Accelerated Lambda Iteration

Motivation

Complete Linearization provides a solution scheme, solving the radiation transfer, the statistical equilibrium and the radiative equilibrium simultaneously.

But, the system is coupled over all depths (via RT) and all frequencies (via SE, RE) → HUGE! ☛

Abbreviations used in this chapter:

RT = Radiation Transfer equations

SE = Statistical Equilibrium equations

RE = Radiative Equilibrium equation

Multi-frequency / multi-gray

Ways around:

Multi-frequency / multi-gray method by Anderson (1985,1989)

- Group all frequency points according to their opacity into bins (typically 5) and solve the RT with mean opacities of these bins. → Only 5 RT equations instead of thousands
- Use a Complete Linearization with the reduced set of equations
- Solve RT alone in between to get all intensities, Eddington-factors, etc.
- Main disadvantage: in principle depth dependent grouping

Lambda Iteration

Split RT and SE+RE:

$$J^{new} = \Lambda S^{old}(n,T)$$ RT formal solution

$$\underline{\underline{A(J,T)}}\,\underline{n}^{new} = \underline{b}$$ SE

$$\int_0^\infty \kappa(\nu,n,T)\big(J_\nu - S_\nu(\nu,n,T)\big)\,d\nu = 0$$ RE

- Good: SE is linear (if a separate T-correction scheme is used)
- Bad: SE contain old values of n,T (in rate matrix A)

Disadvantage: not converging, this is a Lambda iteration!

Accelerated Lambda Iteration (ALI)

Again: split RT and SE+RE but now use ALI

$$J^{new} = \Lambda S^{old}(n^{old},T^{old}) + \Lambda^* S^{new}(n^{new},T^{new}) - \Lambda^* S^{old}(n^{old},T^{old}) \quad \text{RT}$$

$$\underline{\underline{A}}(J^{new},T^{new})\underline{n}^{new} = \underline{b} \quad \text{SE}$$

$$\int_0^\infty \kappa(\nu,n^{new},T^{new})\left(J_\nu^{new} - S_\nu(\nu,n^{new},T^{new})\right)d\nu = 0 \quad \text{RE}$$

- Good: SE contains new quantities n, T
- Bad: Non-Linear equations → linearization (but without RT)

Basic advantage over Lambda Iteration: ALI converges!

Example: ALI working on Thomson scattering problem

$S = (1-\beta_e)B + \beta_e J$ source function with scattering, problem: J unknown→iterate

$$\Rightarrow J^{new} = \left(\Lambda - \Lambda^*\right)S^{old} + \Lambda^* S^{new}$$

$$= \left(\Lambda - \Lambda^*\right)S^{old} + \Lambda^*\left((1-\beta_e)B^{new} + \beta_e J^{new}\right) \qquad J^{FS}\text{:=formal solution on } S^{old}$$

$$= J^{FS} - \Lambda^*\left((1-\beta_e)B^{old} + \beta_e J^{old} - (1-\beta_e)B^{new} - \beta_e J^{new}\right) \qquad B^{old} = B^{new}$$

$$= J^{FS} - \Lambda^*\left(\beta_e J^{old} - \beta_e J^{new}\right) \qquad \text{solve for } J^{new}$$

$$\Rightarrow J^{new} = \left[1 - \Lambda^*\beta_e\right]^{-1}\left(J^{FS} - \Lambda^*\beta_e J^{old}\right) \qquad \text{subtract } J^{old} \text{ on both sides}$$

$$\Rightarrow J^{new} - J^{old} = \left[1 - \Lambda^*\beta_e\right]^{-1}\left(J^{FS} - J^{old}\right)$$

↑ amplification factor

Interpretation: iteration is driven by difference (J^{FS}-J^{old}) but: this difference is amplified, hence, iteration is accelerated.
Example: β_e=0.99; at large optical depth Λ^* almost 1 → strong amplifaction

What is a good Λ*?

The choice of Λ* is in principle irrelevant but in practice it decides about the success/failure of the iteration scheme.

First (useful) Λ* (Werner & Husfeld 1985):

$$\Lambda_\nu^*(\tau,\tau')\,S_\nu(\tau') = \begin{cases} S_\nu(\tau) & \tau > \gamma \\ 0 & \tau \le \gamma \end{cases}$$

A few other, more elaborate suggestions until Olson & Kunasz (1987): Best Λ* is the diagonal of the Λ-matrix

(Λ-matrix is the numerical representation of the integral operator Λ)

We therefore need an efficient method to calculate the elements of the Λ-matrix (are essentially functions of τ_ν).

Could compute directly elements representing the Λ-integral operator, but too expensive (E_1 functions). Instead: use solution method for transfer equation in differential (not integral) form: short characteristics method

In the final lecture tomorrow, we will learn two important methods to obtain numerically the formal solution of the radiation transfer equation.

1. Solution of the differential equation as a boundary-value problem (Feautrier method). [can include scattering]

2. Solution employing Schwarzschild equation on local scale (short characteristics method). [cannot include scattering, must ALI iterate]

The direct numerical evaluation of Schwarzschild equation is much too cpu-time consuming, but in principle possible.

Olson-Kunasz Λ*

Short characteristics with linear approximation of source function

$$I^+(\tau,\mu,\nu) = I^+(\tau_{\max},\mu,\nu)\exp\left(-\frac{\tau_{\max}-\tau}{\mu}\right) + \int_\tau^{\tau_{\max}} S\,(\tau')\exp\left(-\frac{\tau'-\tau}{\mu}\right)\frac{d\tau'}{\mu}$$

$$I^-(\tau,\mu,\nu) = I^-(0,\mu,\nu) \quad \exp\left(-\frac{\tau}{|\mu|}\right) \quad + \int_0^{\tau} S\,(\tau')\exp\left(-\frac{\tau-\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}$$

$$I^+(\tau_i,\mu,\nu) = I^+(\tau_{i+1},\mu,\nu)\exp\left(-\Delta\tau_i\right) + \Delta I_i^+(S,\mu,\nu)$$

$$I^-(\tau_i,\mu,\nu) = I^-(\tau_{i-1},\mu,\nu)\exp\left(-\Delta\tau_{i-1}\right) + \Delta I_i^-(S,\mu,\nu)$$

with $\Delta\tau_{i-1} = \dfrac{(\tau_i - \tau_{i-1})}{|\mu|}$

using a linear interpolation for the spatial variation of S

the intergrals $\Delta I_i^\pm$ can be evaluated as

$$\Delta I_i^\pm = \alpha_i^\pm S_{i-1} + \beta_i^\pm S_i + \gamma_i^\pm S_{i+1}$$

Olson-Kunasz Λ*

Short characteristics with linear approximation of source function

$$\alpha_i^+ = 0 \qquad \alpha_i^- = -e^{-\Delta} + \frac{1-e^{-\Delta}}{\Delta}$$

$$\beta_i^+ = 1 + \frac{e^{-\Delta}-1}{\Delta} \qquad \beta_i^- = 1 - \frac{1-e^{-\Delta}}{\Delta}$$

$$\gamma_i^+ = -e^{-\Delta} - \frac{e^{-\Delta}-1}{\Delta} \qquad \gamma_i^- = 0$$

$$J = \frac{1}{2}\int_0^1 \left(I^+ + I^-\right)d\mu = \frac{1}{2}\int_0^1 \left(\Lambda_\mu^+ S + \Lambda_\mu^- S\right)d\mu$$

use $S = \left(0,\cdots,1,\cdots,0\right)^T$ for $\left(0,\cdots,i,\cdots,0\right)$ to project colums of Λ

Stellar Atmospheres: Accelerated Lambda Iteration

Inward

$$\begin{pmatrix} & i & \\ & \vdots & \\ & 0 & \\ & \vdots & \\ & \Delta\hat{I}^-_{i-1} & \\ 0 & \hat{I}^-_{i-1}\exp(-\Delta\tau_{i-1})+\Delta\hat{I}^-_i & 0 \\ & \hat{I}^-_i\exp(-\Delta\tau_i)+\Delta\hat{I}^-_{i+1} & \\ & \vdots & \\ & \hat{I}^-_{k=i+2\cdots ND} & \\ & \vdots & \end{pmatrix} =$$

$$\begin{pmatrix} & i & \\ & \vdots & \\ & 0 & \\ & \vdots & \\ & \gamma^-_{i-1} & \\ 0 & \gamma^-_{i-1}\exp(-\Delta\tau_{i-1})+\beta^-_i & 0 \\ & \left(\gamma^-_{i-1}\exp(-\Delta\tau_{i-1})+\beta^-_i\right)\exp(-\Delta\tau_i)+\alpha^-_{i+1} & \\ & \vdots & \\ & \hat{I}^-_{k-1}\exp(-\Delta\tau_{k-1}) & \\ & \vdots & \end{pmatrix} = \begin{pmatrix} & i & \\ & \vdots & \\ & 0 & \\ & \vdots & \\ & 0 & \\ 0 & \beta^-_i & 0 \\ & \beta^-_i\exp(-\Delta\tau_i)+\alpha^-_{i+1} & \\ & \vdots & \\ & \hat{I}^-_{k-1}\exp(-\Delta\tau_{k-1}) & \\ & \vdots & \end{pmatrix}$$

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Stellar Atmospheres: Accelerated Lambda Iteration

Outward

$$\begin{pmatrix} & i & \\ & \vdots & \\ & \hat{I}^+_{k=0\cdots i-2} & \\ & \vdots & \\ & \hat{I}^+_i\exp(-\Delta\tau_{i-1})+\Delta\hat{I}^+_{i-1} & \\ 0 & \hat{I}^+_{i+1}\exp(-\Delta\tau_i)+\Delta\hat{I}^+_i & 0 \\ & \Delta\hat{I}^+_{i+1} & \\ & \vdots & \\ & 0 & \\ & \vdots & \end{pmatrix} =$$

$$\begin{pmatrix} & i & \\ & \vdots & \\ & \hat{I}^+_{k+1}\exp(-\Delta\tau_{k-1}) & \\ & \vdots & \\ & \left(\alpha^+_{i+1}\exp(-\Delta\tau_i)+\beta^+_i\right)\exp(-\Delta\tau_{i-1})+\gamma^+_{i-1} & \\ 0 & \alpha^+_{i+1}\exp(-\Delta\tau_i)+\beta^+_i & 0 \\ & \alpha^+_{i+1} & \\ & \vdots & \\ & 0 & \\ & \vdots & \end{pmatrix} = \begin{pmatrix} & i & \\ & \vdots & \\ & \hat{I}^+_{k+1}\exp(-\Delta\tau_{k-1}) & \\ & \vdots & \\ & \beta^+_i\exp(-\Delta\tau_{i-1})+\gamma^+_{i-1} & \\ 0 & \beta^+_i & 0 \\ & 0 & \\ & \vdots & \\ & 0 & \\ & \vdots & \end{pmatrix}$$

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Λ-Matrix

$$\Lambda_{*,i} = \begin{pmatrix} & \vdots & \\ & \frac{1}{2}\int_0^1 d\mu \hat{I}^+_{k+1} \exp(-\Delta\tau_{k-1}) & \\ & \vdots & \\ & \frac{1}{2}\int_0^1 d\mu \left(\beta_i^+ \exp(-\Delta\tau_{i-1}) + \gamma_{i-1}^+\right) & \\ 0 & \frac{1}{2}\int_0^1 d\mu \left(\beta_i^+ + \beta_i^-\right) & 0 \\ & \frac{1}{2}\int_0^1 d\mu \left(\beta_i^- \exp(-\Delta\tau_i) + \alpha_{i+1}^-\right) & \\ & \vdots & \\ & \frac{1}{2}\int_0^1 d\mu \left(\hat{I}^-_{k-1} \exp(-\Delta\tau_{k-1})\right) & \\ & \vdots & \end{pmatrix}$$

Towards a linear scheme

Λ* acts on S, which makes the equations non-linear in the occupation numbers

- Idea of Rybicki & Hummer (1992): use $J=\Delta J+\Psi^*\eta^{new}$ instead
- Modify the rate equations slightly:

$$R_{ij} n_i = 4\pi \int_0^\infty \frac{\sigma_{ij}}{h\nu} n_i J_\nu d\nu = 4\pi \int_0^\infty \frac{\sigma_{ij}}{h\nu} n_i \left(\Psi^* \eta(n) + \Delta J\right) d\nu$$

$$R_{ji} n_j = 4\pi \left(\frac{n_i}{n_j}\right)^* \int_0^\infty \frac{\sigma_{ij}}{h\nu} n_j \left(J_\nu + \frac{2h\nu^3}{c^2}\right) d\nu$$

$$= 4\pi \left(\frac{n_i}{n_j}\right)^* \int_0^\infty \frac{\sigma_{ij}}{h\nu} n_j \left(\Psi^* \eta(n) + \Delta J + \frac{2h\nu^3}{c^2}\right) d\nu$$