Radiative Equilibrium

Energy conservation

Radiative Equilibrium

Assumption:

Energy conservation, i.e., no nuclear energy sources
Counter-example: radioactive decay of $Ni^{56} \rightarrow Co^{56} \rightarrow Fe^{56}$ in supernova atmospheres

Energy transfer predominantly by radiation

Other possibilities:

Convection e.g., H convection zone in outer solar layer

Heat conduction e.g., solar corona or interior of white dwarfs

Radiative equilibrium means, that we have at each location:

Radiation energy absorbed / sec
=
Radiation energy emitted / sec

integrated over all frequencies and angles

Radiative Equilibrium

Absorption per cm² and second: $\oint_{4\pi} d\omega \int_0^\infty d\nu\, \kappa(\nu) I_\nu$

Emission per cm² and second: $\oint_{4\pi} d\omega \int_0^\infty d\nu\, \eta(\nu)$

Assumption: isotropic opacities and emissivities

Integration over dω then yields

$$\int_0^\infty d\nu\, \kappa(\nu) J_\nu = \int_0^\infty d\nu\, \eta(\nu) \quad \boxed{\Rightarrow \int_0^\infty \kappa(\nu)(J_\nu - S_\nu)d\nu = 0}$$

Constraint equation in addition to the radiative transfer equation; fixes temperature stratification ***T(r)***

Conservation of flux

Alternative formulation of energy equation

In plane-parallel geometry: 0-th moment of transfer equation

$$\frac{dH_\nu}{dt} = \kappa(J_\nu - S_\nu)$$

Integration over frequency, exchange integration and differentiation:

$$\frac{d}{dt}\int_0^\infty H_\nu d\nu = \int_0^\infty \kappa\left(J_\nu - S_\nu\right) d\nu = 0 \qquad \text{because of radiative equilibrium}$$

$$\Rightarrow H = \int_0^\infty H_\nu d\nu = \text{const} = \frac{\sigma}{4\pi} T_{\text{eff}}^4 \qquad \text{for all depths. Alternatively written:}$$

$$\int_0^\infty H_\nu d\nu = \frac{\sigma}{4\pi} T_{\text{eff}}^4 = \int_0^\infty \frac{dK_\nu}{d\tau} d\nu \qquad \text{(1st moment of transfer equation)}$$

$$\boxed{\Rightarrow \int_0^\infty \frac{d(f_\nu J_\nu)}{d\tau} d\nu = \frac{\sigma}{4\pi} T_{\text{eff}}^4} \qquad \text{(definiton of Eddington factor)}$$

Which formulation is good or better?

I Radiative equilibrium: local, integral form of energy equation

II Conservation of flux: non-local (gradient), differential form of radiative equilibrium

I / II numerically better behaviour in **small / large** depths

Very useful is a linear combination of both formulations:

$$A \cdot \left[\int_0^\infty \kappa(J_\nu - S_\nu) d\nu \right] + B \cdot \left[\int_0^\infty \frac{d(f_\nu J_\nu)}{d\tau} d\nu - H \right] = 0$$

A,B are coefficients, providing a smooth transition between formulations I and II.

Stellar Atmospheres: Radiative Equilibrium

Flux conservation in spherically symmetric geometry

0-th moment of transfer equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 H_\nu\right) = \kappa(S_\nu - J_\nu)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \int_0^\infty H_\nu d\nu \right) = r^2 \int_0^\infty \kappa(S_\nu - J_\nu) d\nu = 0$$

$$r^2 \int_0^\infty H_\nu d\nu = \text{const} = \frac{1}{16\pi^2} L \quad \text{because} \quad L = 16\pi^2 R^2 H$$

Another alternative, if T de-couples from radiation field

Thermal balance of electrons

$$Q^H - Q^C = 0$$

$$Q^H_{\mathrm{ff}} = 4\pi\, n_{\mathrm{e}} \sum_j N_j \int_0^\infty \alpha_{\mathrm{ff},j}(\nu,T) J_\nu d\nu$$

$$Q^C_{\mathrm{ff}} = 4\pi\, n_{\mathrm{e}} \sum_j N_j \int_0^\infty \alpha_{\mathrm{ff},j}(\nu,T) \left(J_\nu + \frac{2h\nu^3}{c^2} \right) e^{-h\nu/kT} d\nu$$

$$Q^H_{\mathrm{bf}} = 4\pi \sum_{l,k} n_l \int_0^\infty \alpha_{\mathrm{bf},lk}(\nu) J_\nu \left(1 - \frac{\nu_{lk}}{\nu} \right) d\nu$$

$$Q^C_{\mathrm{bf}} = 4\pi \sum_{l,k} n_k \int_0^\infty \alpha_{\mathrm{bf},lk}(\nu) J_\nu \left(1 - \frac{\nu_{lk}}{\nu} \right) \left(J_\nu + \frac{2h\nu^3}{c^2} \right) e^{-h\nu/kT} d\nu$$

$$Q^H_{\mathrm{c}} = n_{\mathrm{e}} \sum_{l,m} n_m q_{lm}(T) h\nu_{lm}$$

$$Q^C_{\mathrm{c}} = n_{\mathrm{e}} \sum_{l,m} n_l q_{lm}(T) h\nu_{lm}$$

The gray atmosphere

Simple but insightful problem to solve the transfer equation together with the constraint equation for radiative equilibrium

Gray atmosphere: $\kappa_\nu = \overline{\kappa}$

Moments of transfer equation

$$(I)\ \frac{dH_\nu}{d\tau} = J_\nu - S_\nu \quad (II)\ \frac{dK_\nu}{d\tau} = H_\nu \quad \text{with} \quad \tau = \overline{\kappa} dt$$

Integration over frequency

$$(I)\ \frac{dH}{d\tau} = J - S \quad (II)\ \frac{dK}{d\tau} = H$$

Radiative equilibrium $\int \overline{\kappa}(J_\nu - S_\nu) d\nu = \overline{\kappa} \int (J_\nu - S_\nu) d\nu = J - S = 0$

$\Rightarrow (I)\ J = S$

and because of conservation of flux $\frac{dH}{d\tau} = 0$

$\Rightarrow (II)\ \frac{d^2K}{d\tau^2} = 0 \Rightarrow K = c_1\tau + c_2$ from (II) follows $c_1 = \frac{dK}{d\tau} = H,\ c_2$ see below

The gray atmosphere

Relations *(I)* und *(II)* represent two equations for three quantities *S,J,K* with pre-chosen *H* (resp. T_{eff})

Closure equation: Eddington approximation

$$K = 1/3J \rightarrow S = J = 3K = 3H\tau + 3c_2 \quad (III)$$

Source function is linear in τ

Temperature stratification?

In LTE:

$$S(\tau) = B(T(\tau)) = \frac{\sigma}{\pi} T^4$$

insert into (III): $\frac{\sigma}{\pi} T^4 = 3H\tau + 3c_2$

with $H = \frac{\sigma}{4\pi} T_{\text{eff}}^4$ we get:

$\frac{\sigma}{\pi} T^4(\tau) = \frac{3}{4\pi} \sigma T_{\text{eff}}^4 \tau + 3c_2 \quad (IV)$ c_2 is now determined from boundary condition (τ=0)

Gray atmosphere: Outer boundary condition

Emergent flux:

$$H(0) = \frac{1}{2} \int_0^\infty S(\tau') E_2(\tau') d\tau' \quad \text{with } S \text{ from } (III)$$

$$= \frac{1}{2} \int_0^\infty (3H\tau' + 3c_2) E_2(\tau') d\tau'$$

$$= \frac{3}{2} \left[H \int_0^\infty \tau' E_2(\tau') d\tau' + c_2 \int_0^\infty E_2(\tau') d\tau' \right]$$

with $\int_0^\infty t^l E_n(t) dt = \frac{l!}{l+n}$ and $E_2(t) = \frac{1}{2-1}\left[e^{-t} - tE_1(t) \right]$

$$H(0) = \frac{3}{2}\left[\frac{1}{3} H + \frac{1}{2} c_2 \right] \rightarrow c_2 = \frac{2}{3} H$$

from (IV): $\Rightarrow T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3} \right)$, $S = 3H\left(\tau + \frac{2}{3} \right)$ (from III)

Avoiding Eddington approximation

Ansatz: $J(\tau) = 3H(\tau + q(\tau))$ generalization of (III)

$q(\tau)$ = Hopf function

$$J(\tau) = \frac{3}{4}\frac{\sigma}{\pi}T_{\text{eff}}^4(\tau + q(\tau))$$

Insert into Schwarzschild equation:

$J(\tau) = \Lambda S = \Lambda J$ integral equation for J

$$\Rightarrow \tau + q(\tau) = \frac{1}{2}\int_0^\infty (\tau' + q(\tau'))E_1(|\tau' - \tau|)d\tau'$$ (*) integral equation for q, see below

Approximate solution for J by iteration ("Lambda iteration")

$J^{(1)} = 3H(\tau + 2/3)$ i.e., start with Eddington approximation

$$J^{(2)} = \Lambda J^{(1)} = \Lambda(3H(\tau + 2/3)) = 3H\left(\tau + \frac{2}{3} - \frac{1}{3}E_2(\tau) + \frac{1}{2}E_3(\tau)\right)$$

(was result for linear S)

At the surface $\tau = 0,\ E_2(0) = 1,\ E_3(0) = \frac{1}{2}$

$$J^{(2)} = 3H\left(\tau + \frac{2}{3} - \frac{1}{3} + \frac{1}{4}\right) = 3H(\tau + 0.58\bar{3})$$

exact: q(0)=0.577….

At inner boundary $\tau = \infty,\ E_2(\infty) = 0,\ E_3(\infty) = 0$

$$J^{(2)} = 3H\left(\tau + \frac{2}{3}\right)$$

Basic problem of Lambda Iteration: Good in outer layers, but does not work at large optical depths, because exponential integral function approaches zero exponentially.

Exact solution of (*) for Hopf function, e.g., by Laplace transformation (Kourganoff, Basic Methods in Transfer Problems)

Analytical approximation (Unsöld, Sternatmosphären, p. 138)

$$q(\tau) \approx 0.6940 - 0.1167e^{-1.972\tau}$$

Gray atmosphere: Interpretation of results

Temperature gradient

$$\frac{d}{d\tau}T^4 = 4T^3\frac{dT}{d\tau} = \frac{3}{4}T_{\text{eff}}^4$$

$$\frac{dT}{d\tau} \sim T_{\text{eff}}^4$$ The higher the effective temperature, the steeper the temperature gradient.

$$\frac{dT}{dt} = \kappa\frac{dT}{d\tau}$$ The larger the opacity, the steeper the (geometric) temperature gradient.

Flux of gray atmosphere

LTE: $S_\nu = B_\nu(T(\tau))$

$$H_\nu(\tau) = \frac{1}{2}\int_\tau^\infty B_\nu(T(\tau))E_2(t-\tau)dt - \frac{1}{2}\int_0^\tau B_\nu(T(\tau))E_2(\tau-t)dt$$

with $\alpha = h\nu/kT_{\text{eff}}$, $T/T_{\text{eff}} = \left[3/4\,(\tau+q(\tau))\right]^{1/4} = p(\tau) \rightarrow h\nu/kT = \alpha\,p(\tau)$

$H_\alpha d\alpha = H_\nu d\nu$ and $H = \sigma/_{4\pi}T_{\text{eff}}^4$

$$\rightarrow H_\alpha(\tau)/H = \frac{H_\nu}{H}\frac{d\nu}{d\alpha} = \frac{4\pi}{\sigma T_{\text{eff}}^4}\frac{kT_{\text{eff}}}{h}H_\nu = \frac{4\pi k^4}{h^3c^2\sigma}\alpha^3\left(\int_\tau^\infty \frac{E_2(t-\tau)}{\exp(\alpha p(\tau))-1}dt - \int_0^\tau \frac{E_2(\tau-t)}{\exp(\alpha p(\tau))-1}dt\right)$$

$$\frac{1}{2}\frac{2h\nu^3}{c^2}\frac{4\pi}{\sigma}\frac{k}{h}\frac{\alpha^3k^3}{h^3\nu^3}$$

Stellar Atmospheres: Radiative Equilibrium

Gray atmosphere: Interpretation of results

Limb darkening of total radiation

$$I(\tau=0,\mu) = S(\tau=\mu) = B(T(\tau=\mu)) = \frac{\sigma}{\pi}T^4(\tau=\mu) = \frac{\sigma}{\pi}T_{\text{eff}}^4\frac{3}{4}\left(\mu+\frac{2}{3}\right)$$

$$\rightarrow \frac{I(0,\mu)}{I(0,1)} = \frac{\mu+2/3}{1+2/3} = \frac{2}{5}(1+\frac{3}{2}\cos\vartheta)$$

i.e., intensity at limb of stellar disk smaller than at center by 40%, good agreement with solar observations

Empirical determination of temperature stratification ☚

measure $I(\tau=0,\mu) \rightarrow S(\tau=\mu) \rightarrow S(\tau) = B(T(\tau)) \rightarrow T$

Observations at different wavelengths yield different T-structures, hence, the opacity must be a function of wavelength

The Rosseland opacity

Gray approximation (κ=const) very coarse, ist there a good mean value $\bar{\kappa}$? What choice to make for a mean value?

	gray	non-gray
transfer equation	$\mu \frac{dI}{dz} = \kappa(S - I)$	$\mu \frac{dI_\nu}{dz} = \kappa(\nu)(S_\nu - I_\nu)$
0-th moment	$\frac{dH}{dz} = \kappa(S - J) = 0$	$\frac{dH_\nu}{dz} = \kappa(\nu)(S_\nu - J_\nu)$
1st moment	$\frac{dK}{dz} = -\kappa H$	$\frac{dK_\nu}{dz} = -\kappa(\nu) H_\nu$

For each of these 3 equations one can find a mean $\bar{\kappa}$, with which the equations for the gray case are equal to the frequency-integrated non-gray equations.

Because we demand flux conservation, the 1st moment equation is decisive for our choice:

→ Rosseland mean of opacity

Stellar Atmospheres: Radiative Equilibrium

The Rosseland opacity

$$\int_0^\infty H_\nu d\nu = \text{const} = \int_0^\infty \frac{1}{\kappa(\nu)} \frac{dK_\nu}{dz} d\nu = \frac{1}{\kappa_R} \frac{dK}{dz}$$

$$\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa(\nu)} \frac{dK_\nu}{dz} d\nu}{\frac{dK}{dz}} \quad \text{with Eddington approximation} \quad K = 1/3J \quad \text{and LTE} \quad J = B:$$

$$\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa(\nu)} \frac{dB_\nu}{dz} d\nu}{\frac{dB}{dz}} \quad \text{with} \quad \frac{dB_\nu}{dz} = \frac{dB_\nu}{dT}\frac{dT}{dz} \quad \text{and} \quad \frac{dB}{dz} = \frac{d}{dz}\left(\frac{\sigma}{\pi}T^4\right) = \frac{4\sigma}{\pi}T^3\frac{dT}{dz}$$

$$\boxed{\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa(\nu)} \frac{dB_\nu}{dT} d\nu}{\frac{4\sigma}{\pi}T^3}}$$

Definition of Rosseland mean of opacity

The Rosseland opacity

The Rosseland mean $\frac{1}{\kappa_R}$ is a weighted mean

of opacity $\frac{1}{\kappa(\nu)}$ with weight function $\frac{dB_\nu}{dT}$

Particularly, strong weight is given to those frequencies, where the radiation flux is large.
The corresponding optical depth is called Rosseland depth

$$\tau_{Ross}(z) = \int_0^z \kappa_R(z')dz'$$

For $\tau_{Ross} >> 1$ the gray approximation with κ_R is very good,

i.e. $T^4(\tau_{Ross}) = \frac{3}{4} T_{\text{eff}}^4 (\tau_{Ross} + q(\tau_{Ross}))$

Convection

Compute model atmosphere assuming

- Radiative equilibrium (Sect. VI) $\rightarrow$ temperature stratification
- Hydrostatic equilibrium $\rightarrow$ pressure stratification

Is this structure stable against convection, i.e. small perturbations?

- Thought experiment

Displace a blob of gas by Δr upwards, fast enough that no heat exchange with surrounding occurs (i.e., adiabatic), but slow enough that pressure balance with surrounding is retained (i.e. << sound velocity)

Inside of blob | outside

$$T+\Delta T_{\mathrm{ad}}=T_{\mathrm{ad}}(r+\Delta r) \qquad T+\Delta T_{\mathrm{rad}}=T_{\mathrm{rad}}(r+\Delta r)$$

$$\rho+\Delta\rho_{\mathrm{ad}}=\rho_{\mathrm{ad}}(r+\Delta r) \qquad \rho+\Delta\rho_{\mathrm{rad}}=\rho_{\mathrm{rad}}(r+\Delta r)$$

$$\uparrow \Delta r$$

$$T(r),\rho(r) \qquad T(r),\rho(r)$$

$\rho_{\mathrm{ad}}(r+\Delta r)<\rho_{\mathrm{rad}}(r+\Delta r)\rightarrow$ further buoyancy, unstable

$\rho_{\mathrm{ad}}(r+\Delta r)>\rho_{\mathrm{rad}}(r+\Delta r)\rightarrow$ gas blob falls back, stable

i.e. $\left|\frac{d\rho_{\mathrm{ad}}}{dr}\right|\begin{Bmatrix}>\\<\end{Bmatrix}\left|\frac{d\rho_{\mathrm{rad}}}{dr}\right|\begin{cases}\text{unstable}\\\text{stable}\end{cases}$

with ideal gas equation $p=\frac{k}{Am_H}\rho T$ and pressure balance $\rho_{\mathrm{ad}}T_{\mathrm{ad}}=\rho_{\mathrm{rad}}T_{\mathrm{rad}}$

$$\frac{dT_{\mathrm{ad}}}{dr}\begin{Bmatrix}<\\>\end{Bmatrix}\frac{dT_{\mathrm{rad}}}{dr}\begin{cases}\text{unstable}\\\text{stable}\end{cases}$$

Stratification becomes unstable, if temperature gradient dT_{ad}/dr rises above critical value.

Alternative notation

Pressure as independent depth variable:

hydrostatic equation: $dp=-\rho g_{\mathrm{eff}}dr=-\frac{Am_H}{k}g_{\mathrm{eff}}\frac{p}{T}dr$ (ideal gas)

$$\rightarrow dr=-dp\frac{kT}{Am_H g_{\mathrm{eff}}p}$$

$$\frac{dT}{dr}=-\frac{Am_H}{k}g_{\mathrm{eff}}\frac{dT/T}{dp/p}=-\frac{Am_H}{k}g_{\mathrm{eff}}\frac{d(\ln T)}{d(\ln p)}$$

$$\frac{d(\ln T_{\mathrm{ad}})}{d(\ln p)}\begin{Bmatrix}<\\>\end{Bmatrix}\frac{d(\ln T_{\mathrm{rad}})}{d(\ln p)}\begin{cases}\text{unstable}\\\text{stable}\end{cases}$$

Schwarzschild criterion

Abbreviated notation

$$\nabla_{\mathrm{ad}}=\frac{d(\ln T_{\mathrm{ad}})}{d(\ln p)};\nabla_{\mathrm{rad}}=\frac{d(\ln T_{\mathrm{rad}})}{d(\ln p)}$$

$\nabla_{\mathrm{ad}}>\nabla_{\mathrm{rad}}$ stable

The adiabatic gradient

$dQ = 0$ (no heat exchange)

$dQ = dE + pdV$ (1st law of thermodynamics)

$dE = c_V dT$ internal energy $\Rightarrow c_V dT + pdV = 0$ (*)

Internal energy of a one-atomic gas excluding effects of ionisation and excitation

$$E = \frac{3}{2} NkT \rightarrow c_V = \frac{3}{2} Nk$$

But if energy can be absorbed by ionization:

$$c_V >> \frac{3}{2} Nk$$

Specific heat at constant pressure

$$c_p = \left.\frac{\partial Q}{\partial T}\right|_{p=const} = \left.\frac{dE}{dT} + p\frac{dV}{dT}\right|_{p=const} = c_V + p\frac{d(NkT/p)}{dT} = c_V + p\frac{Nk}{p}$$

$$\rightarrow c_p - c_V = Nk$$

The adiabatic gradient

Ideal gas: $pV = NkT \Rightarrow Vdp + pdV = NkdT = \left(c_p - c_V\right) dT$

$$dT = \frac{Vdp + pdV}{c_p - c_V} \quad (**)$$

from(*) with (**) $\rightarrow c_V \dfrac{Vdp + pdV}{c_p - c_V} + pdV = 0 \quad \Bigg| \; /pV \; \cdot \dfrac{c_p - c_V}{c_V}$

$$\frac{dp}{p} + \frac{dV}{V} + \frac{dV}{V}\frac{c_p - c_V}{c_V} = 0$$

$$\frac{dp}{p} + \frac{dV}{V}\frac{c_p}{c_V} = 0$$

$$\frac{c_p}{c_V} d(\ln V) = -d(\ln p)$$

definition: $\gamma := \dfrac{c_p}{c_V} \qquad \dfrac{d(\ln V)}{d(\ln p)} = -\dfrac{1}{\gamma}$

The adiabatic gradient

needed: $\left.\frac{d(\ln T)}{d(\ln p)}\right|_{\mathrm{ad}}$

$$T = pV/Nk$$

$$\ln T = \ln p + \ln V - \ln(Nk)$$

$$\frac{d(\ln T)}{d(\ln p)} = 1 + \frac{d(\ln V)}{d(\ln p)}$$

$$\frac{d(\ln T)}{d(\ln p)} = 1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma}$$

$$\nabla_{\mathrm{ad}} = \frac{\gamma - 1}{\gamma}$$

$\nabla_{\mathrm{rad}} < \frac{\gamma - 1}{\gamma}$ stable Schwarzschild criterion

Stellar Atmospheres: Radiative Equilibrium

The adiabatic gradient

- 1-atomic gas $c_{\mathrm{V}} = 3/2\,Nk \quad c_{\mathrm{p}} = c_{\mathrm{V}} + Nk = 5/2\,Nk$
  $\gamma = 5/3 \quad \nabla_{\mathrm{ad}} = 2/5 = 0.4$
- with ionization $\gamma \to 1 \quad \nabla_{\mathrm{ad}} \to 0$ convection starts $\gamma -$effect
- Most important example: Hydrogen (Unsöld p.228)

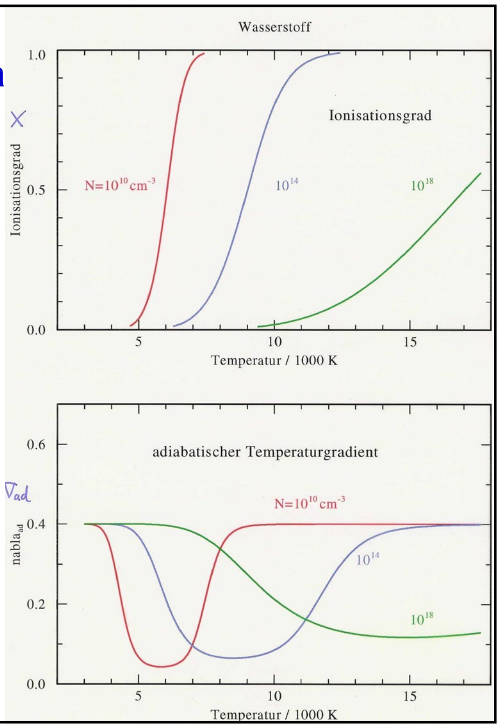
$$\nabla_{\mathrm{ad}} = \frac{2 + (x - x^2)(5/2 + E_{\mathrm{Ion}}/kT)}{5 + (x - x^2)(5/2 + E_{\mathrm{Ion}}/kT)^2}$$

with ionization degree $x = -\frac{f(T)}{2N} + \sqrt{\left(\frac{f(T)}{2N}\right)^2 + \frac{f(T)}{N}}$

The adiaba

$$\nabla_{\mathrm{ad}} = \frac{2+\left(x-x^2\right)\left(5/2+E_{\mathrm{Ion}}/kT\right)}{5+\left(x-x^2\right)\left(5/2+E_{\mathrm{Ion}}/kT\right)^2}$$

$$x = -\frac{f(T)}{2N} + \sqrt{\left(\frac{f(T)}{2N}\right)^2 + \frac{f(T)}{N}}$$



Example: Grey approximation

$$T^4(\tau) = \tfrac{3}{4} T_{\mathrm{eff}}^4 \left(\tau + \tfrac{2}{3}\right)$$

$$4 \ln T = \ln\left(\tfrac{3}{4} T_{\mathrm{eff}}^4\right) + \ln\left(\tau + \tfrac{2}{3}\right)$$

$$\frac{d(\ln T)}{d\tau} = \frac{d\left(\ln\left(\tau+\tfrac{2}{3}\right)\right)}{4 d\tau} = \frac{1}{4\left(\tau+\tfrac{2}{3}\right)}$$

hydrostatic equation: $\frac{dp}{d\tau} = \frac{g}{\kappa}$ Ansatz: $\kappa = Ap^b$ (κ here a mass absorption coefficient)

$$\rightarrow p^b \frac{dp}{d\tau} = \frac{g}{A} \quad \text{integrate} \rightarrow \frac{1}{b+1} p^{b+1} = \frac{g}{A}\tau \quad \rightarrow \frac{g}{Ap^{b+1}} = \frac{1}{(b+1)\tau}$$

$$\frac{d(\ln p)}{d\tau} = \frac{1}{p}\frac{dp}{d\tau} = \frac{1}{p}\frac{g}{Ap^b} = \frac{g}{Ap^{b+1}} = \frac{1}{(b+1)\tau}$$

$$\nabla_{\mathrm{rad}} = \frac{d\ln T/d\tau}{d\ln p/d\tau} = \frac{(b+1)\tau}{4\left(\tau+\tfrac{2}{3}\right)}$$

∇_{rad} becomes large, if opacity strongly increases with depth (i.e. exponent b large).

The absolute value of κ is not essential but the change of κ with depth (gradient)

∇_{rad} large ($>\nabla_{\mathrm{ad}}$): convection starts, κ-Effekt

Hydrogen convection zone in the Sun

κ-effect and γ-effect act together

Going from the surface into the interior: At T~6000K ionization of hydrogen begins

∇_{ad} decreases and κ increases, because a) more and more electrons are available to form H^- and b) the excitation of H is responsible for increased bound-free opacity

In the Sun: outer layers of atmosphere radiative

inner layers of atmosphere convective Video

In F stars: large parts of atmosphere convective

In O,B stars: Hydrogen completely ionized, atmosphere radiative; He I and He II ionization zones, but energy transport by convection inefficient

Transport of energy by convection

Consistent hydrodynamical simulations very costly;

Ad hoc theory: mixing length theory (Vitense 1953)

Model: gas blobs rise and fall along distance *l* (mixing length). After moving by distance *l* they dissolve and the surrounding gas absorbs their energy.

$$l = \alpha H(r) \quad H = \text{pressure scale height}$$

$$\alpha \ \text{mixing length parameter}$$

$$\alpha = 0.5 \cdots 2$$

Gas blobs move without friction, only accelerated by buoyancy; detailed presentation in Mihalas‘ textbook (p. 187-190)

Transport of energy by convection

Again, for details see Mihalas (p. 187-190)

For a given temperature structure

$\rightarrow$ compute $F_{\text{conv}}(r)$

$\rightarrow$ flux conservation including convective flux

$$F_{\text{rad}}(r) = \frac{\sigma}{\pi} T_{\text{eff}}^4 - F_{\text{conv}}(r)$$

$\rightarrow$ new temperature stratification $T(r)$

with $\nabla_{\text{ad}} < \nabla < \nabla_{\text{rad}}$

iterate

Summary: Radiative Equilibrium

Radiative Equilibrium:

$$A \cdot \left[\int_0^\infty \kappa (J_\nu - S_\nu) d\nu \right] + B \cdot \left[\int_0^\infty \frac{d(f_\nu J_\nu)}{d\tau} d\nu - H \right] = 0$$

Schwarzschildt Criterion:

$$\frac{d(\ln T_{\mathrm{ad}})}{d(\ln p)} \begin{Bmatrix} < \\ > \end{Bmatrix} \frac{d(\ln T_{\mathrm{rad}})}{d(\ln p)} \begin{cases} \text{unstable} \\ \text{stable} \end{cases}$$

Temperature of a gray Atmosphere

$$T^4 = \frac{3}{4} T_{\mathrm{eff}}^4 \left(\tau + \frac{2}{3} \right)$$