Solution Strategies

All equations

Radiation Transport	$I_\nu(z), J_\nu(z), H_\nu(z), K_\nu(z)$
Energy Balance	$T(z)$
Hydrostatic Equilibrium	$n_e(z)$
Saha-Boltzmann / Statistical Equilibrium	$n_{ijk}(z)$

Huge system with coupling over depth (RT) and frequency (SE)

Complete Linearisation (Auer Mihalas 1969)
Separate in sub-problems

RT: Short characteristic method

Olson & Kunasz, 1987, JQSRT 38, 325

$$I^+(\tau,\mu,\nu) = I^+(\tau_{\max},\mu,\nu)\exp\left(-\frac{\tau_{\max}-\tau}{\mu}\right) + \int_{\tau}^{\tau_{\max}} S(\tau')\exp\left(-\frac{\tau'-\tau}{\mu}\right)\frac{d\tau'}{\mu}$$

$$I^-(\tau,\mu,\nu) = I^-(0,\mu,\nu) \quad \exp\left(-\frac{\tau}{|\mu|}\right) \quad + \int_{0}^{\tau} S(\tau')\exp\left(-\frac{\tau-\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}$$

Solution on a discrete depth grid $\tau_i,\ i=1,ND$ with boundary conditions:

$$I_1^-(\mu,\nu) = I^-(0,\mu,\nu)$$

$$I_{ND}^+(\mu,\nu) = I^+(\tau_{\max},\mu,\nu)$$

Solution along rays passing through whole plane-parallel slab

Short characteristic method

Rewrite with previous depth point as boundary condition for the next interval:

$$I^+(\tau_i,\mu,\nu) = I^+(\tau_{i+1},\mu,\nu)\exp\left(-\Delta\tau_i\right) + \Delta I_i^+(S,\mu,\nu)$$

$$I^-(\tau_i,\mu,\nu) = I^-(\tau_{i-1},\mu,\nu)\exp\left(-\Delta\tau_{i-1}\right) + \Delta I_i^-(S,\mu,\nu)$$

with

$$\Delta\tau_{i-1} = \frac{(\tau_i - \tau_{i-1})}{|\mu|}$$

using a linear interpolation for the spatial variation of S

the intergrals $\Delta I_i^\pm$ can be evaluated as

$$\Delta I_i^\pm = \alpha_i^\pm S_{i-1} + \beta_i^\pm S_i + \gamma_i^\pm S_{i+1}$$

Short characteristic method

Out-going rays:

$$\Delta I_i^+(S,\mu,\nu)=\int_{\tau_i}^{\tau_{i+1}} S\exp\left(-\frac{\tau'-\tau_i}{\mu}\right)\frac{d\tau'}{\mu}=\exp\left(\frac{\tau_i}{\mu}\right)\int_{\tau_i}^{\tau_{i+1}} S\exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu}$$

$$x=\frac{\tau'}{\mu}\ ,\ g(x)=\exp(-x)\ ,\ a=\tau_i\ ,\ b=\tau_{i+1}\ ,\ \Delta=\frac{\Delta\tau_i}{\mu}$$

$$\Rightarrow \beta_i^+=w_a=e^{a/\mu}\left(e^{-a/\mu}+\frac{1}{\Delta}\left(e^{-b/\mu}-e^{-a/\mu}\right)\right)=1+\frac{e^{-\Delta}-1}{\Delta}$$

$$\Rightarrow \gamma_i^+=w_b=e^{a/\mu}\left(-e^{-b/\mu}-\frac{1}{\Delta}\left(e^{-b/\mu}-e^{-a/\mu}\right)\right)=-e^{-\Delta}-\frac{e^{-\Delta}-1}{\Delta}$$

Short characteristic method

In-going rays:

$$\Delta I_i^-(S,\mu,\nu)=\int_{\tau_{i-1}}^{\tau_i} S\exp\left(-\frac{\tau_i-\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}=\exp\left(-\frac{\tau_i}{|\mu|}\right)\int_{\tau_{i-1}}^{\tau_i} S\exp\left(\frac{\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}$$

$$x=\frac{\tau'}{|\mu|}\ ,\ g(x)=\exp(x)\ ,\ a=\tau_{i-1}\ ,\ b=\tau_i\ ,\ \Delta=\frac{\Delta\tau_{i-1}}{|\mu|}$$

$$\Rightarrow \alpha_i^-=w_a=e^{-b/|\mu|}\left(-e^{a/|\mu|}+\frac{1}{\Delta}\left(e^{b/|\mu|}-e^{a/|\mu|}\right)\right)=-e^{-\Delta}+\frac{1-e^{-\Delta}}{\Delta}$$

$$\Rightarrow \beta_i^-=w_b=e^{-b/|\mu|}\left(e^{b/|\mu|}-\frac{1}{\Delta}\left(e^{b/|\mu|}-e^{a/|\mu|}\right)\right)=1-\frac{1-e^{-\Delta}}{\Delta}$$

Short characteristic method

Also possible: Parabolic instead of linear interpolation

Problem: Scattering $\kappa_e = n_e \sigma_e \;\; , \;\; \eta_e = \kappa_e J = \kappa_e \frac{1}{2} \int_{-1}^{1} I(\mu) d\mu$

Requires iteration

Solution as boundary-value problem Feautrier scheme

Radiation transfer equation along a ray:

$$\pm \frac{dI_\nu^\pm(\tau)}{d\tau} = I_\nu^\pm(\tau) - S_\nu(\tau)$$

$$\text{pp: } d\tau = \kappa \frac{dt}{d\mu}$$

$$\text{sp: } d\tau = -\kappa dZ$$

Two differential equations for inbound and outbound rays

Definitions by Feautrier (1964):

$u = \frac{1}{2}\left(I^+ + I^-\right)$ symmetric, intensity-like

$v = \frac{1}{2}\left(I^+ - I^-\right)$ antisymmetric, flux-like

Feautrier scheme

Addition and subtraction of both DEQs:

$$\frac{dv(\tau)}{d\tau} = u(\tau) - S_\nu(\tau) \qquad (1)$$

$$\frac{du(\tau)}{d\tau} = v(\tau) \qquad (2)$$

$$\Rightarrow \frac{d^2u(\tau)}{d\tau^2} = u(\tau) - S_\nu(\tau)$$

One DEQ of second order instead of two DEQ of first order

Feautrier scheme

Boundary conditions (pp-case)

Outer boundary

$$I^-(\tau=0)=0 \rightarrow u(\tau=0)=v(\tau=0)$$

$$(2) \quad \Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=0} = u(\tau=0)$$

... with irradiation

$$I^-(\tau=0)=I_0^- \rightarrow u-v=I^-$$

$$\Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=0} = u(\tau=0)-I_0^-$$

Inner boundary

$$I^+(\tau=\tau_{\max})=I^+_{\tau_{\max}} \rightarrow u(\tau_{\max})+v(\tau_{\max})=I^+_{\tau_{\max}}$$

$$(2) \quad \Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=\tau_{\max}} = I^+_{\tau_{\max}} - u(\tau_{\max})$$

Schuster boundary-value problem

Finite differences

Approximation of the derivatives by finite differences:

$$u - \frac{d^2u}{d\tau^2} = S \quad \text{discretization on a } \tau-\text{scale}$$

first derivative at intermediate points:

$$\tau_{i+1/2} = \frac{1}{2}(\tau_{i+1} + \tau_i)$$

$$\left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i+1/2}} \approx \frac{u_{i+1} - u_i}{\tau_{i+1} - \tau_i} \quad \text{and} \quad \left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i-1/2}} \approx \frac{u_i - u_{i-1}}{\tau_i - \tau_{i-1}}$$

second derivative:

$$\left.\frac{d}{d\tau}\left(\frac{du(\tau)}{d\tau}\right)\right|_{\tau_i} = \frac{\left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i+1/2}} - \left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i-1/2}}}{\tau_{i+1/2} - \tau_{i-1/2}}$$

$$\left.\frac{du^2(\tau)}{d\tau^2}\right|_{\tau_i} \approx \frac{\frac{u_{i+1} - u_i}{\tau_{i+1} - \tau_i} - \frac{u_i - u_{i-1}}{\tau_i - \tau_{i-1}}}{\frac{1}{2}(\tau_{i+1} - \tau_{i-1})}$$

Finite differences

Approximation of the derivatives by finite differences:

$$u - \frac{d^2u}{d\tau^2} = S \quad \text{discretisation on a } \tau-\text{scale}$$

$$\Rightarrow u_i - \frac{\frac{u_{i+1} - u_i}{\tau_{i+1} - \tau_i} - \frac{u_i - u_{i-1}}{\tau_i - \tau_{i-1}}}{\frac{1}{2}(\tau_{i+1} - \tau_{i-1})} = S_i \quad , \quad i = 2 \cdots ND-1$$

$$\Rightarrow -A_i u_{i-1} + B_i u_i - C_i u_{i+1} = S_i \quad , \quad i = 2 \cdots ND-1$$

$$A_i = \left[\frac{1}{2}(\tau_i - \tau_{i-1})(\tau_{i+1} - \tau_{i-1})\right]^{-1}$$

$$C_i = \left[\frac{1}{2}(\tau_{i+1} - \tau_i)(\tau_{i+1} - \tau_{i-1})\right]^{-1}$$

$$B_i = 1 + A_i + C_i$$

Linear system of equations

$$\begin{bmatrix} B_1 & -C_1 & & & & & \\ -A_2 & B_2 & -C_2 & & 0 & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & 0 & & & \ddots & \ddots & -C_{ND-1} \\ & & & & & -A_{ND} & B_{ND} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{ND-1} \\ u_{ND} \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_{ND-1} \\ W_{ND} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{ND-1} \\ S^*_{ND} \end{bmatrix}$$

Linear system for u_i

Use Gauss-Jordan elimination for solution

Upper diagonal matrix

1st step:

$$\begin{bmatrix} 1 & -\tilde{C}_1 & & & & \\ & 1 & -\tilde{C}_2 & & 0 & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & 0 & & & \ddots & -\tilde{C}_{ND-1} \\ & & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{ND-1} \\ u_{ND} \end{bmatrix} = \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \\ \vdots \\ \tilde{W}_{ND-1} \\ \tilde{W}_{ND} \end{bmatrix}$$

$$i=1 \qquad \tilde{C}_1 = B_1^{-1} C_1 \qquad \tilde{W}_1 = B_1^{-1} W_1$$

$$i=2 \cdots ND \qquad \tilde{C}_i = \left(B_i - A_i \tilde{C}_{i-1}\right)^{-1} C_i \qquad \tilde{W}_i = \left(B_i - A_i \tilde{C}_{i-1}\right)^{-1} \left(W_i + A_i \tilde{W}_{i-1}\right)$$

Back-substitution

2nd step:

$$
\begin{array}{ll}
i = ND & u_{ND} = \tilde{W}_{ND} \\
i = ND-1\cdots 1 & u_i = \tilde{W}_i + \tilde{C}_i u_{i+1}
\end{array}
$$

Solution fulfils differential equation as well as both boundary conditions

Remark: for later generalization the matrix elements are treated as matrices (non-commutative)

Complete Linearization

Auer & Mihalas 1969

Newton-Raphson method in $\mathbb{R}^n$

Solution according to Feautrier scheme

Unknown variables:

$$
\vec{\psi}_i = \begin{bmatrix} \vec{J}_i \\ \vec{n}_i \end{bmatrix} \; , \; i = 1\cdots ND \qquad \psi = [\vec{\psi}_1, \cdots, \vec{\psi}_i, \cdots, \vec{\psi}_{ND}]^T
$$

Equations:

$$
\begin{array}{ll}
-A_{i,k} J_{i-1,k} + B_{i,k} J_{i,k} - C_{i,k} J_{i+1,k} - S_{i,k}(\vec{n}_i) = 0 & \text{NF transfer equations} \\
P(\vec{J}_i)\vec{n}_i - \vec{b}_i = 0 & \text{NL equations for SE}
\end{array}
$$

System of the form:

$$
f_{i,\alpha}(\psi) = 0 \; , \; \alpha = 1\cdots NF + NL
$$

Complete Linearization

Start approximation: $f_{i,\alpha}(\psi^0) \neq 0$

Now looking for a correction so that

$$f_{i,\alpha}(\psi^0 + \delta\psi) = 0 \ \ \forall\ i,\alpha$$

Taylor series:

$$0 = f_{i,\alpha}(\psi) = f_{i,\alpha}(\psi^0 + \delta\psi)$$

$$= f_{i,\alpha}(\psi^0) + \sum_{i=1}^{ND} \left\{ \sum_{k=1}^{NF} \frac{\partial f_{i,\alpha}}{\partial J_{i,k}} \delta J_{i,k} + \sum_{l=1}^{NL} \frac{\partial f_{i,\alpha}}{\partial n_{i,l}} \delta n_{i,l} \right\} \Bigg|_{\psi^0} + \cdots$$

Linear system of equations for ND(NF+NL) unknowns $\delta J_{i,k}$, $\delta n_{i,l}$

Converges towards correct solution

Many coefficients vanish

Complete Linearization - structure

Only neighbouring depth points (2nd order transfer equation with tri-diagonal depth structure and diagonal statistical equations): $f_{i,\alpha}(\psi) = f_{i,\alpha}(\vec\psi_{i-1}, \vec\psi_i, \vec\psi_{i+1})$

Results in tri-diagonal block scheme (like Feautrier)

$$-A_i\delta\vec\psi_{i-1} + B_i\delta\vec\psi_i - C_i\delta\vec\psi_{i+1} = \vec L_i$$

$$-\left(\begin{array}{ccc|c} \ddots & & 0 & \\ & A_{i,k} & & 0 \\ 0 & & \ddots & \\ \hline & 0 & & 0 \end{array}\right)\left[\begin{array}{c} \delta\vec J_{i-1} \\ \hline \delta\vec n_{i-1} \end{array}\right] + \left(\begin{array}{ccc|ccc} \ddots & 0 & & \square & \square & \square \\ & B_{i,k} & & \square & \square & \square \\ 0 & & \ddots & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array}\right)\left[\begin{array}{c} \delta\vec J_i \\ \hline \delta\vec n_i \end{array}\right]$$

$$-\left(\begin{array}{ccc|c} \ddots & & 0 & \\ & C_{i,k} & & 0 \\ 0 & & \ddots & \\ \hline & 0 & & 0 \end{array}\right)\left[\begin{array}{c} \delta\vec J_{i+1} \\ \hline \delta\vec n_{i+1} \end{array}\right] = -\left[\begin{array}{c} \square \\ \square \\ f_{i,\alpha}(\psi^0) \\ \square \\ \square \\ \square \end{array}\right]$$

Complete Linearization - structure

Transfer equations: coupling of $J_{i-1,k}$, $J_{i,k}$, and $J_{i+1,k}$ at the same frequency point:

$\rightarrow$ Upper left quadrants of A_i, B_i, C_i describe 2nd derivative $\frac{d^2 J}{d\tau^2}$

Source function is local:

$\rightarrow$ Upper right quadrants of A_i, C_i vanish

Statistical equations are local

$\rightarrow$ Lower right and lower left quadrants of A_i, C_i vanish

Complete Linearization - structure

Matrix B_i:

Columns: 1 ... NF 1 ... NL; rows: 1 ... NF 1 ... NL

$$
B_i = \left(\begin{array}{ccc|ccc}
\ddots & & 0 & & \vdots & \\
& B_{i,k} & & \cdots & -\dfrac{\partial S_{i,k}}{\partial n_{i,l'}} & \cdots \\
0 & & \ddots & & \vdots & \\
\hline
& \vdots & & & \vdots & \\
\cdots & \displaystyle\sum_{m=1}^{NL} \frac{\partial (P_i)_{l,m}}{\partial J_{i,k'}} n_{i,m} & \cdots & \cdots & (P_i)_{l,l'} & \cdots \\
& \vdots & & & \vdots &
\end{array}\right)
$$

Complete Linearization

Alternative (recommended by Mihalas): solve SE first and linearize afterwards: $P(\vec{J}_i)\vec{n}_i - \vec{b}_i = 0 \;\rightarrow \vec{n}_i = P(\vec{J}_i)^{-1}\vec{b}_i$

Newton-Raphson method:

- Converges towards correct solution
- Limited convergence radius
- In principle quadratic convergence, however, not achieved because variable Eddington factors and τ-scale are fixed during iteration step
- CPU~ND (NF+NL)^3 → simple model atoms only
  - Rybicki scheme is no relief since statistical equilibrium not as simple as scattering integral

Energy Balance

→ Including radiative equilibrium into solution of radiative transfer → Complete Linearization for model atmospheres

→ Separate solution via temperature correction

- \+ Quite simple implementation
- \+ Application within an iteration scheme allows completely linear system → next chapter
- – No direct coupling
- – Moderate convergence properties

Temperature correction – basic scheme

0. start approximation for $T(\tau) \leftarrow T_0(\tau)$
1. formal solution $J_\nu = \Lambda_\nu S_\nu(T)$
2. correction $T(\tau) \leftarrow T(\tau) + \Delta T(\tau)$
3. convergence?

Several possibilities for step 2 based on radiative equilibrium or flux conservation

Generalization to non-LTE not straightforward

With additional equations towards full model atmospheres:

- Hydrostatic equilibrium
- Statistical equilibrium

LTE

Strict LTE $S_\nu(\tau) = B_\nu(T(\tau))$

Scattering $S_\nu(\tau) = (1-\beta_e)B_\nu(T(\tau)) + \beta_e J_\nu(\tau)$

Simple correction from radiative equilibrium:

$$\int_{\nu=0}^{\infty} \kappa(\tau,\nu)\left(J_\nu(\tau,\nu) - B_\nu(T(\tau),\nu)\right) d\nu \neq 0$$

$$\xrightarrow[\Delta T]{} \int_{\nu=0}^{\infty} \kappa(\tau,\nu)\left(J_\nu(\tau,\nu) - B_\nu\left[T(\tau) + \Delta T(\tau)\right]\right) d\nu = 0$$

$$\Rightarrow \int_{\nu=0}^{\infty} \kappa\left(J_\nu - B_\nu - \Delta T \left.\frac{\partial B_\nu}{\partial T}\right|_{T=T(\tau)}\right) d\nu = 0$$

$$\Rightarrow \Delta T = \int_{\nu=0}^{\infty} \kappa\left(J_\nu - B_\nu\right) d\nu \Bigg/ \int_{\nu=0}^{\infty} \kappa \left.\frac{\partial B_\nu}{\partial T}\right|_{T=T(\tau)} d\nu$$

LTE

Problem:

$$\Delta T = \int_{\nu=0}^{\infty} \kappa\left(J_\nu - B_\nu\right) d\nu \Big/ \int_{\nu=0}^{\infty} \kappa \left.\frac{\partial B_\nu}{\partial T}\right|_{T=T(\tau)} d\nu$$

$$J_\nu \xrightarrow[\tau\to\infty]{} B_\nu \quad \text{independent of the temperature } \Rightarrow \Delta T \to 0$$

Gray opacity (κ independent of frequency):

$$\int_{\nu=0}^{\infty} \kappa(\nu)\left(J_\nu - B_\nu\right) d\nu \to \kappa(J-B)$$

$$\to \kappa(J - B - \Delta B) = 0$$

$$\to \kappa(J - B) = \kappa\Delta B$$

$$\xrightarrow[\text{0.Moment equation}]{} \frac{dH}{dt} = \kappa\Delta B$$

deviation from constant flux provides temperature correction

Stellar Atmospheres: Solution Strategies

Unsöld-Lucy correction

Unsöld (1955) for gray LTE atmospheres, generalized by Lucy (1964) for non-gray LTE atmospheres

0-th moment: $\frac{dH_\nu}{dt} = \kappa_\nu\left(J_\nu - B_\nu\right)$

$$\int \cdots d\nu \to \frac{dH}{d\tau} = \frac{\kappa_J}{\kappa_B} J - B \ , \ \kappa_B B = \int_{\nu=0}^{\infty} \kappa_\nu B_\nu d\nu \ , \ \kappa_J J = \int_{\nu=0}^{\infty} \kappa_\nu J_\nu d\nu \ , \ d\tau = \kappa_B dt$$

1st moment: $\frac{dK_\nu}{dt} = \kappa_\nu H_\nu$

$$\int \cdots d\nu \to \frac{dK}{d\tau} = \frac{\kappa_H}{\kappa_B} H \ , \ \kappa_H H = \int_{\nu=0}^{\infty} \kappa_\nu H_\nu d\nu$$

now new quantities J', H', K' fulfilling radiative equilibrium (local) and flux conservation (non local)

radiative equilibrium: $\frac{dH'}{d\tau} = \frac{\kappa_J}{\kappa_B} J' - B' = 0$

flux conservation: $\frac{dK'}{d\tau} = \frac{\kappa_H}{\kappa_B} H' = \frac{\kappa_H}{\kappa_B} \frac{\sigma}{4\pi} T_{\text{eff}}^4$

Unsöld-Lucy correction

Now corrections to obtain new quantities:

$$\Delta X = X' - X$$

$$\frac{d\Delta K}{d\tau} = \frac{\kappa_H}{\kappa_B}\Delta H \quad \text{integrate} \quad \rightarrow \Delta K = \Delta K(0) + \int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau'$$

$$K = \int_0^\infty K_\nu d\nu = \int_0^\infty f_\nu J_\nu d\nu = fJ \ , \ H(0) = \int_0^\infty H_\nu(0) d\nu = \int_0^\infty h_\nu J_\nu(0) d\nu = hJ(0)$$

$$\rightarrow \Delta K = \frac{f(0)\Delta H(0)}{h} + \int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau' = f\Delta J$$

$$\frac{d\Delta H}{d\tau} = \frac{\kappa_J}{\kappa_B}\Delta J - \Delta B \rightarrow \Delta B = -\frac{d\Delta H}{d\tau} + \frac{\kappa_J}{\kappa_B}\left(\frac{f(0)\Delta H(0)}{fh} + \frac{1}{f}\int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau'\right)$$

$$\Delta B = \frac{4\sigma T^3}{\pi}\Delta T = -\frac{dH'}{d\tau} + \frac{dH}{d\tau} + \frac{\kappa_J}{\kappa_B}\left(\frac{f(0)\Delta H(0)}{fh} + \frac{1}{f}\int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau'\right)$$

$$\Delta T = \frac{\pi}{4\sigma T^3}\left[\frac{\kappa_J}{\kappa_B}J - B + \frac{\kappa_J}{\kappa_B}\left(\frac{f(0)\Delta H(0)}{fh} + \frac{1}{f}\int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau'\right)\right]$$

Unsöld-Lucy correction

$$\Delta T = \frac{\pi}{4\sigma T^3}\left[\frac{\kappa_J}{\kappa_B}J - B + \frac{\kappa_J}{\kappa_B}\left(\frac{f(0)\Delta H(0)}{fh} + \frac{1}{f}\int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau'\right)\right]$$

„Radiative equilibrium“ part good at small optical depths but poor at large optical depths $J \rightarrow B$

„Flux conservation“ part good at large optical depths but poor at small optical depths $\frac{dH}{d\tau} \rightarrow 0$

Unsöld-Lucy scheme typically requires damping

Still problems with strong resonance lines, i.e. radiative equilibrium term is dominated by few optically thick frequencies

NLTE Model Atmospheres

Radiation Transport and Sattistical Equilibrium are very closely coupled

Simple separation (Lamda Iteration) does not work

Complete Linearization

Accelerated Lambda Iteration

Lambda Iteration

Split RT and SE+RE:

$$J^{new} = \Lambda S^{old}(n,T)$$ RT formal solution

$$\underline{\underline{A}}(J,T)\underline{n}^{new} = \underline{b}$$ SE

$$\int_0^\infty \kappa(\nu,n,T)\left(J_\nu - S_\nu(\nu,n,T)\right)d\nu = 0$$ RE

- Good: SE is linear (if a separate T-correction scheme is used)
- Bad: SE contain old values of n,T (in rate matrix A)

Disadvantage: not converging, this is a Lambda iteration!

Accelerated Lambda Iteration (ALI)

Again: split RT and SE+RE but now use ALI

$$J^{new} = \Lambda S^{old}(n^{old},T^{old}) + \Lambda^* S^{new}(n^{new},T^{new}) - \Lambda^* S^{old}(n^{old},T^{old}) \quad \text{RT}$$

$$\underline{\underline{A}}(J^{new},T^{new})\underline{n}^{new} = \underline{b} \quad \text{SE}$$

$$\int_0^\infty \kappa(\nu,n^{new},T^{new})\left(J_\nu^{new} - S_\nu(\nu,n^{new},T^{new})\right)d\nu = 0 \quad \text{RE}$$

- Good: SE contains new quantities n, T
- Bad: Non-Linear equations → linearization (but without RT)

Basic advantage over Lambda Iteration: ALI converges!

Example: ALI working on Thomson scattering problem

$S = (1-\beta_e)B + \beta_e J$ source function with scattering, problem: J unknown→iterate

$$\Rightarrow J^{new} = (\Lambda - \Lambda^*)S^{old} + \Lambda^* S^{new}$$

$$= (\Lambda - \Lambda^*)S^{old} + \Lambda^*\left((1-\beta_e)B^{new} + \beta_e J^{new}\right) \qquad J^{FS} := \text{formal solution on } S^{old}$$

$$= J^{FS} - \Lambda^*\left((1-\beta_e)B^{old} + \beta_e J^{old} - (1-\beta_e)B^{new} - \beta_e J^{new}\right) \qquad B^{old} = B^{new}$$

$$= J^{FS} - \Lambda^*\left(\beta_e J^{old} - \beta_e J^{new}\right) \qquad \text{solve for } J^{new}$$

$$\Rightarrow J^{new} = \left[1 - \Lambda^*\beta_e\right]^{-1}\left(J^{FS} - \Lambda^*\beta_e J^{old}\right) \qquad \text{subtract } J^{old} \text{ on both sides}$$

$$\Rightarrow J^{new} - J^{old} = \left[1 - \Lambda^*\beta_e\right]^{-1}\left(J^{FS} - J^{old}\right)$$

amplification factor

Interpretation: iteration is driven by difference (J^{FS}-J^{old}) but: this difference is amplified, hence, iteration is accelerated.
Example: β_e=0.99; at large optical depth Λ^* almost 1 → strong amplification

What is a good Λ*?

The choice of Λ* is in principle irrelevant but in practice it decides about the success/failure of the iteration scheme.

First (useful) Λ* (Werner & Husfeld 1985):

$$\Lambda_\nu^*(\tau,\tau')S_\nu(\tau') = \begin{cases} S_\nu(\tau) & \tau > \gamma \\ 0 & \tau \le \gamma \end{cases}$$

A few other, more elaborate suggestions until Olson & Kunasz (1987): Best Λ* is the diagonal of the Λ-matrix

(Λ-matrix is the numerical representation of the integral operator Λ)

We therefore need an efficient method to calculate the elements of the Λ-matrix (are essentially functions of τ_ν).

Could compute directly elements representing the Λ-integral operator, but too expensive (E_1 functions). Instead: use solution method for transfer equation in differential (not integral) form: short characteristics method

Towards a linear scheme

Λ* acts on S, which makes the equations non-linear in the occupation numbers

- Idea of Rybicki & Hummer (1992): use $J=\Delta J+\Psi^*\eta^{new}$ instead
- Modify the rate equations slightly:

$$R_{ij}n_i = 4\pi\int_0^\infty \frac{\sigma_{ij}}{h\nu} n_i J_\nu d\nu = 4\pi\int_0^\infty \frac{\sigma_{ij}}{h\nu} n_i \left(\Psi^*\eta(n) + \Delta J\right) d\nu$$

$$R_{ji}n_j = 4\pi\left(\frac{n_i}{n_j}\right)^* \int_0^\infty \frac{\sigma_{ij}}{h\nu} n_j \left(J_\nu + \frac{2h\nu^3}{c^2}\right) d\nu$$

$$= 4\pi\left(\frac{n_i}{n_j}\right)^* \int_0^\infty \frac{\sigma_{ij}}{h\nu} n_j \left(\Psi^*\eta(n) + \Delta J + \frac{2h\nu^3}{c^2}\right) d\nu$$

Stellar Atmospheres

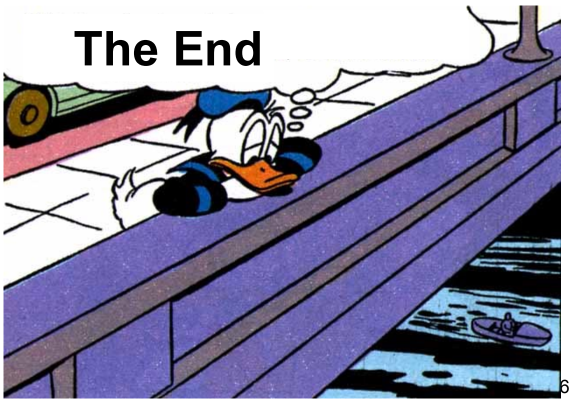
This was the contents of our lecture:

Radiation field
Radiation transfer
Emission and absorption
Energy balance and Radiative equilibrium
Hydrostatic equilibrium
Solution Strategies for Stellar atmosphere models

Stellar Atmospheres

This was the contents of our lecture:

Radiation field
Radiation transfer
Emission and absorption
Radiative equilibrium
Hydrostatic equilibrium
Stellar atmosphere models



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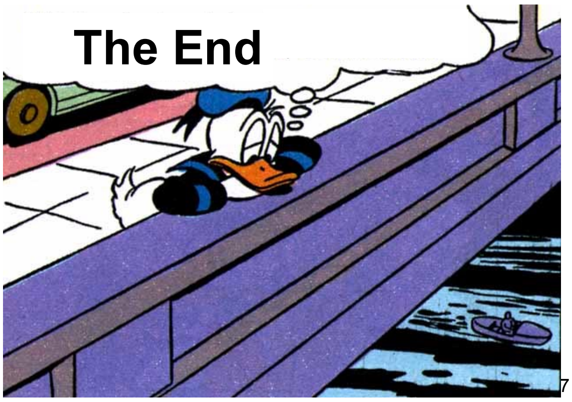

Thank you for listening !