The Radiation Field

Description of the radiation field

Macroscopic description:

Specific intensity $I_\nu(\nu, \vec{n}, \vec{r}, t)$

as function of frequency, direction, location, and time; energy of radiation field (no polarization)

•in frequency interval (ν,ν+dν)

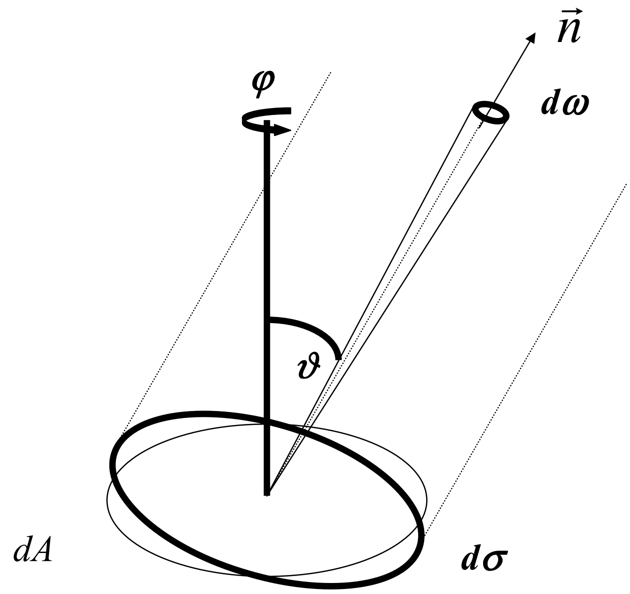
•in time interval (t,t+dt)

•in solid angle dω around n

•through area element dσ at location r ⊥ n

$$I_\nu(\nu, \vec{n}, \vec{r}, t) := \frac{d^4 E}{d\nu\, dt\, d\omega\, d\sigma}$$

The radiation field



Relation $I_\nu \leftrightarrow I_\lambda$

Energy in frequency interval (ν,ν+Δν) $\rightarrow I_\nu$
Energy in wavelength interval (λ,λ+Δλ) $\rightarrow I_\lambda$

i.e. $d^4E = I_\lambda \, dA \cos\vartheta \, dt \, d\lambda \, d\omega$

thus $I_\nu \, |d\nu| = I_\lambda \, |d\lambda|$

with $\nu\lambda = c \Rightarrow \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$ $\boxed{I_\nu = \frac{c}{\nu^2} I_\lambda \quad I_\lambda = \frac{c}{\lambda^2} I_\nu}$

	I_ν	I_λ
Dimension	$\frac{\text{energy}}{\text{area time freq. solid angle}}$	$\frac{\text{energy}}{\text{area time wavelength solid angle}}$
Unit	$\frac{\text{erg}}{\text{cm}^2 \text{ s Hz sterad}}$	$\frac{\text{erg}}{\text{cm}^2 \text{ s Å sterad}}$

Invariance of specific intensity

Irradiated energy:

$$dE = I_\nu(\nu,\vartheta)d\nu \cos\vartheta\, dA\, d\omega$$

dA' as seen from dA subtends solid angle $d\omega$

$$d\omega = \cos\vartheta' dA' / d^2$$

$$\rightarrow dE = I_\nu(\nu,\vartheta)d\nu \frac{\cos\vartheta\, dA \cos\vartheta'\, dA'}{d^2}$$

now, dA as seen from dA'

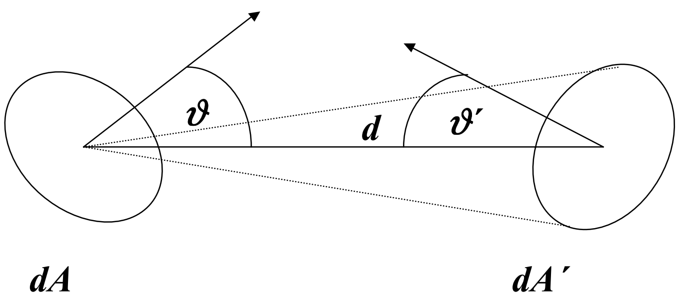
$$dE' = I'_\nu(\nu,\vartheta)d\nu \frac{\cos\vartheta\, dA\ \cos\vartheta'\, dA'}{d^2}$$

if no sources or sinks along d:

$$dE = dE' \Leftrightarrow I'_\nu = I_\nu$$

The specific intensity is distance independent if no sources or sinks are present.

Irradiance of two area elements



Specific Intensity

Specific intensity can only be measured from extended objects, e.g. Sun, nebulae, planets

Detector measures energy per time and frequency interval

$$dE = I_\nu \cos\vartheta \, d\omega \, A$$

e.g. A is the detector area

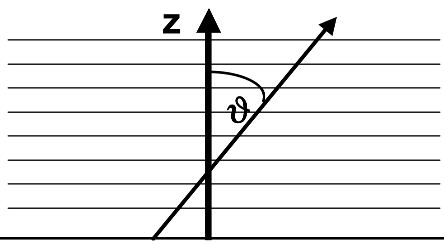
$d\omega \sim (1'')^2$ is the seeing disk

Special symmetries

- Time dependence unimportant for most problems
- In most cases the stellar atmosphere can be described in plane-parallel geometry

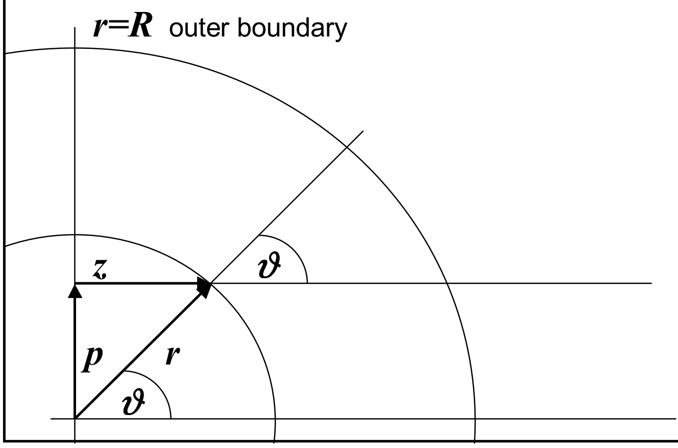
Sun: $$\frac{\text{atmosphere}}{\text{radius}} = \frac{200\text{ km}}{700000\text{ km}} = \frac{1}{3500} << 1$$

$$\mu := \cos\vartheta \quad I_\nu = I_\nu(\nu, \mu, z)$$



- For extended objects, e.g. giant stars (expanding atmospheres) spherical symmetry can be assumed

spherical coordinates: $I_\nu(\nu,\mu,r)$ Cartesian coordinates: $I_\nu(\nu,p,z)$



Integrals over angle, moments of intensity

- The 0-th moment, mean intensity

$$J_\nu = \frac{1}{4\pi}\oint_{4\pi} I_\nu(\vec{n})d\omega \quad \text{with spherical coordinates}$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} I_\nu \sin\vartheta\, d\vartheta\, d\phi \quad \text{with } \mu := \cos\vartheta$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} I_\nu\, d\mu\, d\phi$$

- In case of plane-parallel or spherical geometry

$$\boxed{J_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu\, d\mu}$$

$$\frac{\text{energy}}{\text{area time frequency}} \qquad \frac{\text{erg}}{\text{cm}^2\ \text{s Hz}}$$

J_ν is related to the energy density u_ν

radiated energy through area element dA during time dt:

$$dE = I_\nu \, d\nu \; dt \; d\omega \, dA$$

$$l = c\,dt \Rightarrow dV = l\,dA = c\,dt\,dA$$

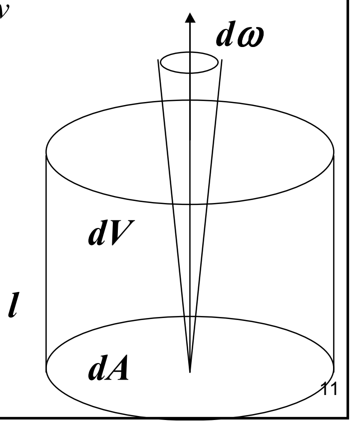
hence, the energy contained in volume element dV per frequency interval is

$$u_\nu dV \; d\nu = \oint_{4\pi} I_\nu \; d\omega \, d\nu \; dt \; dA = 4\pi J_\nu \; dV / c \; d\nu$$

$$u_\nu = \tfrac{4\pi}{c} J_\nu \qquad \frac{\text{energy}}{\text{volume frequency}} \qquad \frac{\text{erg}}{\text{cm}^3 \text{ Hz}}$$

total radiation energy in volume element:

$$u = \int_0^\infty u_\nu \; d\nu = \tfrac{4\pi}{c} \int_0^\infty J_\nu \; d\nu \qquad \frac{\text{energy}}{\text{volume}} \qquad \frac{\text{erg}}{\text{cm}^3}$$

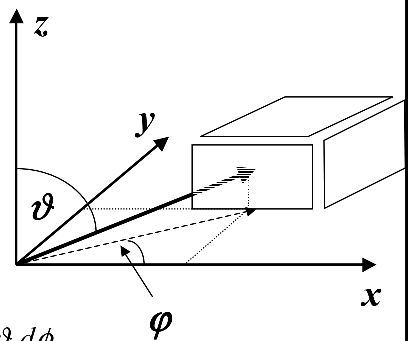


The 1st moment: radiation flux

$$\vec{F}_\nu = \oint_{4\pi} I_\nu(\vec{n}) \; \vec{n} \; d\omega$$

propagation vector in spherical coordinates:

$$\vec{n} = \begin{pmatrix} \sin\vartheta \cos\phi \\ \sin\vartheta \sin\phi \\ \cos\vartheta \end{pmatrix}$$



$$\Rightarrow F_{\nu,x} = \iint I(\vartheta,\phi) \sin\vartheta \cos\phi \sin\vartheta \; d\vartheta \; d\phi$$

in plane-parallel or spherical geometry:

$$F_{\nu,x} = F_{\nu,y} = 0, \; F_{\nu,z} = F_\nu = 2\pi \int_{-1}^{1} I_\nu(\mu)\mu \; d\mu$$

$$\frac{\text{energy}}{\text{area time frequency}} \qquad \frac{\text{erg}}{\text{cm}^2 \text{ s Hz}}$$

Meaning of flux:

Radiation flux = netto energy going through area ⊥ z-axis

Decomposition into two half-spaces:

$$F = 2\pi\int_{-1}^{1} I(\mu)\mu d\mu$$
$$= 2\pi\int_{0}^{1} I(\mu)\mu d\mu + 2\pi\int_{-1}^{0} I(\mu)\mu d\mu$$
$$= 2\pi\int_{0}^{1} I(\mu)\mu d\mu - 2\pi\int_{0}^{1} I(-\mu)\mu d\mu$$
$$= F^{+} - F^{-}$$

netto = outwards - inwards

Special case: isotropic radiation field: $F = 0$

Other definitions:

F_{ν} astrophysical flux

H_{ν} Eddington flux

$F_{\nu} = \pi \mathrm{F}_{\nu} = 4\pi H_{\nu}$

Idea behind definition of Eddington flux

In 1-dimensional geometry the n-th moments of intensity are

0-th moment: $J_{\nu} = \frac{1}{2}\int_{-1}^{1} I(\mu)\, d\mu$

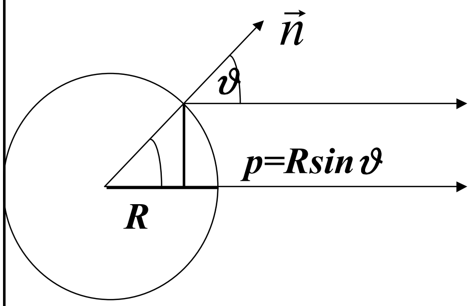
1st moment: $H_{\nu} = \frac{1}{2}\int_{-1}^{1} I(\mu)\, \mu d\mu$

2nd moment: $K_{\nu} = \frac{1}{2}\int_{-1}^{1} I(\mu)\, \mu^{2} d\mu$

n-th moment: $= \frac{1}{2}\int_{-1}^{1} I(\mu)\, \mu^{\mathrm{n}} d\mu$

Idea behind definition of astrophysical flux

Intensity averaged over stellar disk = astrophysical flux



$$p = R\sin\vartheta$$

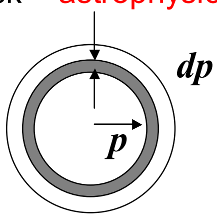
$$p^2 = R^2(1-\mu^2)$$

$$2p\frac{dp}{d\mu} = -2R^2\mu$$

$$pdp = -R^2\mu d\mu$$

Idea behind definition of astrophysical flux

Intensity averaged over stellar disk = astrophysical flux



$$\bar{I}_\nu = \frac{1}{\pi R^2}\int_0^R I_\nu(p)2\pi\, pdp$$

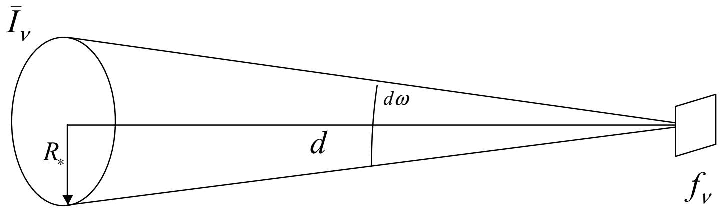
$$= \frac{1}{\pi R^2}\int_0^1 I_\nu(\mu)2\pi R^2 \mu d\mu$$

$$= F_\nu^+/\pi = \mathrm{F}_\nu^+$$

$\mathrm{F}_\nu^- = 0$ no inward flux at stellar surface

$$\boxed{\bar{I}_\nu = \mathrm{F}_\nu}$$

Flux at location of observer



$$\vec{F}_\nu = \oint_{4\pi} I_\nu(\vec{n})\, \vec{n}\, d\omega$$

Flux at distant observer's detector normal to the line of sight:

$$f_\nu = \bar{I}_\nu d\omega = \bar{I}_\nu \pi R_*^2 / d^2 = \pi \mathrm{F}_\nu \frac{R_*^2}{d^2} = F_\nu \frac{R_*^2}{d^2}$$

Total energy radiated away by the star, luminosity

Integral over frequency at outer boundary:

$$F = \int_0^\infty F_\nu^+ d\nu = \int_0^\infty F_\nu\, d\nu$$

Multiplied by stellar surface area yields the luminosity

$$L = 4\pi R_*^2 F = 4\pi^2 R_*^2 \mathrm{F} = 16\pi^2 R_*^2 H$$

$$\frac{\text{energy}}{\text{time}} \quad \frac{\text{erg}}{\text{s}}$$

The photon gas pressure

Photon momentum: $p_\nu = E_\nu / c$

Force: $F = \frac{dp_{\nu\perp}}{dt} = \frac{1}{c}\frac{dE_\nu}{dt}\cos\vartheta$

Pressure: $$dP_\nu = \frac{F}{dA} = \frac{1}{c}\frac{dE_\nu}{dt}\frac{\cos\vartheta}{dA} = \frac{1}{c} I_\nu \cos^2\vartheta \, d\omega \, d\nu$$

$$P(\nu) = \frac{1}{c}\oint_{4\pi} I_\nu \cos^2\vartheta \, d\omega = \frac{2\pi}{c}\int_{-1}^{1} I_\nu \mu^2 d\mu = \frac{4\pi}{c} K_\nu$$

Isotropic radiation field: $I_\nu(\mu) = I_\nu = J_\nu$

$$P(\nu) = \frac{4\pi}{c}\frac{I_\nu}{3} \quad u_\nu = \frac{4\pi}{c} I_\nu \Rightarrow P(\nu) = \frac{1}{3} u_\nu \quad J_\nu = 3K_\nu$$

Special case: black body radiation (Hohlraumstrahlung)

Radiation field in Thermodynamic Equilibrium with matter of temperature T

$$I_\nu(\nu, \vec{n}, \vec{r}, t) = I_\nu(\nu)$$

$$I_\nu = B_\nu(\nu, T) \text{ bzw. } I_\lambda = B_\lambda(\nu, T)$$

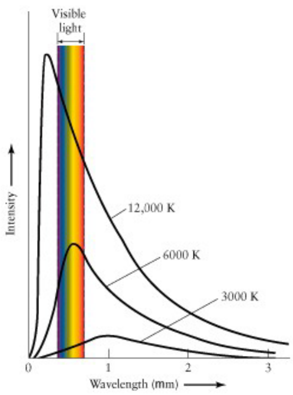
$$\text{in cavity: } \vec{F} = 0 \quad J_\nu = I_\nu = B_\nu$$

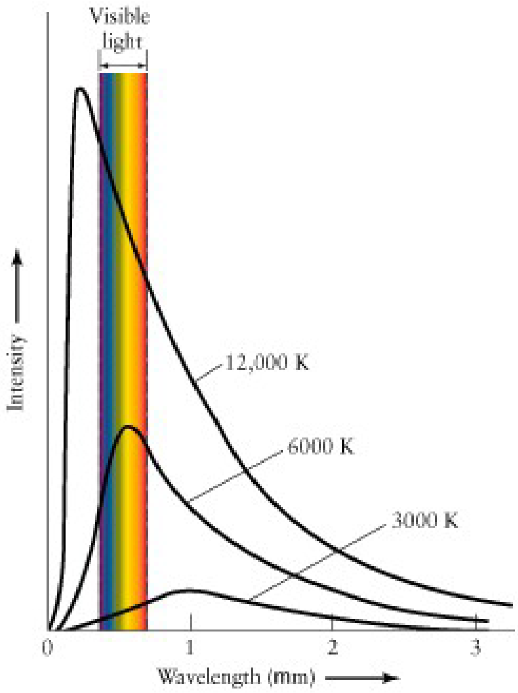
$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1}$$

$$B_\nu(\lambda, T) = \frac{2hc}{\lambda^3}\left[\exp\left(\frac{hc}{\lambda kT}\right) - 1\right]^{-1}$$

$$B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5}\left[\exp\left(\frac{hc}{\lambda kT}\right) - 1\right]^{-1}$$

$$B_\lambda(\nu, T) = \frac{2h\nu^5}{c^3}\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1}$$





Asymptotic behaviour

- In the „red" Rayleigh-Jeans domain

$$\frac{h\nu}{kT} << 1 \qquad \exp\left(\frac{h\nu}{kT}\right) \approx 1 + \frac{h\nu}{kT}$$

$$B_\nu(\nu,T) = \frac{2k\nu^2 T}{c^2}$$

$$\boxed{B_\nu(\lambda,T) = \frac{2ckT}{\lambda^4}}$$

- In the „blue" Wien domain

$$\frac{h\nu}{kT} >> 1 \qquad \exp\left(\frac{h\nu}{kT}\right) - 1 \approx \exp\left(\frac{h\nu}{kT}\right)$$

$$B_\nu(\nu,T) = \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{kT}\right)$$

$$\boxed{B_\nu(\lambda,T) = \frac{2hc}{\lambda^3}\exp\left(-\frac{hc}{\lambda kT}\right)}$$

Wien´s law

$$\frac{d}{d\nu}B_\nu(\nu,T) = \frac{d}{d\nu}\left[\frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}\right]$$

x:=hv/kT

$$= B_\nu\left[\frac{3}{\nu}+\frac{-1}{\mathrm{e}^x-1}\frac{x}{\nu}\mathrm{e}^x\right]$$

$$\frac{d}{d\nu}B_\nu = 0 \rightarrow 3 - x_{\max}\,\mathrm{e}^{x_{\max}}/\left(\mathrm{e}^{x_{\max}}-1\right)=0$$

$$\rightarrow x_{\max} - 3\left(1-\mathrm{e}^{-x_{\max}}\right)=0$$

numerical solution: $x_{\max} = 2.821 = \frac{h\nu_{\max}}{kT}$ $\quad \lambda_{\max}T = 0.5100\,\mathrm{cm\ \ deg}$

$$\frac{d}{d\lambda}B_\lambda = 0 \rightarrow x_{\max} - 5\left(1-\mathrm{e}^{-x_{\max}}\right)=0$$

numerical solution: $x_{\max} = 4.965 = \frac{hc}{\lambda_{\max}kT}$ $\quad \lambda_{\max}T = 0.2897\,\mathrm{cm\ \ deg}$

Integration over frequencies

$$B(T) = \int_0^\infty B_\nu(T)d\nu = \int_0^\infty \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}d\nu$$

$$= \frac{2k^4}{c^2h^3}T^4\underbrace{\int_0^\infty\frac{x^3}{e^x-1}dx}_{=\pi^4/15} = \frac{2}{15}\frac{\pi^4k^4}{c^2h^3}T^4$$

$$= \frac{\sigma}{\pi}T^4 \quad \text{with } \sigma = \frac{2}{15}\frac{\pi^5k^4}{c^2h^3} = 5.669\cdot10^{-5}\,\mathrm{erg\ cm^{-2}\ s^{-1}\ deg^{-4}}$$

Stefan-Boltzmann law

Energy density of blackbody radiation:

$$u = \frac{4\pi}{c}\int_0^\infty J_\nu(\nu)d\nu = \frac{4\pi}{c}B(T) = \frac{4\sigma}{c}T^4$$

Stars as black bodies – effective temperature

Surface as „open" cavity (... physically nonsense)

$$I_\nu^+ = B_\nu \ , I_\nu^- = 0$$

$$I_\nu = \begin{cases} B_\nu & \text{for } \mu > 0 \\ 0 & \text{for } \mu \le 0 \end{cases}$$

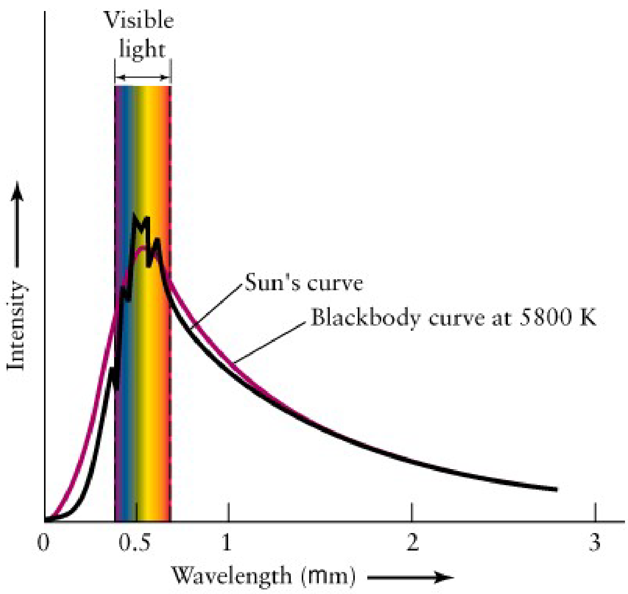
with $F_\nu = B_\nu$ and $F = B(T) = \frac{\sigma}{\pi} T^4$

luminosity: $L = 4\pi^2 R_*^2 F = 4\sigma\pi R_*^2 T^4$

hence, eff. temperature: $T_{\text{eff}} = (4\sigma\pi)^{-1/4} L^{-1/4} R_*^{-1/2}$

Attention: definition dependent on stellar radius!

Stars as black bodies – effective temperature



Examples and applications

- Solar constant, effective temperature of the Sun

$$\int_0^\infty f_\nu(\nu)d\nu = f = 1.36 \text{ kW/m}^2 = 1.36 \text{ erg s}^{-1} \text{ cm}^{-2}$$

$$\mathrm{F} = f\frac{d^2}{\pi R_*^2} \quad \text{with} \quad d = 1.5\cdot 10^{13}\,\text{cm} \quad R_\odot = 6.69\cdot 10^{10}\,\text{cm}$$

$$\mathrm{F}_\odot = 2.01\cdot 10^{10} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ flux at solar surface}$$

$$T_{\text{eff}}^4 = \frac{\pi}{\sigma}\mathrm{F} \Rightarrow T_{\text{eff}}^\odot = 5780 \text{ K}$$

- Sun's center

$$T_c = 1.4\cdot 10^7 \text{ K}$$

$$\Rightarrow \text{Planck maximum at} \quad \lambda_{\max} = 3.4\text{Å} \quad (B_\nu)$$

$$\text{or} \quad \lambda_{\max} = 2.1\text{Å} \quad (B_\lambda)$$

$$\text{with} \quad 1\text{Å} \;\hat{=}\; 12.4 \text{ keV} \quad \text{maximum} \approx 4 \text{ keV}$$

Stellar Atmospheres: The Radiation Field

Examples and applications

- Main sequence star, spectral type O

$$R_* = 10R_\odot \;,\; T_{\text{eff}}^* = 60000 \text{ K}$$

$$\frac{L_*}{L_\odot} = \left(\frac{T_{\text{eff}}^*}{T_{\text{eff}}^\odot}\right)^4 \left(\frac{R_*}{R_\odot}\right)^2 \Rightarrow L_* = 10^6 L_\odot$$

$$\lambda_{\max} = 882\text{Å} \;(B_\nu) \quad \text{or} \quad \lambda_{\max} = 501\text{Å} \;(B_\lambda)$$

- Interstellar dust

$$T = 20\,\text{K} \;,\; \lambda_{\max} = 0.3\,\text{mm}\,(B_\nu)$$

- 3K background radiation

$$T = 2.7\,\text{K} \;,\; \lambda_{\max} = 1.9\,\text{mm}\,(B_\nu)$$

Radiation temperature

... is the temperature, at which the corresponding blackbody would have equal intensity

$$I_\nu(\lambda) = \frac{2hc}{\lambda^3}\left[\exp\left(\frac{hc}{\lambda k T_{\rm rad}}\right) - 1\right]^{-1} \Rightarrow T_{\rm rad} = \frac{hc}{k\lambda}\left[\ln\left(\frac{2hc}{\lambda^3 I_\nu} + 1\right)\right]^{-1}$$

Comfortable quantity with Kelvin as unit

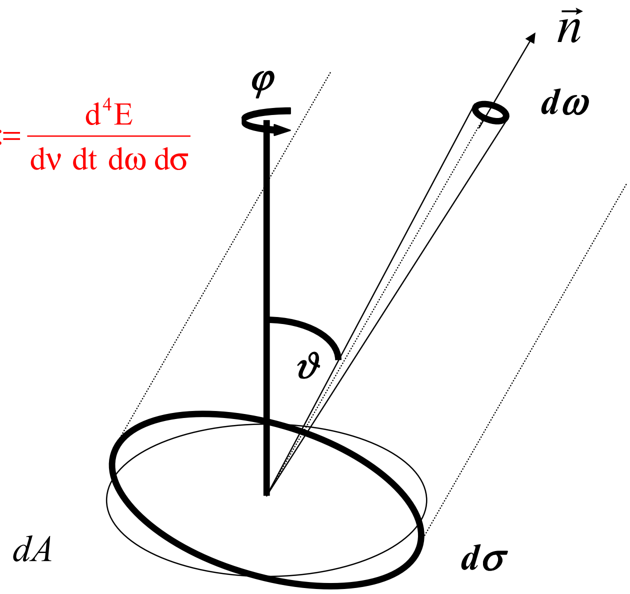
Often used in radio astronomy

The Radiation Field

- Summary -

Summary: Definition of specific intensity

$$I_\nu(\nu,\vec{n},\vec{r},t) := \frac{d^4E}{d\nu\ dt\ d\omega\ d\sigma}$$



Summary: Moments of radiation field

In 1-dim geometry (plane-parallel or spherically symmetric):

0-th moment:	$J_\nu = \frac{1}{2}\int_{-1}^{1} I(\mu)\ d\mu$	Mean intensity
1st moment:	$H_\nu = \frac{1}{2}\int_{-1}^{1} I(\mu)\ \mu d\mu$	Eddington flux
2nd moment:	$K_\nu = \frac{1}{2}\int_{-1}^{1} I(\mu)\ \mu^2 d\mu$	K-integral

$\mathcal{F}_\nu$ =astrophysical flux H_ν=Eddington flux F_ν = flux $F_\nu = \pi\mathcal{F}_\nu = 4\pi H_\nu$

energy density $u = \int_0^\infty u_\nu\ d\nu = \frac{4\pi}{c}\int_0^\infty J_\nu\ d\nu$

total flux at stellar surface $F = \int_0^\infty F_\nu^+ d\nu = \int_0^\infty F_\nu\ d\nu$

stellar luminosity $L = 4\pi R_*^2 F = 4\pi^2 R_*^2 \mathcal{F} = 16\pi^2 R_*^2 H$

Summary: Moments of radiation field

pressure of photon gas $P(\nu) = \frac{4\pi}{c} K_\nu$

blackbody radiation $B_\nu(\nu, T) = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1}$

Wien´s law $\lambda_{max} T = \text{constant}$

Stefan-Boltzmann law $B(T) = \int_0^\infty B_\nu(T) d\nu = \frac{\sigma}{\pi} T^4$

energy density of blackbody radiation $u = \frac{4\sigma}{c} T^4$

effective temperature $L = 4\pi^2 R_*^2 F = 4\pi^2 R_*^2 B = 4\sigma\pi R_*^2 T_{eff}^4$