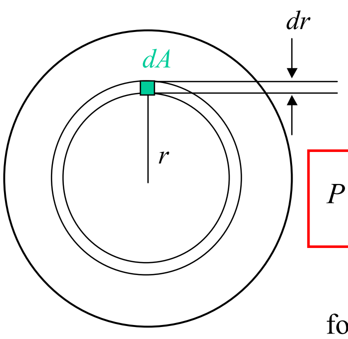
Hydrostatic Equilibrium

Particle conservation

Ideal gas



$$P = \frac{k}{A m_{\mathrm{H}}} \rho \cdot T$$

$P = pressure$
$\rho = \text{mass density}$
$A = \text{atomic weight}$

forces acting on volume element:

$$dV = dAdr \quad dm = \rho dV$$

$$dF_g = -\frac{GM_r dm}{r^2} = -\frac{GM_r \rho}{r^2} dAdr$$

buoyancy:

$$dF_P = -dPdA$$ (pressure difference * area)

Ideal gas

In stellar atmospheres:

$M_r = M_*$ mass of atmosphere negligible

$r = R_*$ thickness of atmosphere << stellar radius

$$\rightarrow dF_g = -\frac{GM_*\rho}{R_*^2} dAdr = g\rho dAdr$$

with $g := \frac{GM_*}{R_*^2}$ surface gravity

usually written as $\log(g/\mathrm{cm\ s^{-2}})$

log *g* is besides T_{eff} the 2nd fundamental parameter of static stellar atmospheres

Type	log *g*
Main sequence star	4.0 .... 4.5
Sun	4.44
Supergiants	0 1
White dwarfs	~8
Neutron stars	~15
Earth	3.0

Hydrostatic equilibrium, ideal gas

buoyancy = gravitational force:

$$dF_P + dF_g = 0$$

$$-dPdA - g\rho dAdr = 0$$

$$\frac{dP}{dr} = -g\rho(r)$$

eliminate $\rho(r)$ with ideal gas equation: $\frac{dP}{dr} = -g\frac{A(r)m_\mathrm{H}}{kT(r)}P(r)$

example:

$T(r) = T = \mathrm{const}$, $A(r) = A = \mathrm{const}$ (i.e., no ionization or dissociation)

$$\frac{dP}{dr} = -g\frac{Am_\mathrm{H}}{kT}P(r) \Rightarrow \frac{1}{P}\frac{dP}{dr} = -g\frac{Am_\mathrm{H}}{kT}$$

solution:

$$\mathrm{P(r)} = \mathrm{P(r_0)}e^{-(r-r_0)gAm_\mathrm{H}/kT}$$

$$\mathrm{P(r)} = \mathrm{P(r_0)}e^{-(r-r_0)/H}$$

$H := \frac{kT}{gAm_\mathrm{H}}$ pressure scale height

Atmospheric pressure scale heights

Earth:

$$\left.\begin{array}{r} A \approx 28\,(\mathrm{N_2}) \\ T \approx 300\ \mathrm{K} \\ \log g = 3 \end{array}\right\} H = 9\ \mathrm{km} \qquad H = \frac{kT}{gAm_{\mathrm{H}}}$$

Sun:

$$\left.\begin{array}{r} A = 1\,(\mathrm{H}) \\ T \approx 6000\ \mathrm{K} \\ \log g = 4.44 \end{array}\right\} H = 180\ \mathrm{km}$$

White dwarf:

$$\left.\begin{array}{r} A = 0.5\,(\mathrm{H}^+ + n_e) \\ T = 15000\ \mathrm{K} \\ \log g = 8 \end{array}\right\} H = 0.25\ \mathrm{km}$$

Neutron star:

$$\left.\begin{array}{r} A = 0.5\,(\mathrm{H}^+ + n_e) \\ T = 10^6\ \mathrm{K} \\ \log g = 15 \end{array}\right\} H = 1.6\ \mathrm{mm}\ !$$

Effect of radiation pressure

2nd moment of intensity $P_R(\nu) = \frac{4\pi}{c} K_\nu$

1st moment of transfer equation (plane-parallel case)

$$\frac{dK_\nu}{d\tau(\nu)} = H_\nu$$

$$\frac{dP_R}{d\tau(\nu)} = \frac{4\pi}{c} H_\nu \quad \text{with} \quad d\tau(\nu) = \kappa(\nu)dr$$

$$\frac{dP_R}{dr} = \frac{4\pi}{c} \kappa(\nu) H_\nu$$

integration over frequencies:

$$\boxed{\frac{dP_R}{dr} = \frac{4\pi}{c} \int_0^\infty \kappa(\nu) H_\nu d\nu}$$

Effect of radiation pressure

Extended hydrostatic equation

$$\frac{dP}{dr} = g\rho(r) - \frac{dP_R}{dr} = g\rho(r) - \frac{4\pi}{c}\int_0^\infty \kappa(\nu)H_\nu d\nu$$

$$= g_{\mathrm{eff}}(r)\rho(r)$$

definition: effective gravity

$$g_{\mathrm{eff}}(r) := g - \frac{4\pi}{c}\frac{1}{\rho(r)}\int_0^\infty \kappa(\nu)H_\nu d\nu = g - g_{\mathrm{rad}} \quad \text{(depth dependent!)}$$

In the outer layers of many stars:

$$g_{\mathrm{eff}} < 0 \quad \text{i.e.} \quad g_{\mathrm{rad}} = \frac{4\pi}{c}\frac{1}{\rho(r)}\int_0^\infty \kappa(\nu)H_\nu d\nu > g$$

Atmosphere is no longer static, hydrodynamical equation

Expanding stellar atmospheres, radiation-driven winds

The Eddington limit

Estimate radiative acceleration

Consider only (Thomson) electron scattering as opacity

$\sigma(\nu) = \sigma_{\mathrm{e}}$ (Thomson cross-section)

$q =$ number of free electrons per atomic mass unit

Pure hydrogen atmosphere, completely ionized

$q = 1$

Pure helium atmosphere, completely ionized

$q = 2/4 = 0.5$

$$g_{\mathrm{rad}}^{\mathrm{e}} = \frac{4\pi}{c}\frac{1}{n_{\mathrm{e}} m_H / q}\int_0^\infty \sigma_{\mathrm{e}} n_{\mathrm{e}} H_\nu d\nu = \frac{4\pi}{c}\frac{q}{m_H}\int_0^\infty \sigma_{\mathrm{e}} H_\nu d\nu = \frac{4\pi}{c}\frac{q\sigma_{\mathrm{e}}}{m_H} H$$

Flux conservation: $H = \frac{\sigma}{4\pi} T_{\mathrm{eff}}^4$

$$\Gamma_{\mathrm{e}} = \frac{g_{\mathrm{rad}}^{\mathrm{e}}}{g} = \frac{4\pi}{c}\frac{q\sigma_{\mathrm{e}}}{\mathrm{m_H}}\frac{\sigma}{4\pi}T_{\mathrm{eff}}^4 \Big/ G\frac{M}{R^2} = \frac{1}{c}\frac{q\sigma_{\mathrm{e}}}{\mathrm{m_H}}\frac{1}{4\pi G}\frac{4\pi\sigma R^2 T_{\mathrm{eff}}^4}{M}$$

$$\boxed{\Gamma_{\mathrm{e}} = \frac{q\sigma_{\mathrm{e}}}{4\pi c \mathrm{m_H} G}\frac{L}{M} = 10^{-4.51} q \frac{L/L_\odot}{M/M_\odot}}$$

The Eddington limit

Consequence: for given stellar mass there exists a maximum luminosity. No stable stars exist above this luminosity limit.

$$L_{max}/L_{\odot} = 10^{-4.51} \cdot 1/q \cdot M/M_{\odot}$$

Sun: $\Gamma_e \ll 1$

Main sequence stars (central H-burning)

Mass luminosity relation: $L/L_{\odot} \approx (M/M_{\odot})^3 \rightarrow M_{max} = 180 M_{\odot}$

Gives a mass limit for main sequence stars

Eddington limit written with effective temperature

and gravity $\Gamma_e = 10^{-15.12} q T_{eff}^4 / g = 1$

$$-15.12 + \log q + 4 \log T_{eff} - \log g = 0$$

Straight line in ($\log T_{eff}$, $\log g$)-diagram

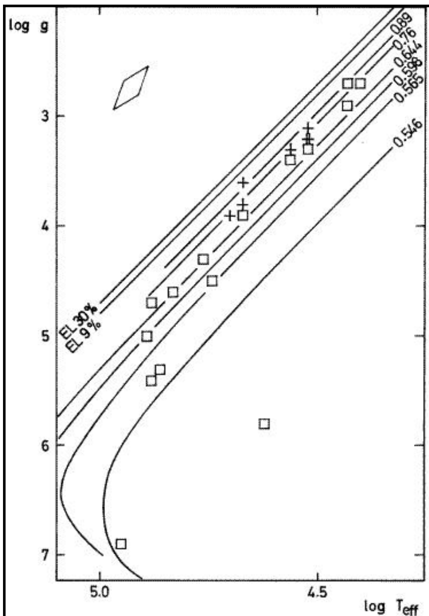


Fig. 3. The log g-log T_{eff} diagram. The two lines labeled EL are Eddington limits for photospheric He abundances of 30% and 9%. We have plotted 6 theoretical post-AGB evolutionary tracks, which are labeled with the corresponding value of the stellar mass, in solar masses. Plus signs and open squares indicate CSPN that show, respectively, He II λ4686 in emission and in absorption. A typical error box can be seen in the upper left corner of the figure

The Eddington limit

Positions of analyzed central stars of planetary nebulae

and

theoretical stellar evolutionary tracks (mass labeled in solar masses)

Computation of electron density

At a given temperature, the hydrostatic equation gives the gas pressure at any depth, or the total particle density N:

$$P_{\mathrm{gas}} = NkT$$

$$N = N_{\mathrm{atoms}} + N_{\mathrm{ions}} + n_{\mathrm{e}} = N_{\mathrm{N}} + n_{\mathrm{e}}$$ N_{N} massive particle density

The Saha equation yields for given (n_e, T) the ion- and atomic densities N_{N}.

The Boltzmann equation then yields for given (N_{N}, T) the population densities of all atomic levels: n_i.

Now, how to get n_e?

We have $\boldsymbol{k}$ different species with abundances α_k

Particle density of species $\boldsymbol{k}$:

$$N_k = \alpha_k N_{\mathrm{N}} = \alpha_k \left(N - n_{\mathrm{e}}\right) \quad \text{, and it is} \quad \sum_{\mathrm{k}=1}^{\mathrm{K}} N_k = N_{\mathrm{N}}$$

Charge conservation

Stellar atmosphere is electrically neutral

Charge conservation electron density=ion density * charge

$$n_{\mathrm{e}} = \sum_{k=1}^{K} \sum_{j=1}^{jk} j \cdot N_{jk} \quad , \; N_{jk} = \text{density of j-th ionization stage of species k}$$

Combine with Saha equation (LTE) by the use of ionization fractions:

$$f_{jk} = \frac{N_{jk}}{N_k} = \frac{\prod_{l=j}^{jk\text{-}1} n_{\mathrm{e}} \Phi_{lk}(T)}{1 + \sum_{m=1}^{jk} \prod_{l=m}^{jk\text{-}1} n_{\mathrm{e}} \Phi_{lk}(T)}$$

We write the charge conservation as

$$n_{\mathrm{e}} = \sum_{k=1}^{K} \sum_{j=1}^{jk} j \cdot N_k f_{jk}(n_{\mathrm{e}}, T) = \sum_{k=1}^{K} \alpha_k (N - n_{\mathrm{e}}) \sum_{j=1}^{jk} j \cdot f_{jk}(n_{\mathrm{e}}, T)$$

$$n_{\mathrm{e}} = (N - n_{\mathrm{e}}) \sum_{k=1}^{K} \alpha_k \sum_{j=1}^{jk} j \cdot f_{jk}(n_{\mathrm{e}}, T) = F(n_{\mathrm{e}})$$

Non-linear equation, iterative solution, i.e., determine zeros of

$$\boxed{F(n_{\mathrm{e}}) - n_{\mathrm{e}} = 0}$$

use Newton-Raphson, converges after 2-4 iterations; yields n_e and f_{ij}, and with Boltzmann all level populations

Summary: Hydrostatic Equilibrium

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Hydrostatic equation including radiation pressure

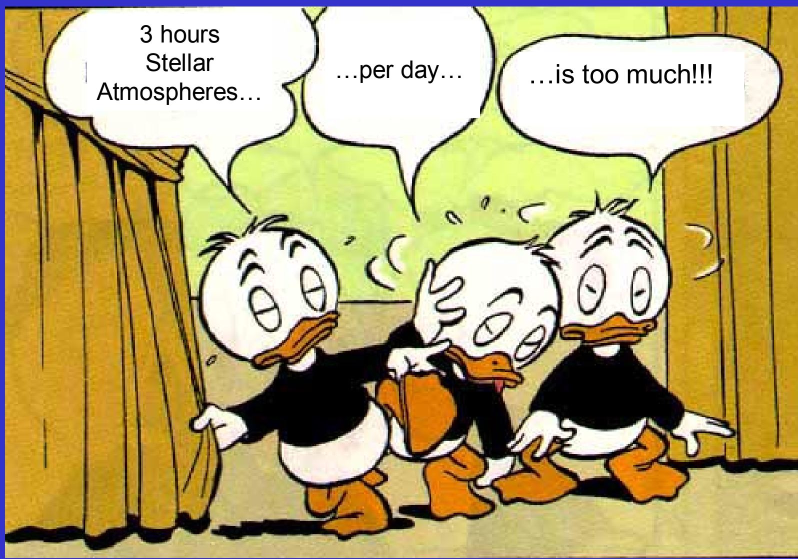
$$\frac{dP}{dr} = g\rho(r) - \frac{dP_R}{dr} = g\rho(r) - \frac{4\pi}{c}\int_0^\infty \kappa(\nu)H_\nu d\nu$$

Photon pressure: Eddington Limit

Hydrostatic equation $\rightarrow N$

Combined charge equation + ionization fraction $\rightarrow n_e$

$\rightarrow$ Population numbers n_{ijk} (LTE) with Saha and Boltzmann equations


3 hours
Stellar
Atmospheres…
…per day…
…is too much!!!