Radiation Transfer

Interaction radiation – matter

Energy can be removed from, or delivered to, the radiation field

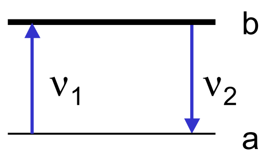
Classification by physical processes:

True absorption:	photon is destroyed, energy is transferred into kinetic energy of gas; photon is thermalized
True emission:	photon is generated, extracts kinetic energy from the gas
Scattering:	photon interacts with scatterer $\rightarrow$ direction changed, energy slightly changed $\rightarrow$ no energy exchange with gas

Examples: true absorption and emission

- photoionization (bound-free) excess energy is transferred into kinetic energy of the released electron $\rightarrow$ effect on local temperature
- photoexcitation (bound-bound) followed by electron collisional de-excitation; excitation energy is transferred to the electron $\rightarrow$ effect on local temperature
- photoexcitation (bound-bound) followed by collisional ionization
- reverse processes are examples for true emission

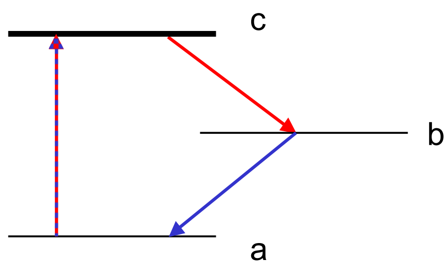
Examples: scattering processes



- 2-level atom absorbs photon with frequency ν_1, re-emits photon with frequency ν_2; frequencies not exactly equal, because
  - levels a and b have non-vanishing energy width
  - Doppler effect because atom moves
- Scattering of photons by free electrons: Compton- or Thomson scattering, (anelastic or elastic) collision of a photon with a free electron

Fluorescence

Neither scattering nor true absorption process



c-b: collisional de-excitation

b-a: radiative

Change of intensity along path element

generally: $\dfrac{dI_\nu}{ds}$

plane-parallel geometry: $\dfrac{dI_\nu}{ds} = -\mu \dfrac{dI_\nu}{dt}$ with $dt = -\mu ds$

spherical geometry:

$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \mu}\frac{d\mu}{ds}$$

$$\Rightarrow \frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\mu + \frac{\partial I_\nu}{\partial \mu}\frac{1-\mu^2}{r}$$

$$dr = ds\cos\vartheta \Rightarrow \frac{dr}{ds} = \mu$$

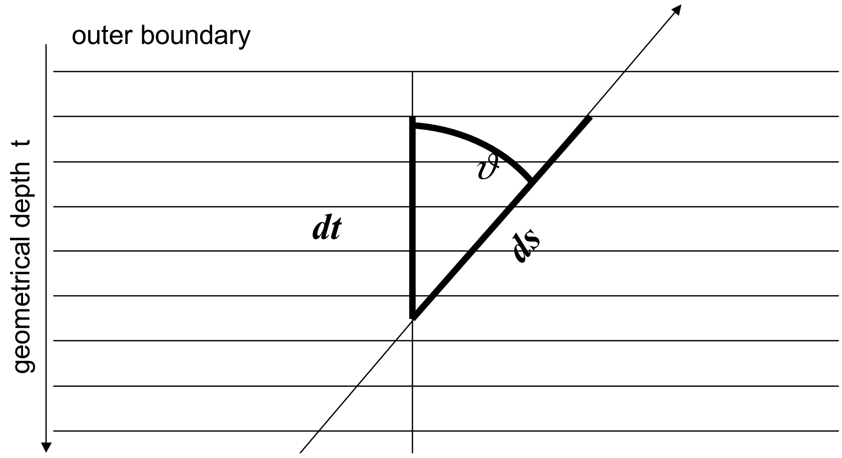
$$\sin(\vartheta + d\vartheta) \approx \sin\vartheta = \frac{-r\,d\vartheta}{ds}$$

$$\Rightarrow \frac{d\mu}{ds} = \frac{d\mu}{d\vartheta}\frac{d\vartheta}{ds} = -\sin\vartheta\frac{1}{r}(-\sin\vartheta)$$

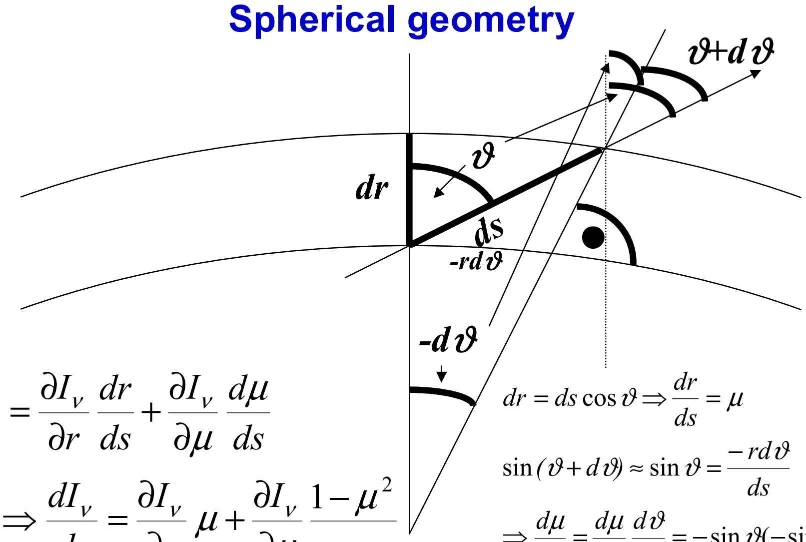
$$= (1-\mu^2)/r$$

Plane-parallel geometry

$$\frac{dI_\nu}{ds} = -\mu \frac{dI_\nu}{dt} \quad \text{with } dt = -\mu ds$$



Spherical geometry



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$$= (1-\mu^2)/r$$

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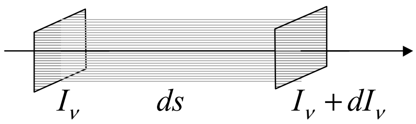
$$= (1-\mu^2)/r$$

Right-hand side of transfer equation

- No absorption (vacuum)

  $\frac{dI_\nu}{ds} = 0 \Rightarrow I_\nu = \text{const.}$ invariance of intensity

- Absorption only, no emission



energy removed from ray: $dE = -dI_\nu \, d\nu \, dt \, d\omega \, d\sigma$

is proportional to energy content in ray: $I_\nu \, d\nu \, dt \, d\omega \, d\sigma$

and to the path element: ds

Absorption coefficient

- thus: $dI_\nu = -\kappa I_\nu \, ds$

κ absorption coefficient, opacity

- dimension: 1/length unit: cm^{-1}
- but also often used: mass absorption coefficient, e.g., per gram matter

κ in general complicated function of physical quantities T, P, and frequency, direction, time...

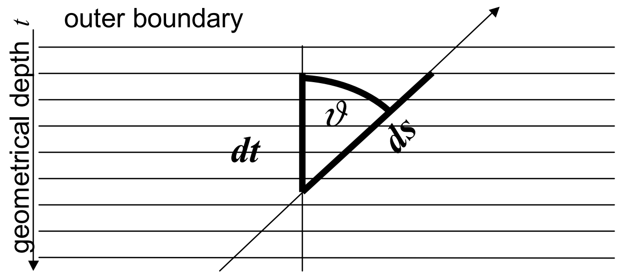
$\kappa = \kappa(\vec{r}, \vec{n}, \nu, t)$

- often there is a coordinate system in which κ isotropic, e.g. co-moving frame in moving atmospheres

$\kappa = \kappa(\vec{r}, \nu)$

- counter-example: magnetic fields (Zeeman effect)

only absorption, plane-parallel geometry



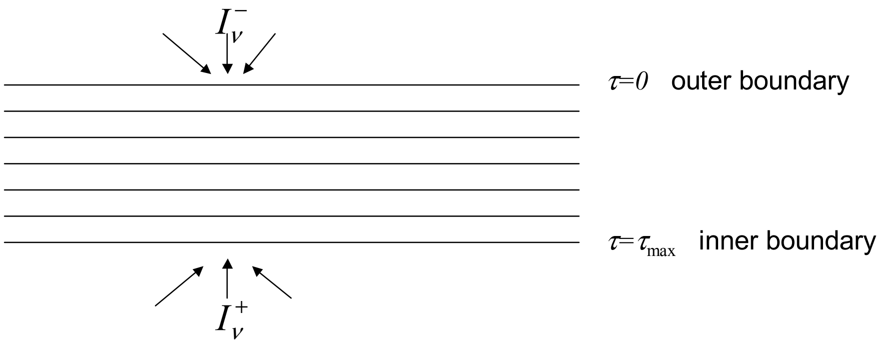
$$\frac{dI_\nu(\mu,t)}{ds} = -\kappa(\nu,t) I_\nu(\mu,t) \Rightarrow -\mu \frac{dI_\nu}{dt}(\mu,t) = -\kappa(\nu,t) I_\nu(\mu,t)$$

with optical depth $d\tau := \kappa dt \rightarrow \tau(\nu,t) = \int_{t=0}^{t} \kappa(\nu,t')dt'$ with $\tau = 0$ at $t = 0$

$$\Rightarrow \frac{dI_\nu(\mu,\tau)}{d\tau} = \frac{1}{\mu} I_\nu(\mu,\tau)$$

$\Rightarrow \boxed{I_\nu(\mu,\tau) = c \cdot e^{\tau/\mu}}$ $\quad c$ integration constant, fixed by boundary values

Schuster boundary-value problem



$$\mu < 0 : I_\nu^-(\mu,\tau=0) = c \cdot e^{0/\mu} = c$$

$$\boxed{I_\nu^-(\mu,\tau) = I_\nu^-(\mu,\tau=0)e^{-\tau/|\mu|}}$$

$$\mu > 0 : I_\nu^+(\mu,\tau=\tau_{\max}) = c \cdot e^{\tau_{\max}/\mu}$$

$$\boxed{I_\nu^+(\mu,\tau) = I_\nu^+(\mu,\tau=\tau_{\max})e^{-(\tau_{\max}-\tau)/\mu}}$$

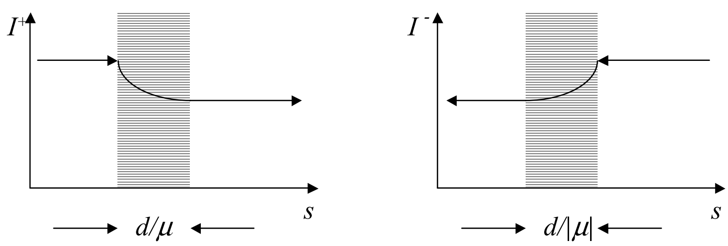
Example: homogeneous medium

e.g. glass filter

$$\kappa(t,\mu) = \kappa \rightarrow \tau = \kappa\, t \rightarrow \tau_{\max} = \kappa\, d \qquad d = \text{thickness of filter}$$

$$I_\nu^+(\mu,\tau=0) \qquad = I_\nu^+(\mu,\tau=\tau_{\max}) \cdot e^{-(\kappa\, d-0)/\mu}$$

$$I_\nu^-(\mu,\tau=\tau_{\max}) = I_\nu^-(\mu,\tau=0) \cdot \qquad e^{-\kappa\, d/|\mu|}$$



Half-width thickness

$$s_{1/2}: \quad e^{-\kappa s_{1/2}} = 1/2$$

Material	$S_{1/2}$ / meter
River water	0.033
Window glass	0.066
City air	330
Glas fiber	6600
Solar atmosphere	200000

Physical interpretation of optical depth

What is the mean penetration depth of photons into medium?

$$\langle \tau \rangle = \int_0^\infty \tau \, p(\tau) \, d\tau \qquad \text{(mathematically: expectation value of probability function } p(\tau)\text{)}$$

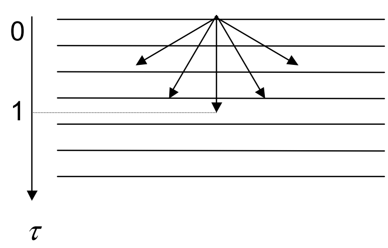
$$p(\tau)\, d\tau := \text{ probability for absorption in interval } [\tau, \tau + d\tau]$$

$$= \underbrace{e^{-\tau/\mu}}_{I_\nu^-(\tau)/I_\nu^-(\tau=0)} \cdot \frac{1}{\mu} d\tau \quad \text{note normalization: } \int_0^\infty p(\tau) d\tau = \int_0^\infty e^{-\tau/\mu} \frac{d\tau}{\mu} = -e^{-x}\Big|_0^\infty = 1$$

$$\langle \tau \rangle = \int_0^\infty \tau \, \mathrm{e}^{-\tau/\mu} \cdot \frac{1}{\mu} d\tau = \mu \int_0^\infty x e^{-x} dx = \mu \cdot 1$$

$$\langle \tau \rangle = \mu \quad \text{mean penetration depth}$$

$$\mathrm{s_f} = \frac{t_f}{\mu} = \frac{1}{\mu} \frac{\langle \tau \rangle}{\kappa} = \boxed{\frac{1}{\kappa} \quad \text{mean free path}}$$



The right-hand side of the transfer equation

- transfer equation including emission

Energy added to the ray: $dE = +dI_\nu \, d\nu \, dt \, d\omega \, d\sigma$

is proportional to path element: ds

emission coefficient η_ν $\boxed{dI_\nu = \eta_\nu ds}$

- dimension: intensity / length unit: erg cm^{-3} sterad $^{-1}$

The right-hand side of the transfer equation

- Transfer equation including emission

η in general a complicated function of physical quantities T,P,..., and frequency $\eta = \eta(\vec{r}, \vec{n}, \nu, t)$

η is not isotropic even in static atmospheres, but is usually assumed to isotropic (complete redistribution)

if constant with time: $\eta = \eta(\vec{r}, \nu)$

The complete transfer equation

$$\frac{dI_\nu}{ds} = \eta_\nu - \kappa(\nu) I_\nu$$

Definition of source function: $S_\nu = \dfrac{\eta_\nu}{\kappa(\nu)}$

$$\boxed{\frac{dI_\nu}{ds} = \kappa(\nu)\left(S_\nu - I_\nu\right)}$$

- Plane-parallel geometry

$$-\mu \frac{dI_\nu(\nu,\mu,t)}{dt} = \kappa(\nu,t)\left(S_\nu(\nu,\mu,t) - I_\nu(\nu,\mu,t)\right)$$

- Spherical geometry

$$\mu \frac{\partial I_\nu(\nu,\mu,r)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_\nu(\nu,\mu,r)}{\partial \mu} = \kappa(\nu,r)\left(S_\nu(\nu,\mu,r) - I_\nu(\nu,\mu,r)\right)$$

Solution with given source function: Formal solution

- Plane-parallel case

$$\frac{dI_\nu}{d\tau} = \frac{1}{\mu}\left(I_\nu - S_\nu\right) \qquad \text{or:} \qquad \frac{dI_\nu}{d\tau} + \frac{1}{-\mu} I_\nu = \frac{1}{-\mu} S_\nu$$

linear 1st-order differential equation of form $y' + f(x)y = g(x)$

has the integrating factor $M(x) = \exp\left(\int_{x_0}^{x} f(x)dx\right)$

und thus the solution $y(x) = \dfrac{1}{M(x)}\left(\int_{x_0}^{x} g(x)M(x)dx + C\right)$ $\quad C = y(x_0)$

(proof by insertion)

in our case:

$$x \to \tau_\nu$$
$$f(x) \to -1/\mu$$
$$g(x) \to -1/\mu\, S_\nu(\tau_\nu)$$
$$y(x) \to I_\nu(\tau_\nu)$$

Formal solution for I^+

Reference point x_0: $\tau=\tau_{\max}$ for I^+ ($\mu > 0$) outgoing radiation

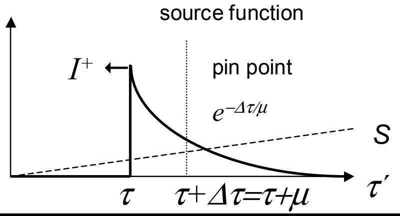
$$M(\tau) = \exp\left(\int_{\tau_{\max}}^{\tau} -\frac{1}{\mu}\,d\tau'\right) = \exp\left(\frac{\tau_{\max}-\tau}{\mu}\right)$$

$$I_\nu^+(\tau) = \exp\left(-\frac{\tau_{\max}-\tau}{\mu}\right)\left[\int_{\tau_{\max}}^{\tau} -\frac{1}{\mu} S_\nu(\tau')\exp\left(\frac{\tau_{\max}-\tau'}{\mu}\right)d\tau' + I_\nu^+(\tau_{\max})\right]$$

$$\boxed{I_\nu^+(\tau) = \int_{\tau}^{\tau_{\max}} S_\nu(\tau')\exp\left(-\frac{\tau'-\tau}{\mu}\right)\frac{d\tau'}{\mu} + I_\nu^+(\tau_{\max})\exp\left(-\frac{\tau_{\max}-\tau}{\mu}\right)}$$

weighted mean over source function

exponentially absorbed ingoing radiation from inner boundary



Hence, as rough approximation:

$$I_\nu^+(\tau) \approx S_\nu(\tau+\mu)$$

Formal solution for I^-

Reference point x_0: $\tau=0$ for I^- ($\mu < 0$) ingoing radiation

$$M(\tau) = \exp\left(\int_{0}^{\tau} -\frac{1}{\mu}\,d\tau'\right) = \exp\left(\frac{\tau}{|\mu|}\right)$$

$$I_\nu^-(\tau) = \exp\left(-\frac{\tau}{|\mu|}\right)\left[\int_{0}^{\tau} -\frac{1}{\mu} S_\nu(\tau')\exp\left(\frac{\tau'}{|\mu|}\right)d\tau' + I_\nu^-(0)\right]$$

$$\boxed{I_\nu^-(\tau) = \int_{0}^{\tau} S_\nu(\tau')\exp\left(-\frac{\tau-\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|} + I_\nu^-(0)\exp\left(-\frac{\tau}{|\mu|}\right)}$$

weighted mean over source function

exponentially absorbed ingoing radiation from outer boundary

Emergent intensity

$$I_\nu^+(0) = \int_0^{\tau_{\max}} S_\nu(\tau') \exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu} + I_\nu^+(\tau_{\max})\exp\left(-\frac{\tau_{\max}}{\mu}\right)$$

for semi-infinite atmospheres: $\tau_{\max} \to \infty$:

$$I_\nu^+(0) = \int_0^{\tau_{\max}} S_\nu(\tau') \exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu}$$

hence, approximately: $I_\nu^+(0) \approx S_\nu(\tau = \mu)$

Eddington-Barbier-Relation

Relation is exactly valid if source function is linear in τ:

i.e. with $S_\nu(\tau) = S_{0\nu} + S_{1\nu} \cdot \tau$ and $x := \tau' / \mu$ we have:

$$I_\nu^+(0) = S_{0\nu}\int_0^\infty e^{-x}dx + S_{1\nu}\int_0^\infty \mu\, xe^{-x}dx = S_{0\nu} + S_{1\nu}\cdot\mu = S_\nu(\mu)$$

The source function

In thermodynamic equilibrium (TE): for any volume element it is:

absorbed energy = emitted energy

per second per second

$$\kappa I_\nu \, ds d\sigma d\omega d\nu = \eta_\nu \, ds d\sigma d\omega d\nu$$

$$\kappa B_\nu = \eta_\nu$$ Kirchhoff´s law

$$S_\nu = \frac{\eta_\nu}{\kappa} = B_\nu$$

The local thermodynamic equilibrium (**LTE**): we assume that

$$S_\nu(\nu,\vec{r}) = B_\nu(\nu, T(\vec{r})) \qquad \text{z.B. } I_\nu^+(0) = \int_0^{\tau_{\max}} B_\nu(T(\tau'))\exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu}$$

Local temperature, unfortunately unknown at the outset

In stellar atmospheres TE is not fulfilled, because

- System is open for radiation
- *T(r)* ≠const (temperature gradient)

Source function with scattering

Example: thermal absorption + continuum scattering

(Thomson scattering of free electrons)

$$\kappa(\nu) = \chi(\nu) + \sigma(\nu) \qquad \eta_\nu = \chi B_\nu + \sigma \oint \frac{d\omega}{4\pi} \int_0^\infty R(\nu', \vec{n}'; \nu, \vec{n}) I_\nu(\nu', \vec{n}') d\nu'$$

true absorption — scattering — redistribution function

isotropic, coherent: $R(\nu', \vec{n}'; \nu, \vec{n}) = \delta(\nu', \nu)$

$$\eta_\nu = \chi B_\nu + \sigma J_\nu$$

$$S_\nu = \frac{\chi B_\nu + \sigma J_\nu}{\chi + \sigma} = \rho J_\nu + (1-\rho) B_\nu \quad \text{with } \rho = \sigma / (\sigma + \chi)$$

Inserting into formal solution:

$$I_\nu^+(0) = \int_0^\infty (1-\rho) B_\nu \exp\left(-\frac{\tau'}{\mu}\right) \frac{d\tau'}{\mu} + \int_0^\infty \rho J_\nu \exp\left(-\frac{\tau'}{\mu}\right) \frac{d\tau'}{\mu}$$

$$\oint \frac{d\omega}{4\pi} I_\nu(\tau, \mu)$$ integral equation for I_ν

The Schwarzschild-Milne equations

Expressions for moments of radiation field obtained by integration of formal solution over angles μ

0-th moment

$$J_\nu(\tau) = \frac{1}{2} \int_{-1}^{1} I_\nu(\tau, \mu) d\mu$$

$$J_\nu(\tau) = \frac{1}{2} \left[\int_0^1 d\mu \int_\tau^\infty S_\nu(\tau') \exp\left(-\frac{\tau'-\tau}{\mu}\right) \frac{d\tau'}{\mu} + \int_{-1}^0 d\mu \int_0^\tau S_\nu(\tau') \exp\left(-\frac{\tau-\tau'}{|\mu|}\right) \frac{d\tau'}{|\mu|} \right]$$

(written for semi-infinite atmosphere without irradiation from outside)

exchange integrals (S, τ independent of μ) $w = \frac{1}{|\mu|}, \frac{dw}{d\mu} = \mp \frac{1}{\mu^2} \rightarrow d\mu = \mp \frac{dw}{w^2}$

$$J_\nu(\tau) = \frac{1}{2} \left[\int_\tau^\infty d\tau' S_\nu(\tau') \int_\infty^1 \exp\left(-w(\tau'-\tau)\right) w \left(-\frac{dw}{w^2}\right) + \int_0^\tau d\tau' S_\nu(\tau') \int_1^\infty \exp\left(-w(\tau-\tau')\right) w \left(\frac{dw}{w^2}\right) \right]$$

$$J_\nu(\tau) = \frac{1}{2} \left[\int_\tau^\infty d\tau' S_\nu(\tau') \int_1^\infty \exp\left(-w(\tau'-\tau)\right) \left(\frac{dw}{w}\right) + \int_0^\tau d\tau' S_\nu(\tau') \int_1^\infty \exp\left(-w(\tau-\tau')\right) \left(\frac{dw}{w}\right) \right]$$

The Schwarzschild-Milne equations

0-th moment

$$J_\nu(\tau)=\frac{1}{2}\left[\int_\tau^\infty d\tau' S_\nu(\tau')\int_1^\infty \exp\left(-w(\tau'-\tau)\right)\left(\frac{dw}{w}\right)+\int_0^\tau d\tau' S_\nu(\tau')\int_1^\infty \exp\left(-w(\tau-\tau')\right)\left(\frac{dw}{w}\right)\right]$$

$$J_\nu(\tau)=\frac{1}{2}\left[\int_\tau^\infty S_\nu(\tau')E_1\left(\tau'-\tau\right)d\tau'+\int_0^\tau S_\nu(\tau')E_1\left(\tau-\tau'\right)d\tau'\right]$$

with $E_1(\mathrm{x}):=\int_1^\infty \mathrm{t}^{-1}e^{-xt}dt$ exponential integral of 1st order

$$\boxed{J_\nu(\tau)=\frac{1}{2}\int_0^\infty S_\nu(\tau')E_1\left(\left|\tau'-\tau\right|\right)d\tau'}$$

Karl Schwarzschild (1914)

Stellar Atmospheres: Radiation Transfer

The Lambda operator

Definition $\Lambda\left[f(t)\right]=\frac{1}{2}\int_0^\infty f(t)E_1\left(\left|t-\tau\right|\right)dt$

$$\Rightarrow \boxed{J_\nu(\tau)=\Lambda\left(S_\nu\right)}$$

In analogy, we obtain the Milne equations for the

1st moment

$$H_\nu(\tau)=\frac{1}{2}\int_\tau^\infty S_\nu(t)E_2\left(t-\tau\right)dt-\frac{1}{2}\int_0^\tau S_\nu(t)E_2\left(\tau-t\right)dt=\frac{1}{4}\Phi\left(S_\nu\right)$$

2nd moment

$$K_\nu(\tau)=\frac{1}{2}\int_0^\infty S_\nu(t)E_3\left(\left|t-\tau\right|\right)dt=\frac{1}{4}\mathrm{X}\left(S_\nu\right)$$

with $E_n(x)=\int_1^\infty \frac{e^{-xt}}{t^n}dt$

LTE

Strict LTE $J_\nu(\tau) = \Lambda B_\nu(T(\tau))$

Including scattering
$$S_\nu = \rho J_\nu + (1-\rho) B_\nu(T(\tau))$$
$$J_\nu(\tau) = \Lambda \rho J_\nu + \Lambda (1-\rho) B_\nu(T(\tau))$$

Integral equation for $J_\nu(\tau)$

Solve $J_\nu(\tau) \rightarrow S_\nu(\tau) \rightarrow I_\nu(\tau)$
$$\rightarrow H_\nu(\tau) = 1/4\, \Phi S_\nu(\tau)$$
$$\rightarrow K_\nu(\tau) = 1/4\, \mathrm{X} S_\nu(\tau)$$

Excursion: exponential integral function

see Chandrasekhar: Radiative Transfer III.18

- For classical LTE atmosphere models, >50% of computation time is needed to calculate $E_n(x)$
- In non-LTE models, $E_n(x)$ is needed to calculate electron collisional rates
- Recursion formula

integration by parts $E_n(x) = \int_1^\infty t^{-n} e^{-xt}\, dt$

with product rule
$$E_n(x) = -\frac{1}{n-1} t^{-(n-1)} e^{-xt} \Big|_1^\infty - \int_1^\infty -\frac{1}{n-1} t^{-(n-1)} (-x) e^{-xt}\, dt$$
$$= 0 + \frac{1^{-(n-1)}}{n-1} e^{-x} - \frac{1}{n-1} x \int_1^\infty t^{-(n-1)} e^{-xt}\, dt$$
$$\boxed{E_n(x) = \frac{1}{n-1}\left[e^{-x} - x E_{n-1}(x)\right] \quad \text{for } n>1}$$
$$E_1(x) \rightarrow E_n(x)$$

Excursion: exponential integral function

- differentiation

$$\frac{d}{dx}E_n(x)=\int_1^\infty t^{-n}\frac{d}{dx}(e^{-xt})dt=\int_1^\infty t^{-n}(-t)e^{-xt}dt=-\int_1^\infty t^{-(n-1)}e^{-xt}dt$$

$$\boxed{\frac{d}{dx}E_n(x)=-E_{n-1}(x)} \qquad n>1$$

$$\frac{d}{dx}E_1(x)=\int_1^\infty t^{-1}\frac{d}{dx}(e^{-xt})dt=\int_1^\infty t^{-1}(-t)e^{-xt}dt=\frac{1}{x}e^{-xt}\Big|_1^\infty=-\frac{e^{-x}}{x}$$

$$\boxed{\frac{d}{dx}E_1(x)=-\frac{e^{-x}}{x}}$$

Stellar Atmospheres: Radiation Transfer

Excursion: exponential integral function

- integrals $\int_0^s x^l E_n(x)dx$ repeated integration by parts

$$\int_0^s x^l E_n(x)dx=\frac{x^{l+1}}{l+1}E_n(x)\Big|_0^s-\int_0^s\frac{x^{l+1}}{l+1}E_n'(x)dx$$

$$=\frac{x^{l+1}}{l+1}E_n(x)\Big|_0^s+\int_0^s\frac{x^{l+1}}{l+1}E_{n-1}(x)dx \quad \text{etc. until} \quad \frac{\mathrm{d}}{\mathrm{dx}}E_1(x)=-\frac{e^{-x}}{x}$$

$$=\frac{s^{l+1}}{l+1}E_n(s)+\frac{s^{l+2}}{(l+1)(l+2)}E_{n-1}(s)+\cdots+\frac{x^{l+n}}{(l+1)(l+2)\cdots(l+n)}E_1(s)$$

$$+\frac{1}{(l+1)(l+2)\cdots(l+n)}\int_0^s x^{l+n-1}e^{-x}dx$$

for $s\to\infty$

$$\int_0^\infty x^l E_n(x)dx=\frac{1}{(l+1)(l+2)\cdots(l+n)}\int_0^\infty x^{l+n-1}e^{-x}dx=\frac{(l+n-1)!}{(l+1)(l+2)\cdots(l+n)}=\frac{(l+n-1)!l!}{(l+n)!}=\frac{l!}{l+n}$$

Excursion: exponential integral function

- asymptotic behaviour

$$x \to \infty : E_1(x) = \int_1^\infty e^{-xt}\frac{1}{t}dt = \frac{e^{-x}}{x} + \int_1^\infty \frac{e^{-xt}}{x}\frac{1}{t^2}dt = \cdots = \frac{e^{-x}}{x}\left[1 - \frac{1}{x} + \frac{2}{x^2} - \frac{6}{x^3} + \cdots\right]$$

$$x \to 0 : E_1(x) = \int_1^\infty e^{-xt}\frac{1}{t}dt = \int_x^\infty e^{-u}\frac{du}{u} = \int_1^\infty e^{-u}\frac{du}{u} + \int_x^1 e^{-u}\frac{du}{u}$$

$$= \int_1^\infty e^{-u}\frac{du}{u} - \int_0^1 (1-e^{-u})\frac{du}{u} + \int_x^1 \frac{du}{u} + \int_0^x (1-e^{-u})\frac{du}{u}$$

$$E_1(x) = \quad -\gamma \qquad\qquad -\ln x + \int_0^x (1-e^{-u})\frac{du}{u}$$

$\gamma = 0.5772156\cdots$ Euler´s constant

series expansion for the integral:

$$E_1(x) = -\gamma - \ln x + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n!n}$$

Values at $x=0$: $E_n(0) = \frac{1}{n-1}\left[e^{-0} - 0 \cdot E_{n-1}(0)\right] = \frac{1}{n-1} \quad n>1 \quad E_2(0)=1, E_3(0)=\frac{1}{2}$

Example: linear source function

$$\mathrm{S}(\tau) = \mathrm{a} + \mathrm{b}\tau$$

$$\mathrm{J}(\tau) = \Lambda \mathrm{S} = \frac{1}{2}\int_0^\infty (\mathrm{a}+\mathrm{b}\tau')\mathrm{E}_1\left(|\tau'-\tau|\right)d\tau'$$

$$= \frac{1}{2}\mathrm{a}\int_0^\infty \mathrm{E}_1\left(|\tau'-\tau|\right)d\tau' + \frac{1}{2}\mathrm{b}\int_0^\infty \tau'\mathrm{E}_1\left(|\tau'-\tau|\right)d\tau'$$

.... $$\mathrm{J}(\tau) = \mathrm{a} + \mathrm{b}\tau + \frac{1}{2}\left[\mathrm{b}\mathrm{E}_3(\tau) - \mathrm{a}\mathrm{E}_2(\tau)\right]$$

$$\mathrm{H}(\tau) = \frac{1}{3}\mathrm{b} + \frac{1}{2}\left[\mathrm{a}\mathrm{E}_3(\tau) - \mathrm{b}\mathrm{E}_4(\tau)\right]$$ ... one can show this

Conclusions: $\tau >> 1 : E_n \approx e^{-x}/x \to 0 \quad J_\nu \to a + b\tau = S_\nu$

The mean intensity approaches the local source function

$$H_\nu \to b/3$$

The flux only depends on the gradient of the source function

Moments of transfer equation

- Plane-parallel geometry

$$\mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu$$

- 0-th moment

$$\frac{1}{2}\int_{-1}^{1} \cdots d\mu$$

$$\frac{d}{d\tau}\left[\frac{1}{2}\int_{-1}^{1} I_\nu(\mu)\mu d\mu\right] = \frac{1}{2}\int_{-1}^{1} I_\nu(\mu) d\mu - \frac{1}{2}\int_{-1}^{1} S_\nu d\mu$$

$$\boxed{\frac{d}{d\tau} H_\nu = J_\nu - S_\nu} \quad \text{(I)}$$

- 1st moment

$$\frac{1}{2}\int_{-1}^{1} \cdots \mu d\mu$$

$$\frac{d}{d\tau}\left[\frac{1}{2}\int_{-1}^{1} I_\nu(\mu)\mu^2 d\mu\right] = \frac{1}{2}\int_{-1}^{1} I_\nu(\mu)\mu d\mu - \frac{1}{2}\int_{-1}^{1} S_\nu \mu d\mu$$

$$\boxed{\frac{d}{d\tau} K_\nu = H_\nu} \quad \text{(II)}$$

Moments of transfer equation

- Spherical geometry

$$\mu \frac{\partial I_\nu(\nu,\mu,r)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_\nu(\nu,\mu,r)}{\partial \mu} = \kappa(\nu,r)\left(S_\nu(\nu,\mu,r) - I_\nu(\nu,\mu,r)\right)$$

- 0-th moment

$$\frac{1}{2}\int_{-1}^{1} \cdots d\mu$$

$$\frac{\partial}{\partial r}\frac{1}{2}\int_{-1}^{1} I_\nu \mu d\mu + \frac{1}{2}\int_{-1}^{1}\frac{1-\mu^2}{r}\frac{\partial I_\nu}{\partial \mu} d\mu = \frac{1}{2}\kappa S_\nu \int_{-1}^{1} d\mu - \frac{1}{2}\kappa \int_{-1}^{1} I_\nu d\mu$$

$$\frac{\partial}{\partial r} H_\nu + \frac{1}{2}\left[\frac{1-\mu^2}{r} I_\nu\right]_{-1}^{1} - \frac{1}{2}\int_{-1}^{1}\frac{-2\mu}{r} I_\nu d\mu = \kappa S_\nu - \kappa J_\nu$$

$$\frac{\partial}{\partial r} H_\nu + 0 + \frac{2}{r} H_\nu = \kappa\left(S_\nu - J_\nu\right)$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 H_\nu\right) = \kappa\left(S_\nu - J_\nu\right)} \quad \text{(I)}$$

Moments of transfer equation

$$\frac{1}{2}\int_{-1}^{1}\cdots\mu d\mu$$

- 1st moment

$$\frac{\partial}{\partial r}\frac{1}{2}\int_{-1}^{1} I_\nu \mu^2 d\mu + \frac{1}{2}\int_{-1}^{1}\frac{\mu-\mu^3}{r}\frac{\partial I_\nu}{\partial \mu}d\mu = \frac{1}{2}\kappa S_\nu \int_{-1}^{1}\mu d\mu - \frac{1}{2}\kappa\int_{-1}^{1} I_\nu \mu d\mu$$

$$\frac{\partial}{\partial r}K_\nu + \frac{1}{2}\left[\frac{\mu-\mu^3}{r}I_\nu\right]_{-1}^{1} - \frac{1}{2}\int_{-1}^{1}\frac{1-3\mu^2}{r}I_\nu d\mu = 0 - \kappa H_\nu$$

$$\frac{\partial}{\partial r}K_\nu + 0 - \frac{1}{r}(J_\nu - 3K_\nu) = -\kappa H_\nu$$

$$\boxed{\frac{\partial}{\partial r}K_\nu + \frac{3K_\nu}{r} - \frac{J_\nu}{r} = -\kappa H_\nu} \quad \text{(II)}$$

Solution of moment equations

Problem: n-th momentum equation contains (n+1)-st moment
$\rightarrow$ always one more unknowns than differential equations
$\rightarrow$ to close the system, another equation has to be found

Closure by introduction of variable Eddington factors

$$K_\nu = f_\nu \cdot J_\nu$$

f_ν Eddington factor, is found by iteration

$$\text{starting estimate for } f_\nu \rightarrow (I)+(II),\ \text{solve} \rightarrow K_\nu$$

$$\rightarrow f_\nu^{\text{new}} = K_\nu / J_\nu$$

Solution of moment equations

$$\left.\begin{aligned}(I)\quad & \frac{dH_\nu}{d\tau} = J_\nu - S_\nu \\ (II)\quad & \frac{d(f_\nu J_\nu)}{d\tau} = H_\nu\end{aligned}\right\}\text{ 2 differential eqs. for } J_\nu, H_\nu$$

Start: approximation for f_ν , assumption: anisotropy small, i.e. substitute I_ν by J_ν (Eddington approximation)

$$K_\nu(\tau) = \frac{1}{2}\int_{-1}^{1} I_\nu \mu^2 d\mu \approx J_\nu \frac{1}{2}\int_{-1}^{1} \mu^2 d\mu = J_\nu \frac{1}{2}\left[\frac{1}{3}\mu^3\right]_{-1}^{1} = J_\nu \frac{1}{3}$$

$$\rightarrow K_\nu = \frac{1}{3} J_\nu$$

$$\rightarrow f_\nu = \frac{1}{3}$$

Eddington approximation

Is exact, if I_ν linear in μ

(one can show by Taylor expansion of S in terms of B that this linear relation is very good at large optical depths)

$$I_\nu(\mu) = I_{0\nu} + \mu I_{1\nu}$$

$$J_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu(\mu) d\mu = I_{0\nu}$$

$$H_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu(\mu)\mu d\mu = \frac{1}{2}\left[I_{1\nu}\frac{\mu^3}{3}\right]_{-1}^{1} = \frac{1}{3} I_{1\nu}$$

$$K_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu(\mu)\mu^2 d\mu = \frac{1}{2}\left[I_{0\nu}\frac{\mu^3}{3}\right]_{-1}^{1} = \frac{1}{3} I_{0\nu}$$

$$\boxed{\Rightarrow K_\nu = \frac{1}{3} J_\nu}$$

Summary: Radiation Transfer

Transfer equation $\frac{dI_\nu}{ds} = \eta_\nu - \kappa(\nu) I_\nu$

Emission and absorption coefficients η_ν , $\kappa(\nu)$

Definitions: source function $S_\nu = \eta_\nu / \kappa(\nu)$
optical depth $d\tau = \kappa \cdot ds$

Formal solution of transfer equation

$$I_\nu^+(\tau) = \int_\tau^{\tau_{\max}} S_\nu(\tau') \exp\left(-\frac{\tau'-\tau}{\mu}\right) \frac{d\tau'}{\mu} + I_\nu^+(\tau_{\max}) \exp\left(-\frac{\tau_{\max}-\tau}{\mu}\right)$$

Eddington-Barbier relation $I_\nu^+(0) \approx S_\nu(\tau = \mu)$

LTE **L**ocal **T**hermodynamic **E**quilibrium

$S_\nu(\nu, \vec{r}) = B_\nu(\nu, T(\vec{r}))$ $T(\vec{r})$ local temperature

Including scattering: $S_\nu = \rho J_\nu + (1-\rho) B_\nu$ with $\rho = \sigma / (\sigma + \chi)$

Schwarzschild-Milne equations

Moment equations of formal solution

$$J_\nu(\tau) = \Lambda(S_\nu) \qquad H_\nu(\tau) = \frac{1}{4}\Phi(S_\nu) \qquad \Lambda,\ \Phi \quad \text{integral operators}$$

Moments of transfer equation (plane-parallel)

$$\frac{dH_\nu}{d\tau} = J_\nu - S_\nu \qquad \frac{d(K_\nu)}{d\tau} = H_\nu$$

Differential equation system (for J,H,K), closed by variable Eddington factor

$$f_\nu := K_\nu / J_\nu$$

Summary: How to calculate I and the moments J,H,K (with given source function S)?

Solve transfer equation $\frac{dI}{d\tau} = \frac{1}{\mu}(I - S)$ (no irradiation from outside, semi-infinite atmosphere, drop frequency index)

Formal solution: $I^+(\tau) = \int_\tau^\infty S(\tau')\exp\left(-\frac{\tau'-\tau}{\mu}\right)\frac{d\tau'}{\mu} \quad (\mu > 0), \quad I^- \text{ analogous}$

How to calculate the higher moments? Two possibilities:

1. Insert formal solution into definitions of J,H,K: $\frac{1}{2}\int_{-1}^{1} I\mu^n d\mu$

$$\rightarrow J(\tau) = \Lambda(S) \quad H(\tau) = \frac{1}{4}\Phi(S) \quad K(\tau) = \frac{1}{4}\mathrm{X}(S)$$ Schwarzschild-Milne equations

2. Angular integration of transfer equation, i.e. 0-th & 1st moment $\frac{1}{2}\int_{-1}^{1} \ldots \mu^n d\mu$

$$\rightarrow \frac{dH}{d\tau} = J - S \qquad \frac{d(K)}{d\tau} = H$$ 2 moment equations for 3 quantities J,H,K

Eliminate K by Eddington factor f: $K = f \cdot J$

$$\rightarrow \boxed{\frac{dH}{d\tau} = J - S \qquad \frac{d(f\cdot J)}{d\tau} = H \quad \text{solve: J,H,K} \leftrightarrows \text{new f (=K/J)}}$$ iteration