\times\mathbf{B} = \frac{1}{\mu_0}\left(-\Delta A - B_y\frac{dB_y}{dA}\right)\nabla A$$
- And the force balance:
  $$-\frac{1}{\mu_0}\left(\Delta A + B_y\frac{dB_y}{dA}\right)\nabla A - \nabla p - \rho\nabla\Psi$$

We now use the following argument. Each of the three vector fields ∇A, ∇p and $\nabla\Psi$ has only two components, namely in the x-z-plane. It follows that only two of these vector fields can be linearly independent.

Including gravity

We now assume that ∇A and $\nabla\Psi$ are linearly independent

$$\nabla p = p_A\nabla A + p_\Psi\nabla\Psi$$

with functions p_A and p_ψ as coefficients. Since

$$\nabla\times\nabla p = \mathbf{0}$$

we get

$$\nabla p_A \times \nabla A + \nabla p_\Psi \times \nabla\Psi = \mathbf{0}$$

This equation is fulfilled for:

$$p = F(A,\Psi),\; p_A = \partial F/\partial A$$
$$p_\Psi = \partial F/\partial\Psi$$

Including gravity

- Inserting this into the last equation:

$$\left(\frac{\partial^2 F}{\partial A^2}\nabla A+\frac{\partial^2 F}{\partial\psi\partial A}\nabla\Psi\right)\times\nabla A+\left(\frac{\partial^2 F}{\partial A\partial\Psi}\nabla A+\frac{\partial^2 F}{\partial A^2}\nabla\Psi\right)\times\nabla\Psi = \frac{\partial^2 F}{\partial A\partial\Psi}\left(\nabla\Psi\times\nabla A+\nabla A\times\nabla\Psi\right)=\mathbf{0}.$$

- And the force balance equation becomes:

$$\left[-\frac{1}{\mu_0}\left(\Delta A+B_y\frac{dB_y}{dA}\right)-\frac{\partial p}{\partial A}\right]\nabla A-\left(\frac{\partial p}{\partial\Psi}+\rho\right)\nabla\Psi=0$$

- The coefficients must vanish because ∇A and $\nabla\Psi$ are linear independent.

Including gravity

- Finally we have to solve:

$$\begin{aligned}-\Delta A &= \mu_0\frac{\partial p}{\partial A}+B_y\frac{dB_y}{dA}\\ \frac{\partial p}{\partial\Psi} &= -\rho.\end{aligned}$$

- To proceed we make now assumptions about the equation of state, e.g. isothermal plasma (simplest approach and the only one we consider here explicitely)
- A polytropic equation of state is also possible.
- Most realistic (and most complicated) would be to solve the energy equation.

Including gravity, isothermal plasma

Ideal gas with $p=R\rho T$ with $T=T(A,\Psi)$ given. It follows that

$$\frac{\partial p}{\partial\psi}=-\frac{p}{RT}$$

and integrating once we get

$$p=p_0(A)\exp\left(-\int_{\Psi_0}^{\Psi}\frac{d\Psi'}{RT(A,\Psi')}\right).$$

In the special case of an isothermal ideal gas this can be written as

$$p=p_0(A)\exp\left(-\frac{\Psi-\Psi_0}{RT}\right).$$

This is the usual barometric formula with different base pressure $p_0(A)$ for each field line !

3D configurations

- Analytic we can find 3D-equilibria without symmetry only in special cases.
- We can use Euler potentials:

$$\mathbf{B}=\nabla\alpha\times\nabla\beta=\nabla\times(\alpha\nabla\beta)$$

- And transform the force balance to:

$$\begin{aligned}\mathbf{j}\times\mathbf{B}-\nabla p-\rho\nabla\Psi &= \mathbf{j}\times(\nabla\alpha\times\nabla\beta)-\nabla p-\rho\nabla\Psi\\ &= (\mathbf{j}\cdot\nabla\beta)\nabla\alpha-(\mathbf{j}\cdot\nabla\alpha)\nabla\beta-\nabla p-\rho\nabla\Psi\\ &= \mathbf{0}.\end{aligned}$$

3D configurations

As in the symmetric case, $\nabla\alpha$, $\nabla\beta$ and $\nabla\Psi$ represent three linearly independent vector fields and we can split the force balance equation into three components along $\nabla\alpha$, $\nabla\beta$ and $\nabla\Psi$:

$$\begin{aligned}\nabla\beta\cdot\nabla\times(\nabla\alpha\times\nabla\beta)-\left(\frac{\partial p}{\partial\alpha}\right)_{\beta,\Psi} &= 0\\ -\nabla\alpha\cdot\nabla\times(\nabla\alpha\times\nabla\beta)-\left(\frac{\partial p}{\partial\beta}\right)_{\alpha,\Psi} &= 0\\ -\left(\frac{\partial p}{\partial\Psi}\right)_{\alpha,\beta} &= \rho\end{aligned}$$

Instead of a single partial differential equation (Grad-Shafranov) we get a system of coupled nonlinear equations => Extremely difficult to solve

Special cases in 3D: Force-free magnetic fields

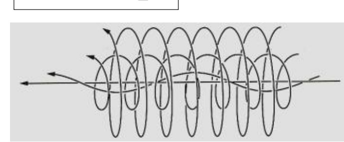
A special equilibrium of ideal MHD (often used in case of the solar corona) occurs if the beta is low, such that the pressure gradient can be neglected. The stationary plasma becomes ***force free***, if the Lorentz force vanishes:

$$\mathbf{j}\times\mathbf{B}=0$$

This condition is guaranteed if the current flows along the field:

$$\mu_0\mathbf{j}=\alpha_L\mathbf{B} \qquad \nabla\times\mathbf{B}=\alpha_L\mathbf{B}$$

By taking the divergence, one finds that $\alpha_L(x)$ is constant along any field line:



$$\mathbf{B}\cdot\nabla\alpha_L=0$$

Magneto-Hydro-Statics (MHS)

- Equilibrium structures (no time dependence, no plasma flow) are important and often approximately a reasonable assumption for space plasmas during quiet times.
- For symmetric configurations, MHS reduces to the Grad-Shafranov equation (GSE), a single (nonlinear) partial differential equation.
- GSE remains valid in kinetic theory.
- 3D MHS-equilibria have usually be computed numerically.

How to proceed?

- Study stationary states with plasma flow.
- The slow evolution of sequences of equilibria is often constraint, e.g., by the assumption of ideal MHD, which does not allow topology changes.
- Plasma waves are ubiquitos in space plasma.
- Discontinuities and current sheets.
- Plasma instabilities cause rapid changes and the equilibrium is lost (ideal and resistive instabilities).