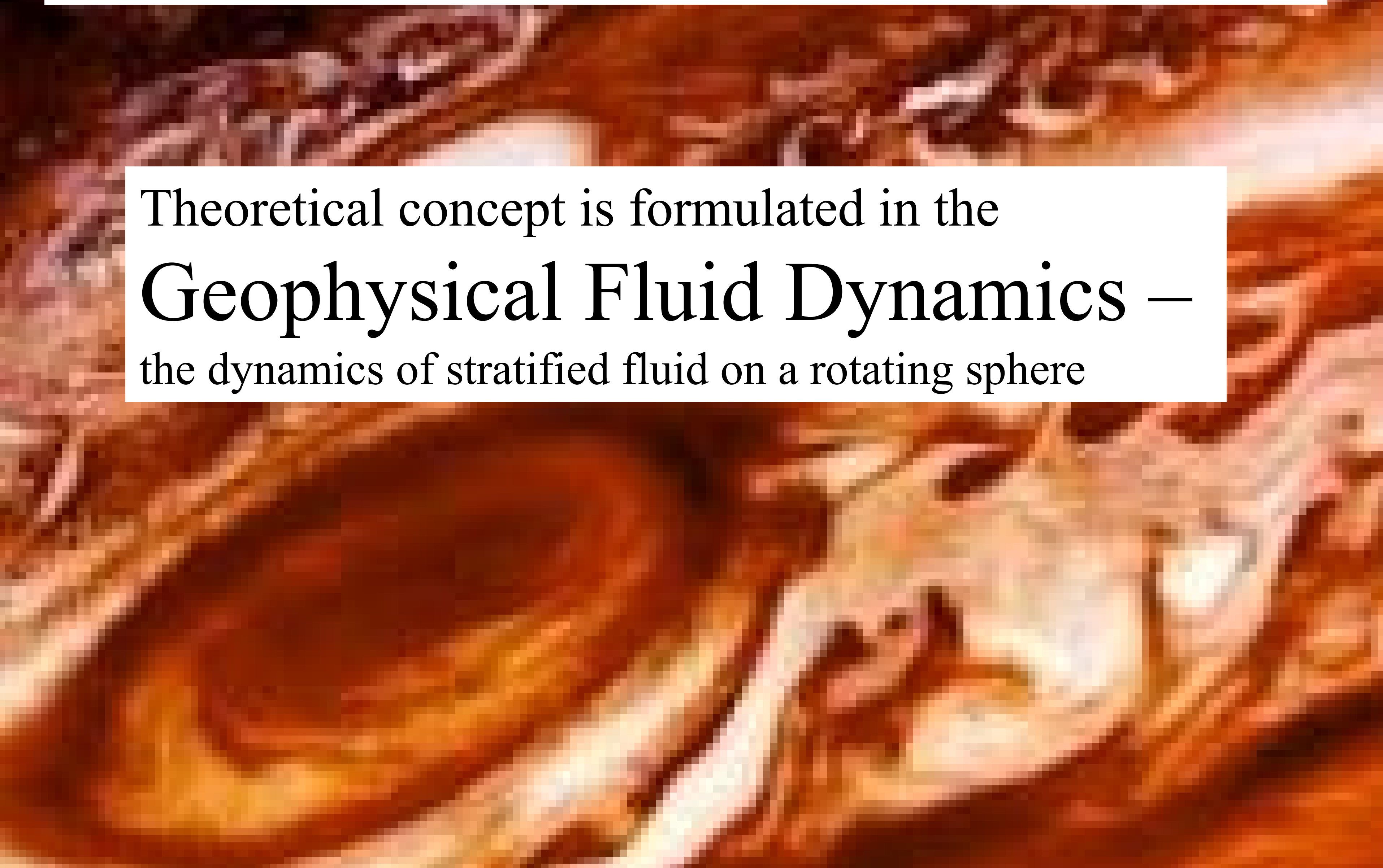
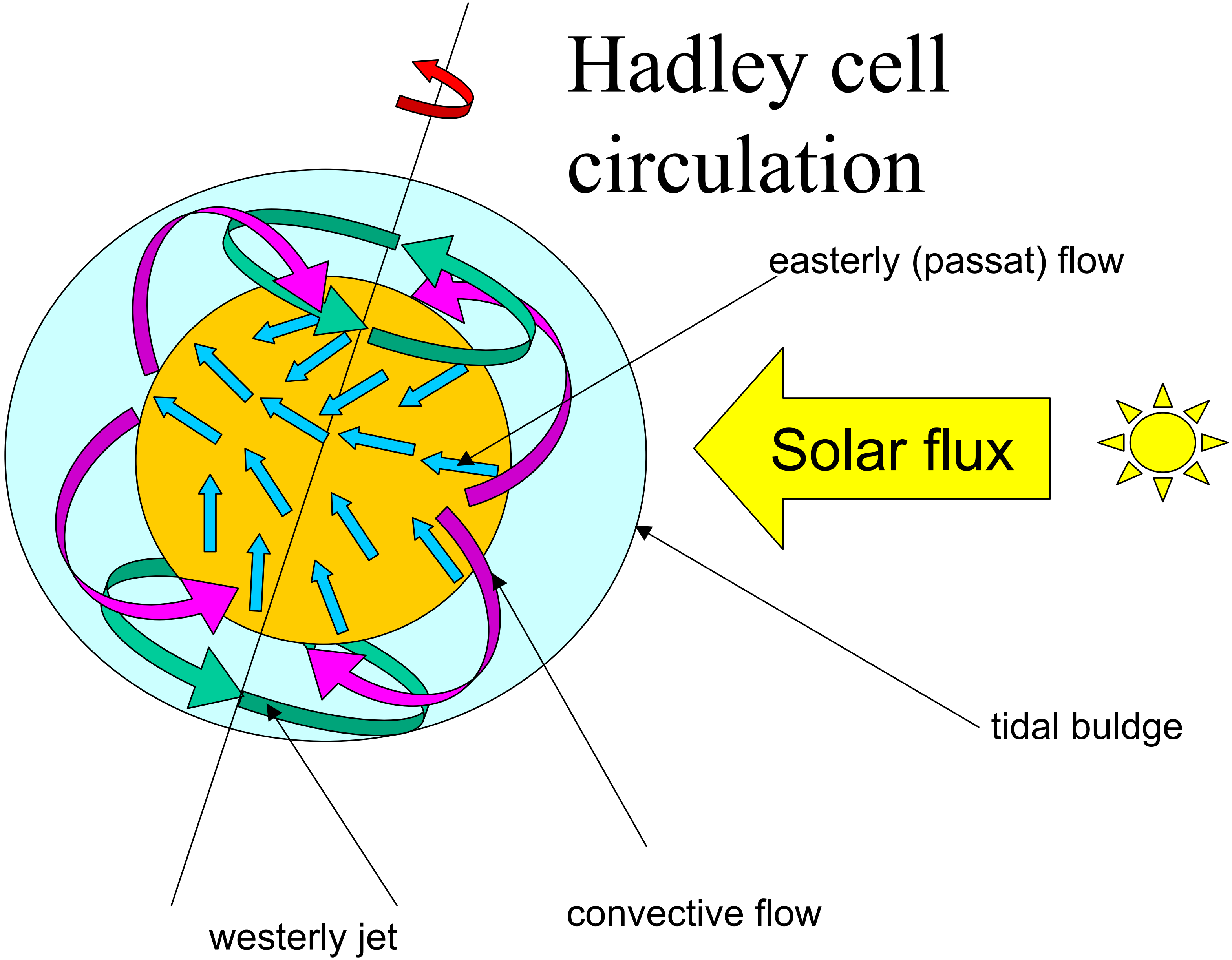
The dynamics of planetary atmospheres



Hadley cell circulation



"Primitive" equations: fluid on a rotating sphere

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\alpha\nabla p + \nu\nabla^2\mathbf{v} - 2\boldsymbol{\omega}\times\mathbf{v} + g,$$

momentum

$$C_V\frac{dT}{dt} + p\frac{d\alpha}{dt} = q + f,$$

energy

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0,$$

continuity

$$p = \rho RT.$$

eqn. of state

modifications

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\rho} + \Phi\right) + (\Omega + 2\omega)\times\mathbf{v} = \nu\nabla^2\mathbf{v}$$

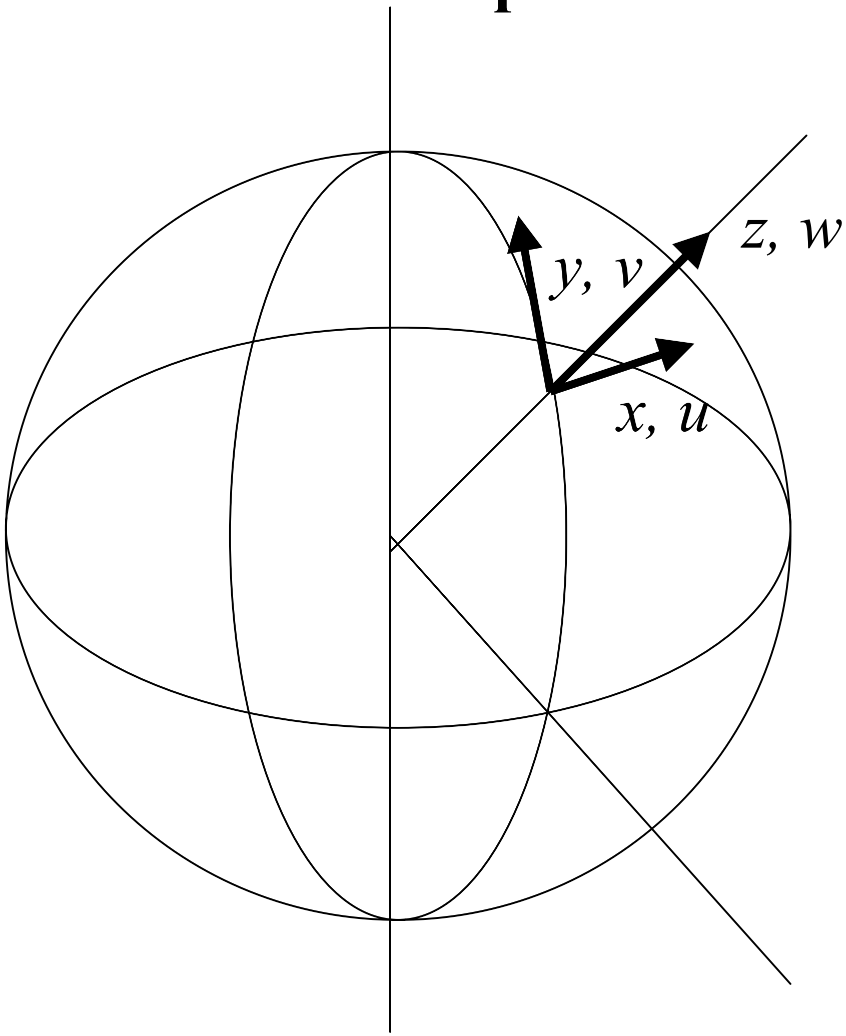
$\nabla\Phi = -g, \Phi -$ **geopotential**

$(\Omega + 2\omega), \Omega = \mathrm{rot}\,\mathbf{v}$ **absolute vorticity**

Boussinesq approximation: $\frac{d\rho}{dt} = 0$ everywhere but gravity

$$\frac{d\mathbf{v}}{dt} = -\nabla p - 2\rho\boldsymbol{\omega}\times\mathbf{v} + \rho g$$

Spherical coordinate system:



$$\frac{du}{dt} = -\alpha \frac{\partial p}{\partial x} - 2\omega(w \sin\theta - v\cos\theta)$$

$$\frac{dv}{dt} = -\alpha \frac{\partial p}{\partial y} - 2\omega u \cos\theta$$

$$\frac{dw}{dt} = -g - \alpha \frac{\partial p}{\partial z} + 2\omega u \cos\theta$$

Useful numbers

$f = 2\omega\cos\theta = 2\omega\sin\varphi$ **Coriolis parameter**

$f \approx 2\omega\sin\varphi_0 + \frac{2\omega}{R}\cos\varphi_0\ y$ **β-plane approximation**

$\beta = \frac{2\omega}{R}\cos\varphi_0$ **Rossby parameter**

Rossby number $\mathrm{Ro} = \frac{V}{fL}$

Adjustments

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + w\frac{\partial \mathbf{u}}{\partial z} + f\mathbf{k} \times \mathbf{u} = -\frac{\nabla p}{\rho}$$

equation of horizontal motion

$$f\,\mathbf{k} \times \mathbf{u} \approx -\frac{\nabla p}{\rho},$$

$$\mathbf{u}_g = \frac{1}{\rho f}\mathbf{k} \times \nabla p$$

geostrophic approximaton

$$\frac{dw}{dt} + fu\,\mathrm{ctg}\,\varphi + g = -\alpha\frac{\partial p}{\partial z},$$

$$g \approx -\alpha\frac{\partial p}{\partial z}.$$

hydrostatic approximation

thermal wind equation

$$\frac{\partial}{\partial z}\mathbf{u}_g = \frac{R}{fp}\left(\frac{\partial T}{\partial z}\mathbf{k}\times\nabla p - \frac{\partial p}{\partial z}\mathbf{k}\times\nabla T\right) \approx \frac{g}{fT}\frac{\partial p}{\partial z}\mathbf{k}\times\nabla T$$

Opposite case: Ro >> 1

$$u_c \approx \left(-\alpha\, R_c\,\frac{\partial p}{\partial n}\right)^{1/2}.$$

cyclostrophic approximation

Wave motion in the atmosphere

•Elasticity	sound
•Buoyancy	gravity waves
•Coriolis force	inertial waves

Planetary-scale forced oscillations of any type are called tides

acoustic waves

$$\frac{d\mathbf{v}}{dt} = -\alpha\,\nabla p,\; \frac{dp}{dt} = -\gamma\; p\nabla\mathbf{v}$$

$$p_0 = const,\, \alpha_0 = const,\, \mathbf{v}_0 = 0$$

$$\frac{\partial^2 p}{dt^2} = -\gamma\,\alpha_0 p_0 \nabla^2 p.$$

$$if \quad u_0 \neq 0,$$

$$\left(\frac{\partial}{dt} + u_0\,\frac{\partial}{dx}\right)^2 p = -\gamma\,\alpha_0 p_0 \nabla^2 p,$$

$$c = u_0\,\frac{k_x}{k} \pm \sqrt{\gamma\,\alpha_0 p_0.}$$

gravity waves

$$\frac{d^2 z}{dt^2} = g\,\frac{\theta - \theta_0}{\theta_0},$$

θ-potential temperature

$$\theta(z) \approx \theta_0(0) + z\,\frac{d\theta}{dz}.$$

$$\frac{d^2 z}{dt^2} + \frac{g}{\theta}\frac{d\theta}{dz} z = 0.$$

$$\omega_g^2 = \frac{g}{\theta}\frac{d\theta}{dz}$$

Brunt-Vajsala frequency

For non-vertically propagating waves, $\omega^2 = \omega_g^2 \frac{k_H^2}{k_V^2}$

Group velocity $c_g = \frac{\omega_g}{k_V^2}\left(-\mathbf{i}k_V + \mathbf{k}k_H\right).$

Rossby waves

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} - fv + \frac{\partial}{\partial x}\frac{p}{\rho} = 0$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + fv + \frac{\partial}{\partial y}\frac{p}{\rho} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Search for the solution in form

$$\frac{u}{\widetilde{u}(y)} = \frac{v}{\widetilde{v}(y)} = \frac{P}{\widetilde{P}(y)} = Ae^{i(\omega t + kx)}.$$

$$i(\omega + kU)\widetilde{u} - f\widetilde{v} + ik\widetilde{P} = 0$$

$$i(\omega + kU)\widetilde{v} + f\widetilde{u} + \frac{\partial \widetilde{P}}{\partial y} = 0$$

$$ik\widetilde{u} + \frac{\partial \widetilde{v}}{\partial y} = 0.$$

Eliminating *P* and use incompressibility,

$$(\omega + kU)\left(-\frac{\partial^2 \tilde{v}}{\partial y^2} + k^2 \tilde{v}\right) - k\beta\,\tilde{v} = 0,$$

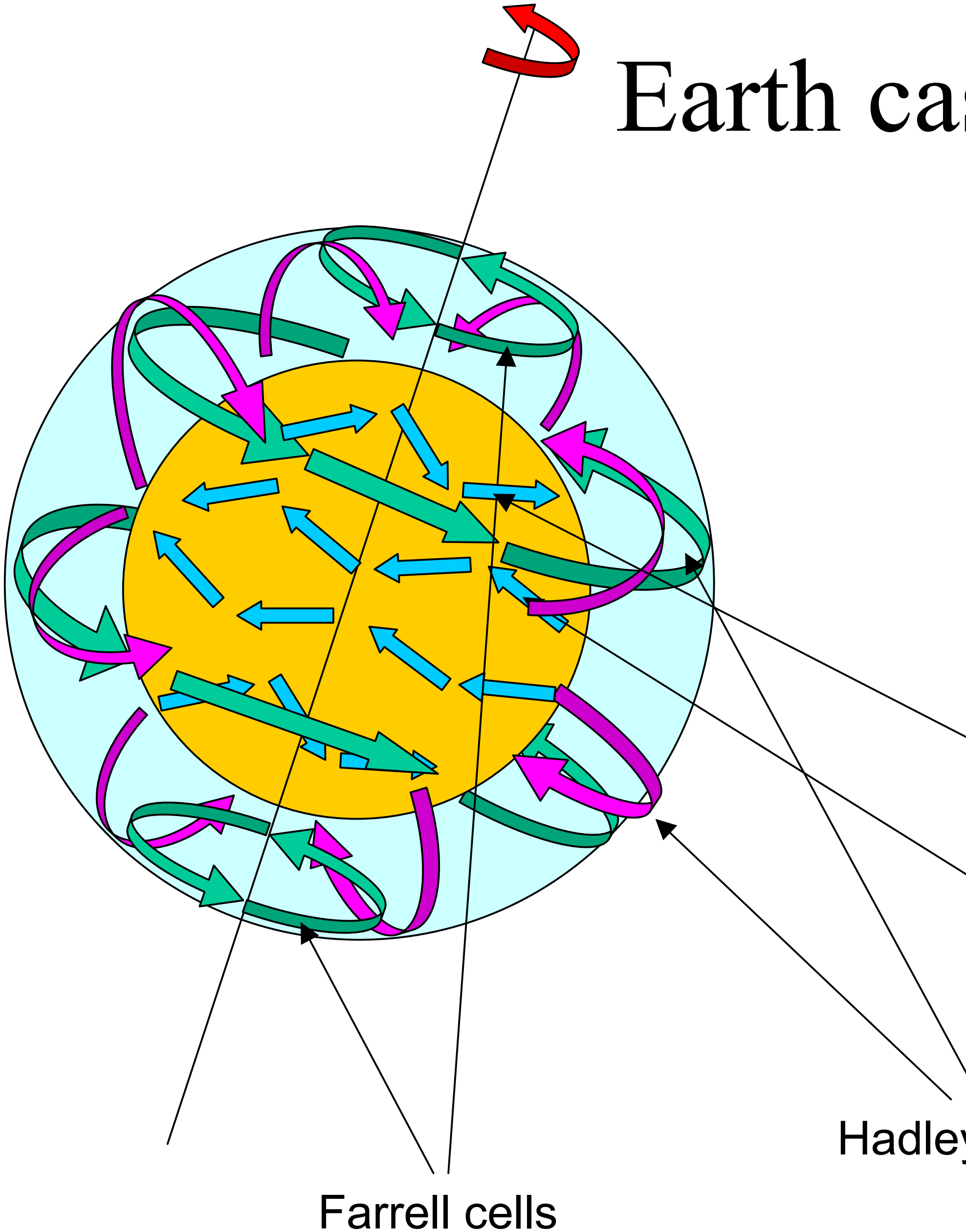
or

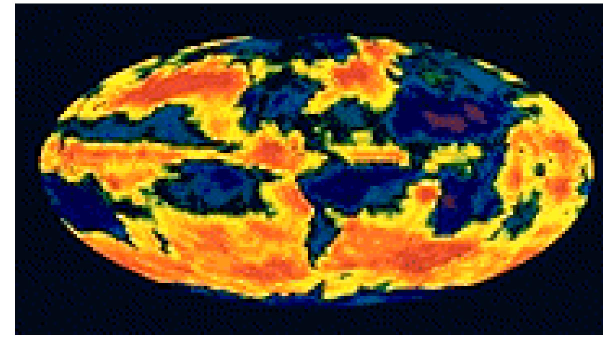
$$\frac{\partial^2 \tilde{v}}{\partial y^2} + \left(\frac{\beta}{U-c} - k^2\right)\tilde{v} = 0,$$

general solution is

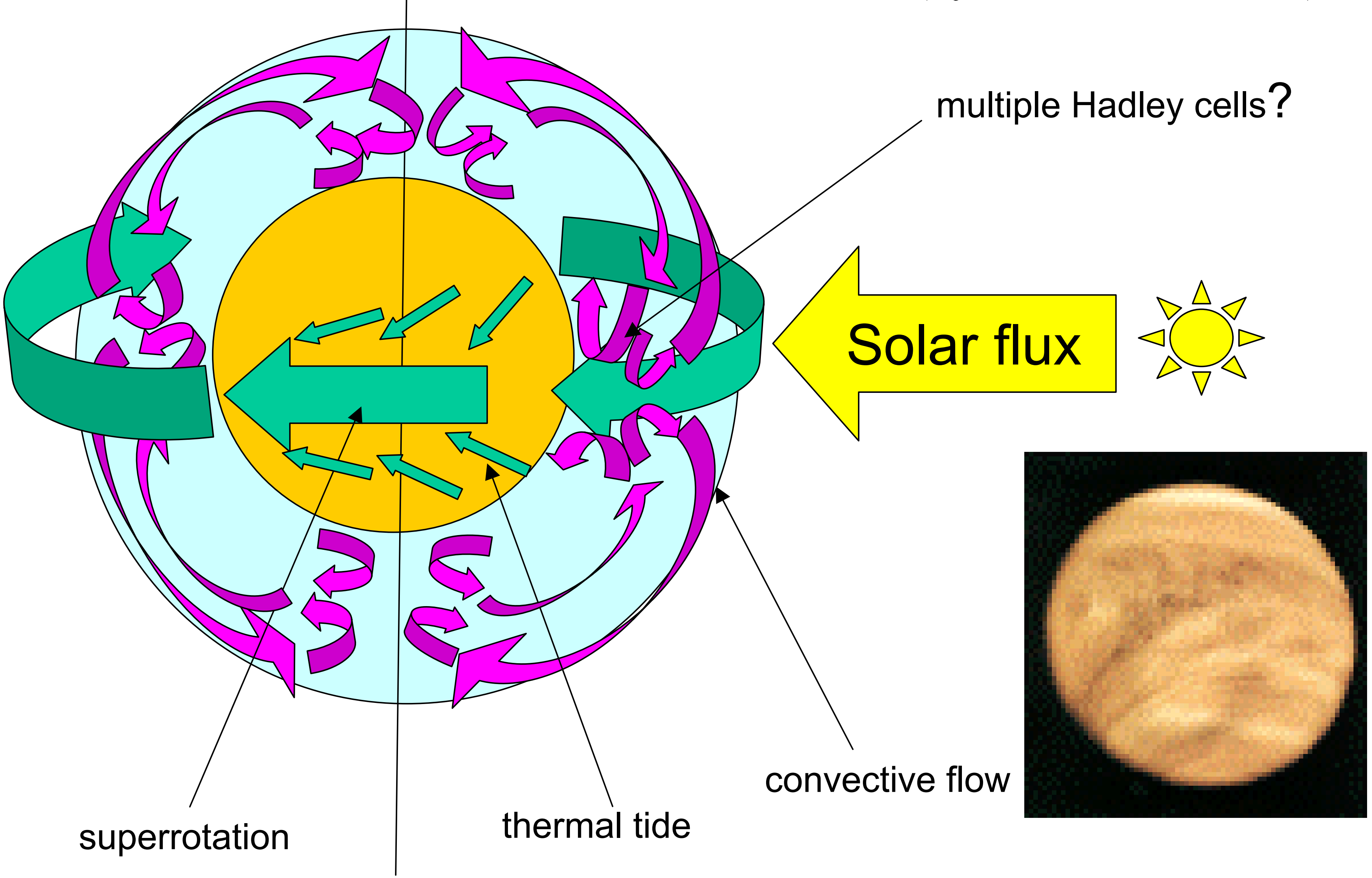
$$\tilde{v} = C_1 \cos\left[\left(\frac{\beta}{U-c} - k^2\right)^{1/2} y\right] + C_2 \sin\left[\left(\frac{\beta}{U-c} - k^2\right)^{1/2} y\right].$$

Earth case (wave mode)

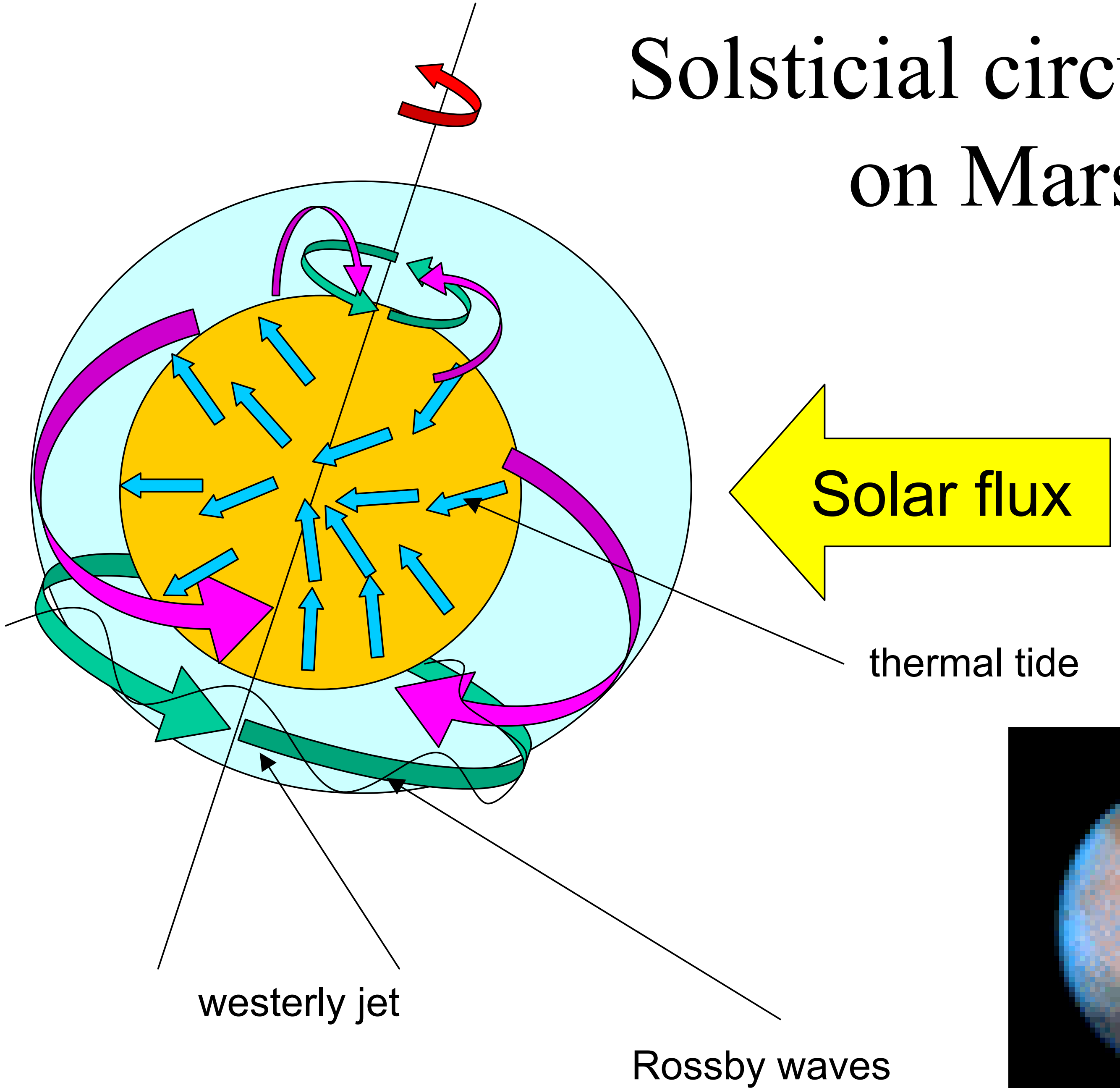


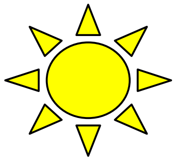


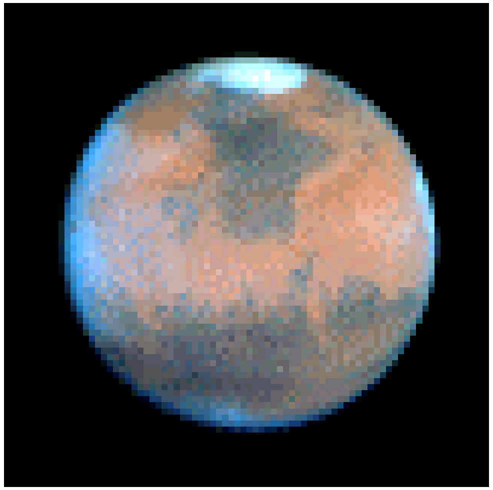
Venus case (symmetric mode)



Solsticial circulation on Mars







Giant planets