Radiation Transport as Boundary-Value Problem of Differential Equations

Solution with given source function

- Formal Solution, applications:
  - Strict LTE, $S_\nu = B_\nu(T)$
  - Step within iterative method
- Numerical integration, short characteristics method
- Algebraic equation, Feautrier method

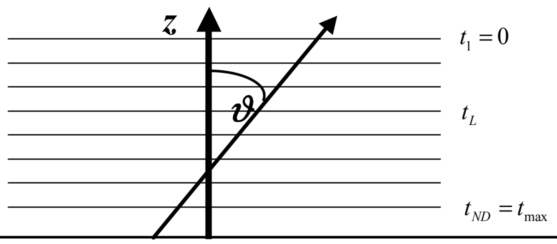
Solution by numerical integration

Emergent intensity, plane-parallel geometry ☚

Depth grid, ND depth points

- Geometrical depth $t_L, L = 1 \cdots ND$
- Optical depth $\tau_L, L = 1 \cdots ND, \tau_L(\nu) = \int_{t'=0}^{t_L} \kappa(\nu, t')dt'$

$$I_\nu(\tau = 0, \mu), \quad \mu = \cos\vartheta$$



Emergent intensity

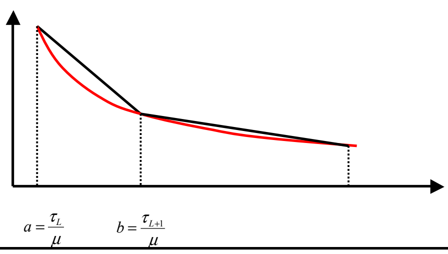
$$I_\nu^+(\tau = 0, \mu) = \int_{\tau'=0}^{\tau_{max}} S_\nu(\tau') \exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu} + I_\nu^+(\tau_{max})\exp\left(-\frac{\tau_{max}}{\mu}\right)$$ ☚

Numerical integration: $I_\nu^+(\tau = 0, \mu) = \sum_{L=1}^{ND} S_L w_L$

Trapezoidal rule,
naive approach:

$$I = \left(S_a e^{-a} + S_b e^{-b}\right)\frac{b-a}{2}$$

Not very smart, systematic summation of approximation errors



Proper integration weights

Problem: $I = \int_a^b f(x) g(x) dx$

Interval: (a, b)

Integrand: $f(x)$

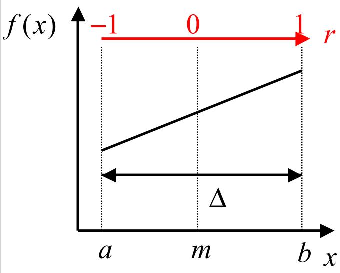
Weight function: $g(x)$

Including weight function in integration weights is smart if:

$f(x)$ is less curved than $f(x)\cdot g(x)$

$g(x)$ and $x\cdot g(x)$ have handy antiderivatives

Determination of proper weights



$$\Delta = b - a$$

$$m = \frac{(b+a)}{2}$$

$$x = m + r\frac{\Delta}{2}$$

$$dx = \frac{\Delta}{2} dr$$

$$f(x) = \frac{f(b) + f(a)}{2} + r\frac{f(b) - f(a)}{2} \quad \text{, even and odd}$$

$$I = \int_a^b f(x) g(x) dx = \frac{\Delta}{4} \int_{r=-1}^{1} \left(f(b) + f(a) + r\left(f(b) - f(a) \right) \right) g\left(m + r\Delta/2 \right) dr$$

like wise:

$$g(x) = g^e(r) + g^o(r)$$

$$g^e(r) = \frac{1}{2}\left(g\left(m + r\Delta/2 \right) + g\left(m - r\Delta/2 \right) \right)$$

$$g^o(r) = \frac{1}{2}\left(g\left(m + r\Delta/2 \right) - g\left(m - r\Delta/2 \right) \right)$$

Determination of proper weights

Integration over symmetric interval leaves only even integrands:

$$I=\frac{\Delta}{4}\left[\left(f(b)+f(a)\right)\int_{r=-1}^{1} g^{e}(r)dr \;+\; \left(f(b)-f(a)\right)\int_{r=-1}^{1} rg^{o}(r)dr\right]$$

$$=\frac{\Delta}{2}\left[\left(f(b)+f(a)\right)\int_{r=0}^{1} g^{e}(r)dr \;+\; \left(f(b)-f(a)\right)\int_{r=0}^{1} rg^{o}(r)dr\right]$$

$$=\frac{\Delta}{2}\left[\left(f(b)+f(a)\right)\mathrm{G} \;+\; \left(f(b)-f(a)\right)H\right]=w_a f(a)+w_b f(b)$$

$$\Rightarrow w_a=\frac{\Delta}{2}(G-H)$$

$$\Rightarrow w_b=\frac{\Delta}{2}(G+H)$$

Examples

$g(x)=x$

$$G=m, H=\frac{\Delta}{6}$$

$$w_a=\frac{\Delta}{2}\left(m-\frac{\Delta}{6}\right)$$

$$w_b=\frac{\Delta}{2}\left(m+\frac{\Delta}{6}\right)$$

$g(x)=x^2$

$$G=m^2+\frac{\Delta^2}{12}, H=m\frac{\Delta}{3}$$

$$w_a=\frac{\Delta}{2}\left(m^2+\frac{\Delta^2}{12}-m\frac{\Delta}{3}\right)$$

$$w_b=\frac{\Delta}{2}\left(m^2+\frac{\Delta^2}{12}+m\frac{\Delta}{3}\right)$$

$g(x)=e^{-x}$

$$G=\frac{1}{\Delta}\left(e^{-a}-e^{-b}\right)$$

$$H=\frac{1}{\Delta}\left(-e^{-a}-e^{-b}+\frac{2}{\Delta}\left(e^{-a}-e^{-b}\right)\right)$$

$$w_a=e^{-a}-\frac{1}{\Delta}\left(e^{-a}-e^{-b}\right)$$

$$w_b=-e^{-b}+\frac{1}{\Delta}\left(e^{-a}-e^{-b}\right)$$

$g(x)=e^{x}$

$$G=\frac{1}{\Delta}\left(-e^{a}+e^{b}\right)$$

$$H=\frac{1}{\Delta}\left(e^{a}+e^{b}+\frac{2}{\Delta}\left(e^{a}-e^{b}\right)\right)$$

$$w_a=-e^{a}-\frac{1}{\Delta}\left(e^{a}-e^{b}\right)$$

$$w_b=e^{-b}+\frac{1}{\Delta}\left(e^{a}-e^{b}\right)$$

Short characteristic method

Olson & Kunasz, 1987, JQSRT 38, 325

$$I^+(\tau,\mu,\nu) = I^+(\tau_{\max},\mu,\nu)\exp\left(-\frac{\tau_{\max}-\tau}{\mu}\right) + \int_{\tau}^{\tau_{\max}} S\,(\tau')\exp\left(-\frac{\tau'-\tau}{\mu}\right)\frac{d\tau'}{\mu}$$

$$I^-(\tau,\mu,\nu) = I^-(0,\mu,\nu)\quad\exp\left(-\frac{\tau}{|\mu|}\right)\quad + \int_0^{\tau} S\,(\tau')\exp\left(-\frac{\tau-\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}$$

Solution on a discrete depth grid $\tau_i,\ i=1,ND$ with boundary conditions:

$$I_1^-(\mu,\nu) = I^-(0,\mu,\nu)$$

$$I_{ND}^+(\mu,\nu) = I^+(\tau_{\max},\mu,\nu)$$

Solution along rays passing through whole plane-parallel slab

Short characteristic method

Rewrite with previous depth point as boundary condition for the next interval:

$$I^+(\tau_i,\mu,\nu) = I^+(\tau_{i+1},\mu,\nu)\exp\left(-\Delta\tau_i\right) + \Delta I_i^+(S,\mu,\nu)$$

$$I^-(\tau_i,\mu,\nu) = I^-(\tau_{i-1},\mu,\nu)\exp\left(-\Delta\tau_{i-1}\right) + \Delta I_i^-(S,\mu,\nu)$$

with

$$\Delta\tau_{i-1} = \frac{(\tau_i - \tau_{i-1})}{|\mu|}$$

using a linear interpolation for the spatial variation of S

the intergrals $\Delta I_i^{\pm}$ can be evaluated as

$$\Delta I_i^{\pm} = \alpha_i^{\pm} S_{i-1} + \beta_i^{\pm} S_i + \gamma_i^{\pm} S_{i+1}$$

Short characteristic method

Out-going rays:

$$\Delta I_i^+(S,\mu,\nu) = \int_{\tau_i}^{\tau_{i+1}} S \exp\left(-\frac{\tau' - \tau_i}{\mu}\right)\frac{d\tau'}{\mu} = \exp\left(\frac{\tau_i}{\mu}\right)\int_{\tau_i}^{\tau_{i+1}} S \exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu}$$

$$x = \frac{\tau'}{\mu} \ , \ g(x) = \exp(-x) \ , \ a = \tau_i \ , \ b = \tau_{i+1} \ , \ \Delta = \frac{\Delta\tau_i}{\mu}$$

$$\Rightarrow \beta_i^+ = w_a = e^{a/\mu}\left(e^{-a/\mu} + \frac{1}{\Delta}\left(e^{-b/\mu} - e^{-a/\mu}\right)\right) = 1 + \frac{e^{-\Delta} - 1}{\Delta}$$

$$\Rightarrow \gamma_i^+ = w_b = e^{a/\mu}\left(-e^{-b/\mu} - \frac{1}{\Delta}\left(e^{-b/\mu} - e^{-a/\mu}\right)\right) = -e^{-\Delta} - \frac{e^{-\Delta} - 1}{\Delta}$$

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Short characteristic method

In-going rays:

$$\Delta I_i^-(S,\mu,\nu) = \int_{\tau_i - 1}^{\tau_i} S \exp\left(-\frac{\tau_i - \tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|} = \exp\left(-\frac{\tau_i}{|\mu|}\right)\int_{\tau_{i-1}}^{\tau_i} S \exp\left(\frac{\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}$$

$$x = \frac{\tau'}{|\mu|} \ , \ g(x) = \exp(x) \ , \ a = \tau_{i-1} \ , \ b = \tau_i \ , \ \Delta = \frac{\Delta\tau_{i-1}}{|\mu|}$$

$$\Rightarrow \alpha_i^- = w_a = e^{-b/|\mu|}\left(-e^{a/|\mu|} + \frac{1}{\Delta}\left(e^{b/|\mu|} - e^{a/|\mu|}\right)\right) = -e^{-\Delta} + \frac{1 - e^{-\Delta}}{\Delta}$$

$$\Rightarrow \beta_i^- = w_b = e^{-b/|\mu|}\left(e^{b/|\mu|} - \frac{1}{\Delta}\left(e^{b/|\mu|} - e^{a/|\mu|}\right)\right) = 1 - \frac{1 - e^{-\Delta}}{\Delta}$$

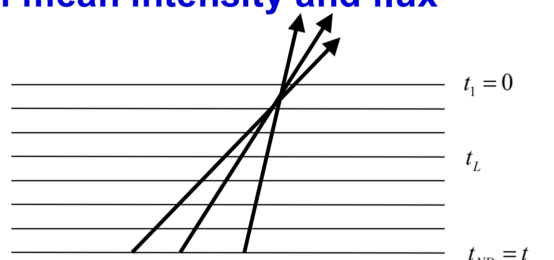
Short characteristic method

Also possible: Parabolic instead of linear interpolation

Problem: Scattering $\kappa_e = n_e \sigma_e \ , \ \eta_e = \kappa_e J = \kappa_e \frac{1}{2} \int_{-1}^{1} I(\mu) d\mu$

Requires iteration

Determination of mean intensity and flux



Discrete angular points

$\mu_j \ , \ j = 1 \cdots NA$

Solution along each ray

$I_j^{\pm} = \sum_{L=1}^{ND} S_L w_{Lj}$ weights depend on μ_j

Angular integration

$J = \frac{1}{2} \sum_{j=1}^{NA} I_j w_j \ , \ \mathrm{H} = \frac{1}{2} \sum_{j=1}^{NA} I_j \mu_j w_j$

Gauss integration with 3 points sufficient for pp-RT

Alternative: numerical integration of moment equation

Problem: numerical approximation of $E_2(\tau)$

Spherical geometry

Impact parameters

$P_j\ ,\ j=1\cdots NP\ ,\ NP=ND+NC$

$P_1=0,\cdots,P_{NC}$ intersecting the core

$P_{NC+1}=1,\cdots,P_{NP}=R_{\max}$

Z_i points

$Z_i=+\sqrt{r_l^2-P_j^2}\ ,\ i=1\cdots ND$ intersecting the core

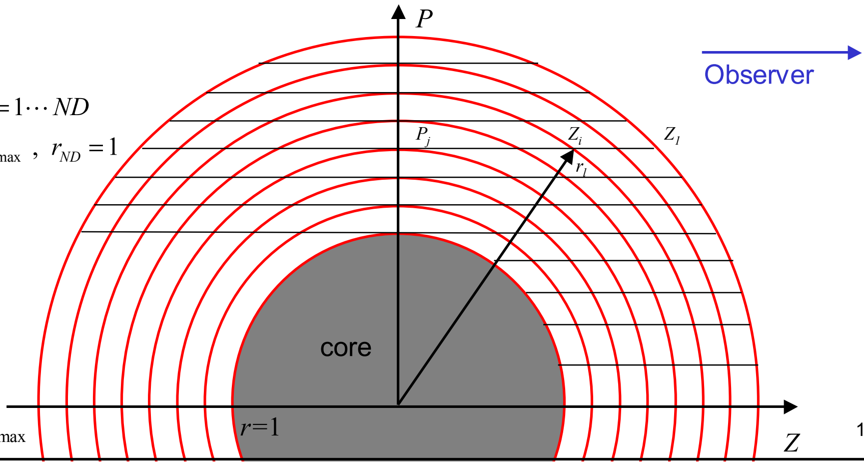
$Z_i=+\sqrt{r_l^2-P_j^2}\ ,\ l=1\cdots NP+1\text{-}j, i=l$

$Z_i=-\sqrt{r_l^2-P_j^2}\ ,\ l=1\cdots NP+1\text{-}j, i=2(NP+1-j)-l$

Radii

$r_l\ ,\ l=1\cdots ND$

$r_1=R_{\max}\ ,\ r_{ND}=1$



Spherical geometry

Numerical integration on this grid, e.g.:

Optical depth

$$\tau_i=\int_{Z_1}^{Z_i}\kappa\left(r=\sqrt{Z_i^2+P_j^2}\right)dZ$$

Emergent intensity

$$I_j^+=\int_{\tau'=0}^{\tau_{\max}}S(\tau')e^{-\tau'}d\tau'=\sum_{i=1}^{i_{\max}}S_l w_i$$

Emergent flux

$$H^+=\frac{1}{2}\int_{P=0}^{R_{\max}}I^+(P)PdP=\frac{1}{2}\sum_{j=1}^{NP}I_j^+w_j$$

Solution as boundary-value problem
Feautrier scheme

Radiation transfer equation along a ray:

$$\pm\frac{dI_\nu^\pm(\tau)}{d\tau} = I_\nu^\pm(\tau) - S_\nu(\tau)$$

$$\text{pp: } d\tau = \kappa \frac{dt}{d\mu}$$

$$\text{sp: } d\tau = -\kappa dZ$$

Two differential equations for inbound and outbound rays

Definitions by Feautrier (1964):

$$u = \frac{1}{2}\left(I^+ + I^-\right) \quad \text{symmetric, intensity-like}$$

$$v = \frac{1}{2}\left(I^+ - I^-\right) \quad \text{antisymmetric, flux-like}$$

Feautrier scheme

Addition and subtraction of both DEQs:

$$\frac{dv(\tau)}{d\tau} = u(\tau) - S_\nu(\tau) \qquad (1)$$

$$\frac{du(\tau)}{d\tau} = v(\tau) \qquad (2)$$

$$\Rightarrow \frac{d^2u(\tau)}{d\tau^2} = u(\tau) - S_\nu(\tau)$$

One DEQ of second order instead of two DEQ of first order

Feautrier scheme

Boundary conditions (pp-case)

Outer boundary

$$I^-(\tau=0)=0 \rightarrow u(\tau=0)=v(\tau=0)$$

$$(2) \quad \Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=0} = u(\tau=0)$$

... with irradiation

$$I^-(\tau=0)=I_0^- \rightarrow u-v=I^-$$

$$\Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=0} = u(\tau=0)-I_0^-$$

Inner boundary

$$I^+(\tau=\tau_{\max})=I^+_{\tau_{\max}} \rightarrow u(\tau_{\max})+v(\tau_{\max})=I^+_{\tau_{\max}}$$

$$(2) \quad \Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=\tau_{\max}} = I^+_{\tau_{\max}} - u(\tau_{\max})$$

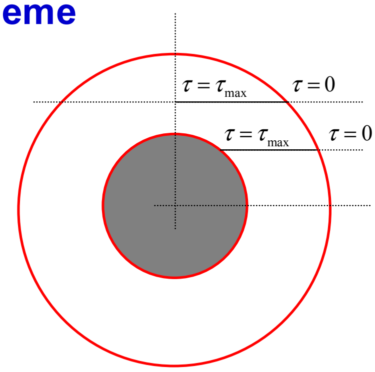
Schuster boundary-value problem

Feautrier scheme

Boundary conditions (spherical)

Core rays:

Like pp-case



Non-core rays:

Restrict to one quadrant (symmetry) inner boundary at Z=0:

$$I^+(\tau=\tau_{\max})=I^-(\tau=\tau_{\max}) \rightarrow v(\tau_{\max})=0$$

$$\Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=\tau_{\max}} = 0$$

Finite differences

Approximation of the derivatives by finite differences:

$$u-\frac{d^2u}{d\tau^2}=S \quad \text{discretization on a } \tau-\text{scale}$$

first derivative at intermediate points:

$$\tau_{i+1/2}=\frac{1}{2}\left(\tau_{i+1}+\tau_i\right)$$

$$\left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i+1/2}}\approx\frac{u_{i+1}-u_i}{\tau_{i+1}-\tau_i} \quad \text{and} \quad \left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i-1/2}}\approx\frac{u_i-u_{i-1}}{\tau_i-\tau_{i-1}}$$

second derivative:

$$\left.\frac{d}{d\tau}\left(\frac{du(\tau)}{d\tau}\right)\right|_{\tau_i}=\frac{\left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i+1/2}}-\left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i-1/2}}}{\tau_{i+1/2}-\tau_{i-1/2}}$$

$$\left.\frac{du^2(\tau)}{d\tau^2}\right|_{\tau_i}\approx\frac{\frac{u_{i+1}-u_i}{\tau_{i+1}-\tau_i}-\frac{u_i-u_{i-1}}{\tau_i-\tau_{i-1}}}{\frac{1}{2}\left(\tau_{i+1}-\tau_{i-1}\right)}$$

Finite differences

Approximation of the derivatives by finite differences:

$$u-\frac{d^2u}{d\tau^2}=S \quad \text{discretisation on a } \tau-\text{scale}$$

$$\Rightarrow u_i-\frac{\frac{u_{i+1}-u_i}{\tau_{i+1}-\tau_i}-\frac{u_i-u_{i-1}}{\tau_i-\tau_{i-1}}}{\frac{1}{2}\left(\tau_{i+1}-\tau_{i-1}\right)}=S_i \quad , \quad i=2\cdots ND-1$$

$$\Rightarrow -A_iu_{i-1}+B_iu_i-C_iu_{i+1}=S_i \quad , \quad i=2\cdots ND-1$$

$$A_i=\left[\frac{1}{2}\left(\tau_i-\tau_{i-1}\right)\left(\tau_{i+1}-\tau_{i-1}\right)\right]^{-1}$$

$$C_i=\left[\frac{1}{2}\left(\tau_{i+1}-\tau_i\right)\left(\tau_{i+1}-\tau_{i-1}\right)\right]^{-1}$$

$$B_i=1+A_i+C_i$$

Discrete boundary conditions

Outer boundary: first order

$$\left.\frac{du}{d\tau}\right|_{\tau_1} = u_1 \rightarrow \frac{u_2 - u_1}{\tau_2 - \tau_1} = u_1$$

$$\rightarrow u_1 - \frac{u_2 - u_1}{\tau_2 - \tau_1} = 0$$

$$\Rightarrow B_1 u_1 - C_1 u_2 = 0$$

$$C_1 = [\tau_2 - \tau_1]^{-1} \quad , \quad B_1 = 1 + C_1$$

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Discrete boundary conditions

Numerically better is a second-order condition:
Taylor expansion of u(τ) around τ_1:

$$u_2 = u_1 + (\tau_2 - \tau_1)\left.\frac{du}{d\tau}\right|_{\tau_1} + \frac{1}{2}(\tau_2 - \tau_1)^2 \left.\frac{d^2u}{d\tau^2}\right|_{\tau_1}$$

boundary condition DEQ

$$u_2 = u_1 + (\tau_2 - \tau_1) u_1 + \frac{1}{2}(\tau_2 - \tau_1)^2 (u_1 - S_1)$$

$$u_1 + 2\frac{u_1}{\tau_2 - \tau_1} + 2\frac{u_1 - u_2}{(\tau_2 - \tau_1)^2} = S_1 \quad \Rightarrow \quad \boxed{B_1 u_1 - C_1 u_2 = S_1}$$

$$C_1 = 2[\tau_2 - \tau_1]^{-2} \quad , \quad B_1 = 1 + 2[\tau_2 - \tau_1]^{-1} + 2[\tau_2 - \tau_1]^{-2}$$

Discrete boundary conditions

Inner boundary: first order

$$\left.\frac{du(\tau)}{d\tau}\right|_{\tau=\tau_{\max}} = \begin{cases} I^+ - u_{ND} & \text{pp or core rays} \\ 0 & \text{non-core rays} \end{cases} \rightarrow \frac{u_{ND-1} - u_{ND}}{\tau_{ND-1} - \tau_{ND}} = \begin{cases} I^+ - u_{ND} \\ 0 \end{cases}$$

$$\rightarrow \left.\begin{matrix} u_{ND} \\ 0 \end{matrix}\right\} + \frac{u_{ND} - u_{ND-1}}{\tau_{ND} - \tau_{ND-1}} = \begin{cases} I^+ \\ 0 \end{cases}$$

$$\Rightarrow -A_{ND} u_{ND-1} + B_{ND} u_{ND} = S^*_{ND}$$

$$A_{ND} = \left[\tau_{ND} - \tau_{ND-1}\right]^{-1} \quad , \quad B_{ND} = \begin{cases} 1 + A_{ND} \\ A_{ND} \end{cases} \quad , \quad S^*_{ND} = \begin{cases} I^+ \\ 0 \end{cases}$$

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Discrete boundary conditions

Outer boundary: second order

$$u_{ND-1} = u_{ND} + \left(\tau_{ND-1} - \tau_{ND}\right) \left.\frac{du}{d\tau}\right|_{\tau_{ND}} + \frac{1}{2}\left(\tau_{ND-1} - \tau_{ND}\right)^2 \left.\frac{d^2u}{d\tau^2}\right|_{\tau_{ND}}$$

boundary condition DEQ

$$u_{ND-1} = u_{ND} - \left(\tau_{ND} - \tau_{ND-1}\right) \left\{\begin{matrix} I^+ - u_{ND} & \text{core} \\ 0 & \text{non-core} \end{matrix}\right\}$$

$$+ \frac{1}{2}\left(\tau_{ND} - \tau_{ND-1}\right)^2 \left(u_{ND} - S_{ND}\right)$$

$$u_{ND} - 2\frac{I^+ - u_{ND}}{\tau_{ND} - \tau_{ND-1}} + 2\frac{u_{ND} - u_{ND-1}}{\left(\tau_{ND} - \tau_{ND-1}\right)^2} = S^*_{ND} \Rightarrow \boxed{B_{ND} u_{ND} - A_{ND} u_{ND-1} = S^*_{ND}}$$

$$A_{ND} = 2\left[\tau_{ND} - \tau_{ND-1}\right]^{-2} \quad , \quad B_{ND} = 1 + 2\left[\tau_{ND} - \tau_{ND-1}\right]^{-1} + 2\left[\tau_{ND} - \tau_{ND-1}\right]^{-2}$$

$$S^*_{ND} = S_{ND} + 2\left[\tau_{ND} - \tau_{ND-1}\right]^{-1} I^+$$

Linear system of equations

$$\begin{bmatrix} B_1 & -C_1 & & & & & \\ -A_2 & B_2 & -C_2 & & 0 & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & 0 & & & \ddots & \ddots & -C_{ND-1} \\ & & & & & -A_{ND} & B_{ND} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \\ \\ u_{ND-1} \\ u_{ND} \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ \\ \\ W_{ND-1} \\ W_{ND} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ \\ \\ S_{ND-1} \\ S^*_{ND} \end{bmatrix}$$

Linear system for u_i

Use Gauss-Jordan elimination for solution

Upper diagonal matrix

1st step:

$$\begin{bmatrix} 1 & -\tilde{C}_1 & & & & \\ & 1 & -\tilde{C}_2 & & 0 & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \\ & 0 & & & \ddots & -\tilde{C}_{ND-1} \\ & & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \\ \\ u_{ND-1} \\ u_{ND} \end{bmatrix} = \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \\ \vdots \\ \\ \\ \tilde{W}_{ND-1} \\ \tilde{W}_{ND} \end{bmatrix}$$

$$i=1 \qquad \tilde{C}_1 = B_1^{-1} C_1 \qquad \tilde{W}_1 = B_1^{-1} W_1$$

$$i=2\cdots ND \qquad \tilde{C}_i = \left(B_i - A_i \tilde{C}_{i-1}\right)^{-1} C_i \qquad \tilde{W}_i = \left(B_i - A_i \tilde{C}_{i-1}\right)^{-1} \left(W_i + A_i \tilde{W}_{i-1}\right)$$

Back-substitution

2nd step:

$$i = ND \qquad u_{ND} = \tilde{W}_{ND}$$
$$i = ND-1\cdots 1 \quad u_i = \tilde{W}_i + \tilde{C}_i u_{i+1}$$

Solution fulfils differential equation as well as both boundary conditions

Remark: for later generalization the matrix elements are treated as matrices (non-commutative)

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Solution with linear dependent source function

Coherent scattering:

General form (complete redistribution)

$$S_\nu = \alpha J_\nu + \beta \quad , \quad J_\nu = \frac{1}{2}\int_{\mu=-1}^{1} I_\nu(\mu)d\mu = \int_{\mu=0}^{1} u_\nu(\mu)d\mu$$

Thomson scattering

$$\kappa_e = n_e\sigma_e \quad , \quad \eta_e = \kappa_e J$$

$$\Rightarrow S^{tot} = \frac{\eta+\eta_e}{\kappa+\kappa_e} = \frac{\kappa}{\kappa+\kappa_e}\frac{\eta}{\kappa} + \frac{\kappa_e}{\kappa+\kappa_e}\frac{\eta_e}{\kappa_e} = \frac{\kappa+\kappa_e-\kappa_e}{\kappa+\kappa_e}S + \frac{\kappa_e}{\kappa+\kappa_e}J$$

$$S^{tot} = (1-\beta_e)S + \beta_e J \quad , \quad \beta_e = \frac{\kappa_e}{\kappa+\kappa_e}$$

Results in coupling of equations for all directions

$$u(\tau,\mu) - \beta_e(\tau)\int_{\mu=0}^{1} u(\tau,\mu)\,d\mu - \frac{d^2u(\tau,\mu)}{d\tau^2} = [1-\beta_e(\tau)]S$$

Discretization

Generalization of Feautrier scheme to a block-matrix scheme:

Angular discretization: $\int\limits_{\mu=0}^{1} u(\tau,\mu)\,d\mu \approx \sum_{j=1}^{NA} u_{ji} w_j$

Depth discretization as previously:

$$-A_i u_{i-1} + B_i u_i - C_i u_{i+1} = W_i$$

equations for vector u_i

Discretization

Identical lines

pp-case identical in all depths

$$u(\tau,\mu) - \beta_e(\tau)\int\limits_{\mu=0}^{1} u(\tau,\mu)\,d\mu - \frac{d^2 u(\tau,\mu)}{d\tau^2} = [1-\beta_e(\tau)]S$$

$$\rightarrow \quad -A_i u_{i-1} + B_i u_i - C_i u_{i+1} = W_i$$

equations for vector u_i

$$u_i = \begin{bmatrix} u_1 \\ \vdots \\ u_j \\ \vdots \\ u_{NA} \end{bmatrix}, A_i = \begin{pmatrix} A_i & & \\ & \ddots & & 0 \\ & & A_i & \\ & 0 & & \ddots \\ & & & & A_i \end{pmatrix}, \ C_i = \begin{pmatrix} C_i & & \\ & \ddots & & 0 \\ & & C_i & \\ & 0 & & \ddots \\ & & & & C_i \end{pmatrix}$$

$$B_i = \begin{pmatrix} B_i & & \\ & \ddots & & 0 \\ & & B_i & \\ & 0 & & \ddots \\ & & & & B_i \end{pmatrix} - \beta_e(i)\begin{pmatrix} w_1 & \cdots & \cdots & \cdots & w_{NA} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ w_1 & \cdots & \cdots & \cdots & w_{NA} \end{pmatrix}, \ W_i = \begin{bmatrix} (1-\beta_e(i))\,S_i \\ \vdots \\ (1-\beta_e(i))\,S_i \\ \vdots \\ (1-\beta_e(i))\,S_i \end{bmatrix}$$

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Block matrix

$$\begin{pmatrix} \begin{bmatrix} B_{1;1,1} & \cdots & B_{1;1,NA} \\ \vdots & \ddots & \vdots \\ B_{1;NA,1} & \cdots & B_{1;NA,NA} \end{bmatrix} & \begin{bmatrix} -C_1 & & \\ & \ddots & \\ & & -C_1 \end{bmatrix} & & & \\ \begin{bmatrix} -A_2 & & \\ & \ddots & \\ & & -A_2 \end{bmatrix} & \begin{bmatrix} B_{2;1,1} & \cdots & B_{2;1,NA} \\ \vdots & \ddots & \vdots \\ B_{2;NA,1} & \cdots & B_{2;NA,NA} \end{bmatrix} & \begin{bmatrix} -C_2 & & \\ & \ddots & \\ & & -C_2 \end{bmatrix} & & \\ & & \ddots & & \\ & & & \begin{bmatrix} B_{ND-1;1,1} & \cdots & B_{ND-1;1,NA} \\ \vdots & \ddots & \vdots \\ B_{ND-1;NA,1} & \cdots & B_{ND-1;NA,NA} \end{bmatrix} & \begin{bmatrix} -C_{ND-1} & & \\ & \ddots & \\ & & -C_{ND-1} \end{bmatrix} \\ & & & \begin{bmatrix} -A_{ND} & & \\ & \ddots & \\ & & -A_{ND} \end{bmatrix} & \begin{bmatrix} B_{ND;1,1} & \cdots & B_{ND;1,NA} \\ \vdots & \ddots & \vdots \\ B_{ND;NA,1} & \cdots & B_{ND;NA,NA} \end{bmatrix} \end{pmatrix} \cdot \begin{bmatrix} \begin{bmatrix} u_{1;1} \\ \vdots \\ u_{1;NA} \end{bmatrix} \\ \begin{bmatrix} u_{2;1} \\ \vdots \\ u_{2;NA} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} u_{ND-1;1} \\ \vdots \\ u_{ND-1;NA} \end{bmatrix} \\ \begin{bmatrix} u_{ND;1} \\ \vdots \\ u_{ND;NA} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} W_{1;1} \\ \vdots \\ W_{1;NA} \end{bmatrix} \\ \begin{bmatrix} W_{2;1} \\ \vdots \\ W_{2;NA} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} W_{ND-1;1} \\ \vdots \\ W_{ND-1;NA} \end{bmatrix} \\ \begin{bmatrix} W_{ND;1} \\ \vdots \\ W_{ND;NA} \end{bmatrix} \end{bmatrix}$$

$$W_{i;j} = (1-\beta_e(i))S_i \quad , \quad i = 1 \cdots ND-1$$

$$W_{ND;j} = (1-\beta_e(ND))S_{ND} + 2\left(\tau_{ND} - \tau_{ND-1}\right)^{-1} I_j^+$$

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Feautrier scheme

Thermal source function

- Decoupled equation for each direction
- Separate Feautrier scheme for each ray
- ND•5 multiplications

Thermal source function + coherent scattering (e.g. Thomson)

- Coupled equations for all directions
- Feautrier scheme in block matrix form
- ND•NA^3 multiplications (matrix inversions)

Spherical geometry

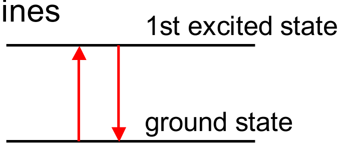
- Radius dependent angular integration $\mu_{j,l} = \left(1 - \frac{P_j^2}{r_l^2}\right)^{1/2} \quad , \; j = 1 \cdots JMAX(l) \quad , \; JMAX(l) = NP + 1 - l$
- Block matrix size depends on radius
- ~ND^4 multiplications

Solution with line scattering

Two-level atom, complete redistribution

Photon conservation within a spectral line

Approximately realized for resonance lines



$$\kappa(\nu) = \kappa_L \varphi(\nu) \quad , \quad \eta(\nu) = \eta_L \varphi(\nu)$$

photon conservation:

$$\int_{Line} \eta(\nu) d\nu = \int_{Line} \kappa(\nu) J_\nu(\nu) d\nu$$

$$\eta_L = \kappa_L \int_{Line} J_\nu(\nu) \varphi_\nu d\nu$$

$$\boxed{S_L = \int_{Line} J_\nu(\nu) \varphi_\nu d\nu}$$

(generalized) scattering intergral $S_L = \alpha \bar{J}_L + \beta$

The frequency independent line source function (☚) is a weighted mean of the mean intensity

Solution with line scattering (non-coherent scattering)

Transfer equation:

$$\mu \frac{dI_\nu(\nu,\mu,\tau)}{d\tau(\nu)} = I_\nu(\nu,\mu,\tau) - S_L(\tau)$$

$$\mu \frac{dI_\nu(\nu,\mu,\tau)}{d\tau(\nu)} = I_\nu(\nu,\mu,\tau) - \frac{1}{2} \int_{Line} \varphi(\nu) \int_{\mu=-1}^{+1} I_\nu(\nu,\mu,\tau) d\mu d\tau$$

Each one equation for each angular and frequency point

Each equation contains intensities from all other points

⇒coupling of all transfer equations to an integro-differential-equation

This system is linear with respect to the intensity

⇒solution with Gauss-Jordan elimination

Solution with line scattering

Discretization:

$$u(\nu,\mu,\tau)-\alpha(\tau)\int_{Line}\varphi(\nu)\int_{\mu=-1}^{+1}u(\nu,\mu,\tau)d\mu d\nu-\frac{d^2u(\nu,\mu,\tau)}{d\tau^2(\nu)}=\beta(\tau)$$

Analogous solution as with coherent scattering

Scattering integral:

$$\bar{J}_L=\int_{\nu_0-\Delta\nu}^{\nu_0+\Delta\nu}J(\nu)\varphi(\nu)d\nu\rightarrow\sum_{k=1}^{NF}J_k w'_k$$

With appropriate weights including the line profile function (e.g. Gauss function)

$$\bar{J}_L\rightarrow\sum_{j=1}^{NA}\sum_{k=1}^{NF}u_{jk}w_j w'_k$$

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Linear system of equations

$$\begin{bmatrix} B_1 & -C_1 & & & & 0 \\ -A_2 & B_2 & -C_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots & -C_{ND-1} \\ & & & & -A_{ND} & B_{ND} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{ND-1} \\ u_{ND} \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_{ND-1} \\ W_{ND} \end{bmatrix}$$

Linear system for u_i

Use Gauss-Jordan elimination for solution

Block matrix

$$u_i - \alpha_i \sum_{j=1}^{NA}\sum_{k=1}^{NF} u_{jk} w_j w'_k - \frac{\dfrac{u_{i+1}-u_i}{\tau_{i+1}-\tau_i} - \dfrac{u_i-u_{i-1}}{\tau_i-\tau_{i-1}}}{\frac{1}{2}\left(\tau_{i+1}-\tau_{i-1}\right)} = \beta_i$$

no longer identical, depend on k but not on j

$$u_i = \begin{bmatrix} u_1 \\ \vdots \\ u_j \\ \vdots \\ u_{NA} \end{bmatrix} \leftarrow u_j = \begin{bmatrix} u_{j,1} \\ \vdots \\ u_{j,k} \\ \vdots \\ u_{j,NF} \end{bmatrix}, \quad A_i = \begin{pmatrix} A_{ijk} & & \\ & \ddots & 0 \\ & A_{ijk} & \\ 0 & \ddots & \\ & & A_{ijk} \end{pmatrix}, \quad C_i = \begin{pmatrix} C_{ijk} & & \\ & \ddots & 0 \\ & C_{ijk} & \\ 0 & \ddots & \\ & & C_{ijk} \end{pmatrix}$$

$$B_i = \begin{pmatrix} B_{ijk} & & \\ & \ddots & 0 \\ & B_{ijk} & \\ 0 & \ddots & \\ & & B_{ijk} \end{pmatrix} - \alpha(i) \left(\begin{bmatrix} w_1 w'_1 & \cdots & w_1 w'_{NF} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ w_1 w'_1 & \cdots & w_1 w'_{NF} \end{bmatrix} \cdots \begin{bmatrix} w_{NA} w'_1 & \cdots & w_{NA} w'_{NF} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ w_{NA} w'_1 & \cdots & w_{NA} w'_{NF} \end{bmatrix} \right), \quad W_i = \begin{bmatrix} \beta(i) \\ \vdots \\ \beta(i) \\ \vdots \\ \beta(i) \end{bmatrix}$$

$$B_{ijk} = 1 + A_{jk} + C_{ijk}$$

Identical lines

Solution with line scattering

The modified Feautrier scheme is not very efficient

pp: inversion of matrices of rank NA•NF ~ 3 •5

  ND•NA^3 •NF^3 multiplications

sp: inversion of matrices of rank NA•NF ~ 70 •5 ⚡

  ND•ND^3 •NF^3 multiplications

Repeated calculation of identical scattering integrals at each frequency point

Rybicki scheme

Rybicki (1971)

Coherent scattering, e.g. Thomson scattering $S_\nu^{tot} = (1-\beta_e) S_\nu + \beta_e J_\nu$

Again: Feautrier transfer equation:

$$u(\tau,\mu) - \beta_e(\tau) J(\tau) - \frac{d^2 u(\tau,\mu)}{d\tau^2} = [1-\beta_e(\tau)] S(\tau)$$

Discretization:

$$-A_{i,j} u_{i-1,j} + B_{i,j} u_{i,j} - C_{i,j} u_{i+1,j} - \beta_i^e J_i = \left[1-\beta_i^e\right] S_i$$

$$j = 1 \cdots NA \text{ (pp)} \;,\; j = 1 \cdots JMAX \text{ (sp)}$$

$$i = (1), 2 \cdots ND-1, (ND)$$

A_i, B_i, C_i like in usual Feautrier scheme

But now: additional variable J_i

With additional equations $J_i = \sum_j u_{i,j} w_{i,j}$

Use alternative grouping

Rybicki scheme

$$\begin{pmatrix} T_1 & & & & U_1 & \\ & \ddots & 0 & & \vdots & \\ & & T_j & & U_j & \\ & 0 & & \ddots & \vdots & \\ & & & T_{NA} & U_{NA} & \\ & & & & -1 & 0 \\ W_1 & \cdots & W_j & \cdots\; W_{NA} & & \ddots \\ & & & & 0 & -1 \end{pmatrix} \begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_j \\ \vdots \\ \vec{u}_{NA} \\ J_1 \\ \vdots \\ J_{ND} \end{bmatrix} = \begin{bmatrix} \vec{K}_1 \\ \vdots \\ \vec{K}_j \\ \vdots \\ \vec{K}_{NA} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vec{u}_j = \begin{bmatrix} u_{1,j} \\ \vdots \\ u_{i,j} \\ \vdots \\ u_{ND,j} \end{bmatrix} \qquad \vec{K}_j = \begin{bmatrix} (1-\beta_1^e) S_1 \\ \vdots \\ (1-\beta_i^e) S_i \\ \vdots \\ (1-\beta_{ND}^e) S_{ND} \end{bmatrix}$$

$$W_j = \begin{pmatrix} w_{1,j} & & & \\ & \ddots & 0 & \\ & & w_{i,j} & \\ & 0 & & \ddots \\ & & & & w_{ND,j} \end{pmatrix}$$

$$T_j = \begin{pmatrix} B_{1,j} & -C_{1,j} & & \\ \ddots & & 0 & \\ -A_{i,j} & B_{i,j} & -C_{i,j} & \\ 0 & & \ddots & \\ & & -A_{ND,j} & B_{ND,j} \end{pmatrix} \qquad U_j = \begin{pmatrix} -\beta_1^e & & & \\ & \ddots & 0 & \\ & & -\beta_i^e & \\ & 0 & & \ddots \\ & & & -\beta_{ND}^e \end{pmatrix}$$

$T_j \vec{u}_j$ represents $u - \frac{d^2 u}{d\tau^2}$, $U_j \vec{J}$ describes $-\beta_e J$ for the ray $j \rightarrow T_j \vec{u}_j + U_j \vec{J} = \vec{K}_j$

Scattering integral is represented by last block line $\sum_{j=1}^{NA} W_j \vec{u}_j - \vec{J} = 0$

Rybicki scheme

1st step: diagonal elements to unit matrixes

$$\left(\begin{array}{ccccc|c} 1 & & & & & \tilde{U}_1 \\ & \ddots & 0 & & & \vdots \\ & & 1 & & & \tilde{U}_j \\ & 0 & & \ddots & & \vdots \\ & & & & 1 & \tilde{U}_{NA} \\ \hline & & & & & -1 \qquad 0 \\ W_1 & \cdots & W_j & \cdots & W_{NA} & \ddots \\ & & & & & 0 \qquad -1 \end{array}\right) \begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_j \\ \vdots \\ \vec{u}_{NA} \\ J_1 \\ \vdots \\ J_{ND} \end{bmatrix} = \begin{bmatrix} \vec{\tilde{K}}_1 \\ \vdots \\ \vec{\tilde{K}}_j \\ \vdots \\ \vec{\tilde{K}}_{NA} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\tilde{U}_j = T_j^{-1} U_j \ , \ \vec{\tilde{K}}_j = T_j^{-1} \vec{K}_j$$

Corresponds to the solution of NA tri-diagonal systems of equations of rank ND → cpu-time ~ NA•ND^2

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Rybicki scheme

2nd step: W_j to zero

$$\begin{pmatrix} 1 & & & & & \tilde{U}_1 \\ & \ddots & 0 & & & \vdots \\ & & 1 & & & \tilde{U}_j \\ & 0 & & \ddots & & \vdots \\ & & & & 1 & \tilde{U}_{NA} \\ 0 & \cdots & 0 & \cdots & 0 & W \end{pmatrix} \begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_j \\ \vdots \\ \vec{u}_{NA} \\ \vec{J} \end{bmatrix} = \begin{bmatrix} \vec{\tilde{K}}_1 \\ \vdots \\ \vec{\tilde{K}}_j \\ \vdots \\ \vec{\tilde{K}}_{NA} \\ \vec{Q} \end{bmatrix}$$

$$W = -1 - \sum_{j=1}^{NA} W_j \tilde{U}_j \ , \ \vec{Q} = -\sum_{j=1}^{NA} W_j \vec{\tilde{K}}_j$$

Corresponds to cpu-time ~ NA•ND^2

Rybicki scheme

3rd step: solve for J

$$W\vec{J} = \vec{Q} \rightarrow \vec{J} = W^{-1}\vec{Q}$$

Corresponds to cpu-time ~ ND^3
Finished if only J is required

4th step: solve for Feautrier variables u:

$$\vec{u}_j = T_j^{-1}\vec{K}_j - T_j^{-1}\vec{U}_j\vec{J}$$

Comparison Rybicki vs. Feautrier

Thomson scattering

	Feautrier	Rybicki
Plane-parallel	C NA^3 ND	C_1 NA ND^2+ $C_2 ND^3$
Spherical	C ND^4	C_1 NP ND^2+ $C_2 ND^3$

Few angular points: take Feautrier
Many angular points: take Rybicki

Solution with line scattering (non-coherent scattering)

Two-level atom or Compton scattering $S_L = \alpha \bar{J}_L + \beta$

Each one block line $T_{jk}\vec{u}_{jk} + U_{jk}\vec{J} = \vec{K}_{jk}$

describes transfer equation for each ray j,k

(direction and frequency dependent!)

$\rightarrow$ Huge system of equations

$$\begin{pmatrix} T_1 & & & & & & U_1 & \\ & \ddots & & 0 & & & \vdots & \\ & & T_{jk} & & & & U_{jk} & \\ & 0 & & \ddots & & & \vdots & \\ & & & & T_{NANF} & & U_{NANF} & \\ & & & & & -1 & & 0 \\ W_1 & \cdots & W_{jk} & \cdots & W_{NANF} & & \ddots & \\ & & & & & 0 & & -1 \end{pmatrix} \begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_{jk} \\ \vdots \\ \vec{u}_{NANF} \\ J_1 \\ \vdots \\ J_{ND} \end{bmatrix} = \begin{bmatrix} \vec{K}_1 \\ \vdots \\ \vec{K}_{jk} \\ \vdots \\ \vec{K}_{NANF} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Solution with line scattering (non-coherent scattering)

$$\begin{pmatrix} T_1 & & & & & & U_1 & \\ & \ddots & & 0 & & & \vdots & \\ & & T_{jk} & & & & U_{jk} & \\ & 0 & & \ddots & & & \vdots & \\ & & & & T_{NANF} & & U_{NANF} & \\ & & & & & -1 & & 0 \\ W_1 & \cdots & W_{jk} & \cdots & W_{NANF} & & \ddots & \\ & & & & & 0 & & -1 \end{pmatrix} \begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_{jk} \\ \vdots \\ \vec{u}_{NANF} \\ J_1 \\ \vdots \\ J_{ND} \end{bmatrix} = \begin{bmatrix} \vec{K}_1 \\ \vdots \\ \vec{K}_{jk} \\ \vdots \\ \vec{K}_{NANF} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vec{u}_{jk} = \begin{bmatrix} u_{1,j} \\ \vdots \\ u_{ij} \\ \vdots \\ u_{ND,j} \end{bmatrix} \qquad \vec{K}_{jk} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_i \\ \vdots \\ \beta_{ND} \end{bmatrix}$$

$$T_{jk} = \begin{pmatrix} B_{1,jk} & -C_{1,jk} & & \\ & \ddots & 0 & \\ -A_{i,jk} & B_{i,jk} & -C_{i,jk} & \\ 0 & & \ddots & \\ & & -A_{ND,jk} & B_{ND,jk} \end{pmatrix}$$

$$U_{jk} = \begin{pmatrix} -\alpha_1 & & \\ \ddots & & 0 \\ & -\alpha_i & \\ 0 & \ddots & \\ & & -\alpha_{ND} \end{pmatrix}$$

$$W_{jk} = \begin{pmatrix} w_{i,j}w'_k & & \\ \ddots & & 0 \\ & w_{i,j}w'_k & \\ 0 & \ddots & \\ & & w_{ND,j}w'_k \end{pmatrix}$$

$$J_i = \int J_\nu(\tau_i)\varphi(\nu)d\nu$$

Comparison Rybicki vs. Feautrier

Line scattering or non-coherent scattering, e.g. Compton scattering

	Feautrier	Rybicki
Plane-parallel	$C\ NA^3\ NF^3\ ND$	$C_1\ NA\ NF\ ND^2 + C_2 ND^3$
Spherical	$C\ NF^3\ ND^4$	$C_1\ NP\ NF\ ND^2 + C_2 ND^3$

Few frequency points: take Feautrier or Rybicki
Many frequency points: take Rybicki
Spherical symmetry: take Rybicki

Variable Eddington factors

0-th moment $J = \frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)\,d\mu = \int_{\mu=0}^{1} u(\mu)\,d\mu$

1st moment $H = \frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)\mu\,d\mu = \int_{\mu=0}^{1} v(\mu)\mu\,d\mu$

2nd moment $K = \frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)\mu^2\,d\mu = \int_{\mu=0}^{1} u(\mu)\mu^2\,d\mu$

Eddington factor: $f = K / J$

0-th moment of RT (pp) $\frac{dH_\nu(\nu,\tau)}{d\tau(\nu)} = J_\nu(\nu,\tau) - S_\nu(\nu,\tau)$

1st moment of RT (pp) $\frac{dK_\nu(\nu,\tau)}{d\tau(\nu)} = H_\nu(\nu,\tau)$

Variable Eddington factors

Plane-parallel $\frac{d^2 K_\nu(\nu,\tau)}{d\tau^2(\nu)} = J_\nu(\nu,\tau) - S_\nu(\nu,\tau)$

With variable Eddington factor $\frac{d^2\left[f(\nu,\tau)J_\nu(\nu,\tau)\right]}{d\tau^2(\nu)} = J_\nu(\nu,\tau) - S_\nu(\nu,\tau)$

With given f and S 2nd-order DEQ for J

Outer boundary:

$$h(\tau=0) = H(\tau=0)/J(\tau=0)$$

$$I^-(\tau=0) \rightarrow u(\tau=0) = v(\tau=0)$$

$$\rightarrow h(\tau=0) = \int_{\mu=0}^{1} u(\mu)\mu d\mu \Big/ \int_{\mu=0}^{1} u(\mu) d\mu$$

$$\left.\frac{d\left[f(\nu,\tau)J_\nu(\nu,\tau)\right]}{d\tau(\nu)}\right|_{\tau=0} = h(\tau=0)J_\nu(\nu,\tau=0)$$

Variable Eddington factors

Inner boundary:

$$h(\tau=\tau_{\max}) = H(\tau=\tau_{\max})/J(\tau=\tau_{\max})$$

$$h(\tau=\tau_{\max}) = \underbrace{\int_{\mu=0}^{1} u(\mu)\mu d\mu}_{\neq H!} \Big/ \underbrace{\int_{\mu=0}^{1} u(\mu) d\mu}_{=J}$$

$$\tau=\tau_{\max}: \quad H = \int_{\mu=0}^{1} v(\mu)\mu d\mu = \int_{\mu=0}^{1} I^+(\mu)\mu d\mu - \int_{\mu=0}^{1} u(\mu)\mu d\mu = H^+ - hJ$$

$$\left.\frac{d\left[f(\nu,\tau)J_\nu(\nu,\tau)\right]}{d\tau(\nu)}\right|_{\tau=\tau_{\max}} = H_\nu^+(\tau=\tau_{\max}) - h(\tau=\tau_{\max})J_\nu(\nu,\tau=\tau_{\max})$$

2nd-order boundary conditions from Taylor series

Solution

Discretization → algebraic equation

$$J_\nu(\nu,\tau) - \frac{d^2\left[f(\nu,\tau)J_\nu(\nu,\tau)\right]}{d\tau^2(\nu)} = S_\nu(\nu,\tau)$$

$$\rightarrow -A_i J_{i-1} + B_i J_i - C_i J_{i+1} = S_i$$

Tri-diagonal system, solution analogous to Feautrier scheme ☚

Thomson scattering:

$$S^{tot} = (1-\beta_e)S + \beta_e J$$

$$\rightarrow (1-\beta_e)J_\nu(\nu,\tau) - \frac{d^2\left[f(\nu,\tau)J_\nu(\nu,\tau)\right]}{d\tau^2(\nu)} = (1-\beta_e)S_\nu(\nu,\tau)$$

$$\rightarrow -A_i J_{i-1} + (B_i - \beta_i^e)J_i - C_i J_{i+1} = (1-\beta_i^e)S_i$$

Possible without extra costs ☺

Variable Eddington factors

But the Eddington factors are unknown → iteration

1. Formal solution u with S=B
2. Start value $f_i \equiv 1/3$
3. Solution of the moment equation for $J \rightarrow S^{tot}$
4. Formal solution u with given S $\rightarrow K$
5. New Eddington factors f

   $$f_i = \sum_{j=1}^{NA} u_{i,j} w_j \Big/ \sum_{j=1}^{NA} u_{i,j} w'_j$$

   w_j contains μ^2 , w'_j contains μ
6. Converged?

Variable Eddington factors

0-th moment $J=\frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)d\mu=\int_{\mu=0}^{1} u(\mu)d\mu \ , \ \tilde{J}=r^2 J$

1st moment $H=\frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)\mu d\mu=\int_{\mu=0}^{1} v(\mu)\mu d\mu \ , \ \tilde{H}=r^2 H$

2nd moment $K=\frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)\mu^2 d\mu=\int_{\mu=0}^{1} u(\mu)\mu^2 d\mu \ , \ \tilde{K}=r^2 K$

0-th moment of RT (sp)

$$\frac{d\tilde{H}_\nu(\nu,r)}{dr}=\kappa(\nu,r)\left(\underbrace{\tilde{S}_\nu(\nu,r)}_{=r^2 S}-\tilde{J}_\nu(\nu,r)\right)$$

1st moment of RT (sp)

$$\frac{dK_\nu(\nu,r)}{dr}+\frac{1}{r}\left[3K_\nu(\nu,r)-J_\nu(\nu,r)\right]=-\kappa(\nu,r)H_\nu(\nu,r)$$

Variable Eddington factors

1st moment of RT (sp)

$$\frac{dK_\nu(\nu,r)}{dr}+\frac{1}{r}\left[3K_\nu(\nu,r)-J_\nu(\nu,r)\right]=-\kappa(\nu,r)H_\nu(\nu,r)$$

Eddington factor

$$\frac{d(fJ)}{dr}+\frac{J}{r}\left[3f-1\right]=-\kappa H$$

Introduction of the sphericality factor (Auer 1971),

$$r^2 q(r)=\exp\left(\int_1^r \frac{3f(r')-1}{r'f(r')}dr'\right) \ , \ \frac{d\left(r^2 q(r)\right)}{dr}=r^2 q(r)\frac{3f(r)-1}{rf(r)}$$

which corresponds to the integrating factor for the DEQ

$$y'+f(x)y=g(x) \ , \ M(x)=\exp\left(\int f(x)dx\right)$$

Variable Eddington factors

$$\frac{d\left(r^2 q(r) K\right)}{dr} = \frac{d\left(r^2 q(r)\right)}{dr} K + r^2 q(r) \frac{dK}{dr}$$

$$= r^2 q(r) \left[\frac{3f-1}{rf} fJ + \frac{d(fJ)}{dr} \right] = r^2 q(r) \left(-\kappa H \right)$$

1st moment equation:

$$\frac{d\left(r^2 q(r) K\right)}{r^2 q(r) dr} = -\kappa H \rightarrow \frac{d\left(qf\tilde{J}\right)}{q\kappa dr} = -\tilde{H}$$

$$\xrightarrow[dx=-q\kappa dr]{}$$

$$\left.\begin{array}{l} \dfrac{d\tilde{H}}{dx} = \dfrac{1}{q}\left(\tilde{J} - \tilde{S}\right) \\ \dfrac{d\left(qf\tilde{J}\right)}{dx} = \tilde{H} \end{array}\right\} \frac{d^2\left(qf\tilde{J}\right)}{dx^2} = \frac{1}{q}\left(\tilde{J} - \tilde{S}\right)$$

Solution

Iteration of Eddington factors like in pp case

Additional integration of sphericality factor

$$q_i = r_i^{-2} \exp\left(\sum_{l=i}^{ND} \frac{3f_l - 1}{f_l} w_l \right) \; , \; w_l \text{ include weights for } \frac{dr}{r}$$

f=1/3 is a bad starting point, f→1 for r →∞

Computational demand much smaller than for formal solution

Non-coherent scattering and moment equation

Two-level atom or Compton scattering $S_L = \alpha \bar{J} + \beta$

For each frequency point one moment equation of 2nd order for mean intensity and Eddington factor $J_\nu(\nu_k)\ ,\ f_{ik}(\tau_i,\nu_k)$

Coupled by frequency integral $\bar{J} = \int J_\nu \varphi(\nu) d\nu \rightarrow \bar{J} = \sum_{k=1}^{NF} J_k w_k$

pp $$J_\nu(\tau,\nu) - \frac{d^2\left(f(\tau,\nu)J_\nu(\tau,\nu)\right)}{d\tau^2(\nu)} - \alpha\int_\nu J_\nu \varphi(\nu)d\nu = \beta(\tau)$$

sp $$\tilde{J}_\nu(r,\nu) - q(r,\nu)\frac{d^2\left(q(r,\nu)f(r,\nu)\tilde{J}_\nu(r,\nu)\right)}{dx^2(\nu)} - \alpha\int_\nu \tilde{J}_\nu \varphi(\nu)d\nu$$

$$= \tilde{\beta}(r)$$

Non-coherent scattering and moment equation

Feautrier: $$\vec{J}_i = [J_{i1},\cdots,J_{ik},\cdots,J_{iNF}]^T \ ,\ i = 1\cdots ND$$

cpu-time~ND•NF³ $$-A_i\vec{J}_{i-1} + B_i\vec{J}_i - C_i\vec{J}_{i+1} = \vec{\beta}_i$$

$$\frac{d^2(fJ)}{d\tau^2} \qquad \bar{J}_i = \sum_{k=1}^{NF} J_{ik} w_k$$

Rybicki: $$\vec{J}_k = [J_{1k},\cdots,J_{ik},\cdots,J_{NDk}]^T \ ,\ k = 1\cdots NF$$

$$\vec{\bar{J}} = \left[\bar{J}_1,\cdots,\bar{J}_i,\cdots,\bar{J}_{ND}\right]^T$$

$$T_k\vec{J}_k + U_k\vec{\bar{J}} = K_k \ ,\ \sum_{k=1}^{NF} W_k\vec{J}_k - \vec{\bar{J}} = 0$$

cpu-time~NF•ND²+ND³

Multi-level atom

Atmospheric structure assumed to be given
(or accounted for by iteration)

Two sets of equations:

Radiative transfer equation for mean intensity

$$J_\nu(\tau,\nu) - \frac{d^2\left(f(\tau,\nu)J_\nu(\tau,\nu)\right)}{d\tau^2(\nu)} = S_\nu(\tau,\nu)$$

$$S_\nu(\tau,\nu) = \sum_{lu} \eta_\nu^{lu} \Big/ \sum_{lu} \kappa^{lu}(\nu) \quad \text{sum over all bb-, bf-, and ff-transitions}$$

Statistical equilibrium

$$P(J_\nu)\vec{n} = \vec{b} \quad , \quad \vec{n} = \left[n_1 \cdots n_{NL}\right]^T$$

Both equations are coupled via radiative rates in matrix P

Multi-level atom

Coupling of J twofold:

- Over frequency via SE
- Over depth via RT

$\rightarrow$ Simultaneous solution

$\rightarrow$ non-linear in J

Lambda Iteration:

0. start approximation for n
1. formal solution with given S
2. solution of statistical equilibrium with given J
3. converged?

Not convergent for large optical depths

Complete Linearization

Auer & Mihalas 1969

Newton-Raphson method in $\mathbb{R}^n$

Solution according to Feautrier scheme

Unknown variables:

$$\vec{\psi}_i = \begin{bmatrix} \vec{J}_i \\ \vec{n}_i \end{bmatrix} \;,\; i = 1 \cdots ND \qquad \psi = [\vec{\psi}_1, \cdots, \vec{\psi}_i, \cdots, \vec{\psi}_{ND}]^T$$

Equations:

$$-A_{i,k} J_{i-1,k} + B_{i,k} J_{i,k} - C_{i,k} J_{i+1,k} - S_{i,k}(\vec{n}_i) = 0 \qquad \text{NF transfer equations}$$

$$P(\vec{J}_i)\vec{n}_i - \vec{b}_i = 0 \qquad \text{NL equations for SE}$$

System of the form:

$$f_{i,\alpha}(\psi) = 0 \;,\; \alpha = 1 \cdots NF + NL$$

Complete Linearization

Start approximation: $f_{i,\alpha}(\psi^0) \neq 0$

Now looking for a correction so that

$$f_{i,\alpha}(\psi^0 + \delta\psi) = 0 \;\; \forall \; i, \alpha$$

Taylor series:

$$0 = f_{i,\alpha}(\psi) = f_{i,\alpha}(\psi^0 + \delta\psi)$$

$$= f_{i,\alpha}(\psi^0) + \sum_{i=1}^{ND} \left\{ \sum_{k=1}^{NF} \frac{\partial f_{i,\alpha}}{\partial J_{i,k}} \delta J_{i,k} + \sum_{l=1}^{NL} \frac{\partial f_{i,\alpha}}{\partial n_{i,l}} \delta n_{i,l} \right\} \Bigg|_{\psi^0} + \cdots$$

Linear system of equations for ND(NF+NL) unknowns $\delta J_{i,k} \;,\; \delta n_{i,l}$

Converges towards correct solution

Many coefficients vanish

Complete Linearization - structure

Only neighbouring depth points (2nd order transfer equation with tri-diagonal depth structure and diagonal statistical equations): $f_{i,\alpha}(\psi) = f_{i,\alpha}(\vec{\psi}_{i-1}, \vec{\psi}_i, \vec{\psi}_{i+1})$

Results in tri-diagonal block scheme (like Feautrier)

$$-A_i \delta\vec{\psi}_{i-1} + B_i \delta\vec{\psi}_i - C_i \delta\vec{\psi}_{i+1} = \vec{L}_i$$

$$-\left(\begin{array}{ccc|c} \ddots & 0 & & \\ & A_{i,k} & & 0 \\ 0 & & \ddots & \\ \hline & 0 & & 0 \end{array}\right)\left[\begin{array}{c} \delta\vec{J}_{i-1} \\ \hline \delta\vec{n}_{i-1} \end{array}\right] + \left(\begin{array}{ccc|ccc} \ddots & 0 & & \square & \square & \square \\ & B_{i,k} & & \square & \square & \square \\ 0 & & \ddots & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array}\right)\left[\begin{array}{c} \delta\vec{J}_{i} \\ \hline \delta\vec{n}_{i} \end{array}\right]$$

$$-\left(\begin{array}{ccc|c} \ddots & 0 & & \\ & C_{i,k} & & 0 \\ 0 & & \ddots & \\ \hline & 0 & & 0 \end{array}\right)\left[\begin{array}{c} \delta\vec{J}_{i+1} \\ \hline \delta\vec{n}_{i+1} \end{array}\right] = -\left[\begin{array}{c} \square \\ \square \\ f_{i,\alpha}(\psi^0) \\ \square \\ \square \\ \square \end{array}\right]$$

Complete Linearization - structure

Transfer equations: coupling of $J_{i-1,k}$, $J_{i,k}$, and $J_{i+1,k}$ at the same frequency point:

$\rightarrow$ Upper left quadrants of A_i, B_i, C_i describe 2nd derivative $\frac{d^2 J}{d\tau^2}$

Source function is local:

$\rightarrow$ Upper right quadrants of A_i, C_i vanish

Statistical equations are local

$\rightarrow$ Lower right and lower left quadrants of A_i, C_i vanish

Complete Linearization - structure

Matrix B_i:

$$
B_i = \left(\begin{array}{ccc|ccc}
\ddots & & 0 & & \vdots & \\
 & B_{i,k} & & \cdots & -\dfrac{\partial S_{i,k}}{\partial n_{i,l'}} & \cdots \\
0 & & \ddots & & \vdots & \\
\hline
 & \vdots & & & \vdots & \\
\cdots & \displaystyle\sum_{m=1}^{NL} \frac{\partial \left(P_i\right)_{l,m}}{\partial J_{i,k'}} n_{i,m} & \cdots & \cdots & \left(P_i\right)_{l,l'} & \cdots \\
 & \vdots & & & \vdots &
\end{array}\right)
$$

Columns: 1 … NF, 1 … NL; rows: 1 … NF, 1 … NL

Complete Linearization

Alternative (recommended by Mihalas): solve SE first and linearize afterwards: $P(\vec{J}_i)\vec{n}_i - \vec{b}_i = 0 \;\rightarrow \vec{n}_i = P(\vec{J}_i)^{-1}\vec{b}_i$

Newton-Raphson method:

- Converges towards correct solution
- Limited convergence radius
- In principle quadratic convergence, however, not achieved because variable Eddington factors and τ-scale are fixed during iteration step
- CPU~ND (NF+NL)^3 $\rightarrow$ simple model atoms only
  - Rybicki scheme is no relief since statistical equilibrium not as simple as scattering integral